2004-2005 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 4: Optics

Problem 1.

A beam of light from a point source *S* passes the motionless glass prism along the side with the length ***l*** and hit the wall.

1) Will the beam reach the wall faster or slower if the prism moves to the wall with speed ***v***<<***c***, where ***c*** is the speed of light in free space.

2) What is the difference in times of propagation of a beam in the cases of motionless and moving prism if index of refraction of the glass is ***n***?

3) ***Bonus question (3 additional points)***. What is the solution when speed ***v*** is of the same order of magnitude as ***c*** (relativistic problem)?

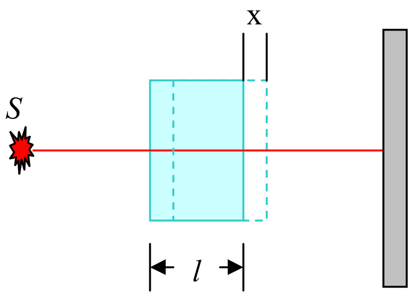


Fig.1

Solution

(a) Rough approximation for nonrelativistic solution

Speed of light in a glass is

$$c_p = \frac{c}{n}$$

where c is speed of light in vacuum, n is the index of refraction of a glass. Light passes the glass prism in time

$$t = \frac{l}{c_p} = \frac{l \cdot n}{c}$$

As the speed of the prism is much less than the speed of light in free space, the same time is elapsed in the laboratory frame of reference.

In the same time interval, the prism is displaced by:

$$x = vt = \frac{nvl}{c} \qquad (1)$$

This is a decrease in the distance of light in the air in comparison with the motionless prism problem. Thus, when the prism moves to the wall, the beam of light reaches the wall faster by the time:

$$\Delta t = \frac{x}{c} = \frac{nvl}{c^2} \qquad (2)$$

(b) More accurate nonrelativistic solution

We need to start with the relativistic formula for the speed of light in the laboratory frame of reference when light propagates in the medium that moves with speed v relatively to the laboratory frame of reference. In general it is not correct to replace a speed v' of any object in the moving frame of reference by the speed of light in this frame of reference in the formula:

$$v_{lab} = \frac{v' + v}{1 + \frac{v'v}{c^2}}$$

It is possible only if the speed of light in the medium is nonrelativistic! It means that the index of refraction of the medium $n >> 1$. Unfortunately this condition was missed in the original text of this problem. We will consider it for both nonrelativistic and relativistic solutions.

For the speed of light in the prism relatively to the laboratory frame of reference we can write:

$$c_{lab} = \frac{c_p + v}{1 + \frac{c_p v}{c^2}} = \frac{\frac{c}{n} + v}{1 + \frac{v}{nc}}$$

Due to the nonrelativistic speed *v* and condition for *n* the second term in the denominator is much less than 1, and we can take two first terms in the Taylor('s) series for the denominator:

$$c_{lab} = \left(\frac{c}{n} + v\right)\left(1 + \frac{v}{nc}\right)^{-1} = \left(\frac{c}{n} + v\right)\left(1 - \frac{v}{nc}\right) = \frac{c}{n} + v - \frac{v}{n^2} - \frac{v^2}{nc} \approx \frac{c}{n} + v$$

Actually, for our conditions we obtained the classic velocity addition in one dimension. To find *x* we have to solve an equation similar to the eq.(1):

$$t = \frac{x}{v} = \frac{l + x}{\frac{c}{n} + v}$$

The results coincide with formulae (1) and (2):

$$x = \frac{nvl}{c}, \text{ and the time decrease for the moving prism is } \Delta t = \frac{nvl}{c^2}$$

(c) Relativistic solution

Let us consider that in the moving frame of reference (prism) a beam of light enters the prism at the moment $t'_1 = 0$ at the point with $x'_1 = 0$, and exits the prism at the moment t'_2 at the point $x'_2 = l$.

In the laboratory (motionless) frame of reference the corresponding time moments are:

$$t_1 = \frac{t'_1 + \frac{v}{c^2} \cdot 0}{\sqrt{1 - \beta^2}} = 0; \qquad t_2 = \frac{t'_2 + \frac{v}{c^2} l}{\sqrt{1 - \beta^2}}; \qquad \beta = \frac{v}{c}$$

The corresponding positions are

$$x_1 = 0; \qquad x_2 = \frac{x'_2 + vt'_2}{\sqrt{1 - \beta^2}} = \frac{l + n\beta l}{\sqrt{1 - \beta^2}} = \frac{l(1 + n\beta)}{\sqrt{1 - \beta^2}}$$

The time elapsed by the beam of light to pass the prism in the laboratory frame of reference is

$$t = t_2 - t_1 = \frac{t'_2 - t'_1 + \frac{v}{c^2} l}{\sqrt{1 - \beta^2}} = \frac{\frac{nl}{c} + \frac{vl}{c^2}}{\sqrt{1 - \beta^2}} = \frac{nl + l\beta}{c\sqrt{1 - \beta^2}} = \frac{l(n + \beta)}{c\sqrt{1 - \beta^2}}$$

In this case the difference in time is:

$$\Delta t = \frac{nl}{c} + t_{air} - t - \frac{t_{air}c - x}{c} = \frac{nl}{c} - t + \frac{x_2 - l}{c} = \frac{l(n-1)}{c}\left(1 - \sqrt{\frac{1-\beta}{1+\beta}}\right),$$

where t_{air} is the time elapsed by the beam to cover the distance between the motionless prism and the wall. For the moving prism the time is again less than for the motionless one.

Problem 2.

A cylindrical container with mercury uniformly rotates around the vertical axis with angular speed ω. Astronomers use the surface of mercury in this experiment as a mirror.

1) What is the shape of the mercury surface?

2) At what position should a photo film be put to get a clear picture of a distant star?

Solution

1) A surface of a liquid in a rotating cylinder has a shape of a paraboloid. Let us examine a little piece of surface with mass m that is at the distance x from the axis of rotation. (Fig.2). This part of liquid experiences a force of gravity and a force of “reaction” $\boldsymbol{R}$ from the liquid. This force is perpendicular to the surface, because the pressure on the surface is constant.

For the chosen piece of liquid a centripetal force is:

$$F_c = m\omega^2 x$$

From the vector diagram for forces we can find the tangent of the denoted angle α :

$$\tan(\alpha) = \frac{\omega^2 x}{g}$$

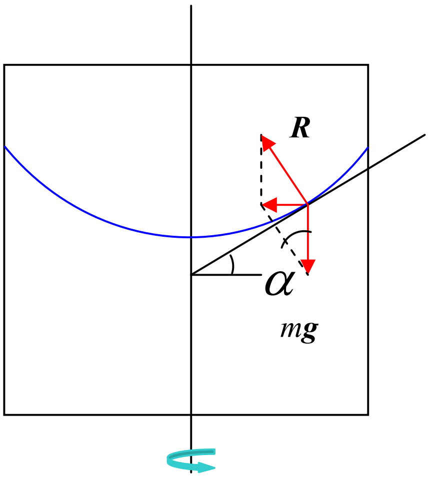


Fig.2

The same tangent has a tangent line to the curve in the same point of surface with coordinates x and y. That is why:

$$y' = \frac{dy}{dx} = \frac{\omega^2 x}{g}$$

A general solution for this equation is: $y = (\omega^2 / 2g)x^2 + const$. We obtained an equation for parabola.

2) Evidently, we must place a photo film at the focus of parabola. It is convenient to choose an origin for the system of coordinates at the vertex of parabola. In this case, the equation is following:

$$y = (\omega^2 / 2g)x^2$$

and the position of the focus on y-axis is upper than the vertex of parabola by $\frac{g}{2\omega^2}$.

Problem 3.

A plane wave with the wavelength λ = 0.5 mm is incident perpendicularly on a barrier in which there are four parallel slits of the width d = 0.2 mm each. The distance between slits is D = 5 mm.

1) Find the direction on the first-order minimum.

2) Suppose one of the internal slits was closed. Calculate the new direction on the first-order minimum.

Solution

1) Fig.3 (a) shows a diagram of an experiment. Any diffraction grid pattern gives main maxima, between which one can find *N-1* supplementary minima separated by the secondary maxima; *N* is the number of slits, terms "maximum" and "minimum" mean the increase and decrease of intensity at these positions.

The direction on the main minima is given by:

$$d\sin\varphi = \pm n\lambda,\ n= 1, 2, 3,\dots \qquad (1)$$

The direction on the supplementary minima is given by:

$$(d+D)\sin\varphi = \pm\frac{\lambda}{N}, \pm\frac{2\lambda}{N}, \pm\frac{3\lambda}{N}, \dots \pm\frac{(N-1)\lambda}{N}, \pm\frac{(N+1)\lambda}{N}, \dots \qquad (2)$$

The first-order minimum will be a supplementary minimum, as it is closer to the central maximum. The direction on the first-order minimum is given by the equation (2)

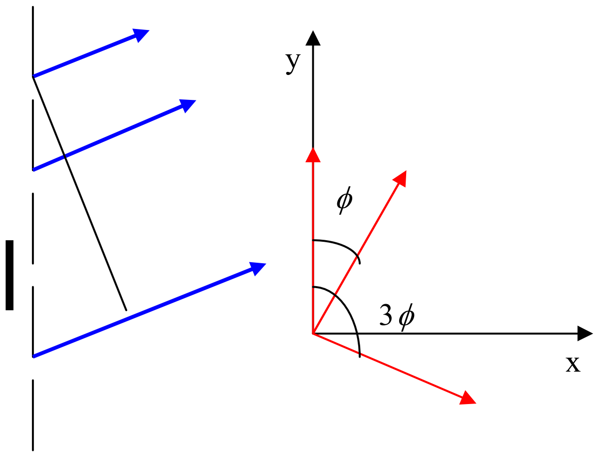
$$\varphi_1 = \sin^{-1}\left(\frac{\lambda}{(d+D)N}\right) = \sin^{-1}(0.024) = 1.38^\circ$$

2) Let us close one of the slits, for instance the third one, as it is shown in the diagram.

We can consider each slit a point source of light. The condition for the destructive or constructive interference can be written using the path difference for two or mare light beams and it can also be expressed through the phase difference between two waves. Suppose the phase difference for two waves from the first and from the second slit is ϕ. The phase difference from the first and the forth slit is 3ϕ. The phasor diagram for the waves is shown in fig.3 (b). To calculate whether the intensity increases or decreases at any point, we will consider the intensity of each slit the same, say 1. For any point we have to calculate a sum of vectors of the wave amplitude. Let us add them with their x- and y-components:

x: $E_x = \sin\phi + \sin 3\phi$

y: $E_y = 1 + \cos\phi + \cos 3\phi$



(a) (b)

Fig.3

Intensity is directly proportional to the E^2. For the intensity we obtain:

$$I = (1+\cos\phi+\cos 3\phi)^2 + (\sin\phi+\sin 3\phi)^2 =$$
$$= 1+\cos^2\phi+\cos^2 3\phi+2\cos\phi+2\cos 3\phi+2\cos\phi\cos 3\phi+\sin^2\phi+\sin^2 3\phi+2\sin\phi\sin 3\phi =$$
$$= 3+2(\cos\phi+\cos 2\phi+\cos 3\phi).$$

We want to find a minimum of this expression. It is possible to do graphically or to calculate a derivative and equate it with zero:

$$(\cos\phi+\cos 2\phi+\cos 3\phi)' = -\sin\phi-2\sin 2\phi-3\sin 3\phi = 0$$

If $\sin\phi$ is not zero, we obtain the next equation:

$6\cos^2\phi+2\cos\phi-1=0$

Its solution is:

$$\cos\phi = \frac{-1\pm\sqrt{7}}{6}$$

$$\cos\phi_1 = -0.6;\ \phi_1 = 127^\circ$$

$$\cos\phi_2 = 0.275;\ \phi_2 = 74^\circ$$

The second angle is closer to the zero direction that is why we should choose it for the first minimum. For the obtained phase difference (angle ϕ_2) we can calculate the spatial angle to the corresponding direction:

$$\varphi_2 = \sin^{-1}\left(\frac{\phi_2\lambda}{2\pi(d+D)}\right) = 1.13^\circ$$

Problem 4.

A light wave, as any other electromagnetic wave, is called the linearly or plane-polarized, if vector $\vec{E}$ and the direction of propagation of the wave always lie in the same plane. This plane is called the *plane of polarization*. A material is said to be optically active if it rotates the plane of polarization of the light wave transmitted through the material. One of such substances is a water solution of an ordinary sugar. Angle of rotation of the plane of polarization φ is directly proportional to the length l of the light path in such solution:

$$\varphi = \alpha l$$

α is a rotation constant for this substance. If a beam of plane-polarized light after passing through the optically active substance reflects back and passes through the same substance in the opposite direction, its plane of polarization returns to the initial state.

There is another source of rotation of the plane of polarization – magnetic field in some substances. It is called a Faraday rotation. The angle of rotation ϕ in the magnetic field B is also proportional to the length of the path l of a light beam in a magnetic field:

$$\phi = \beta l B$$

β is a constant. The direction of rotation depends on the direction of magnetic field and does not depend on the direction of propagation of the light wave.

A beam of plane-polarized light passes through the right-rotating substance that exists in the magnetic field ***B***, as it is shown in Fig.4. The length of the container in the direction of propagation is l. Constants α and β are known.

Find the total angle of rotation of the plane of polarization for the exit beam.

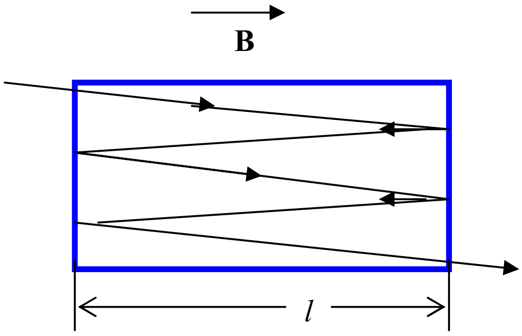


Fig.4

Solution

In the vessel with the optically active solution the beam of light has five tracks. On the “odd” tracks from the left to the right the solution rotates the plane of polarization in the beam clockwise by the angle of $\Delta\varphi_{sol}$. On the “even” tracks the solution also rotates the plane of polarization clockwise, but relatively to the opposite direction of propagation. That is why, for the exit beam the angle of turn of the plane of polarization in the substance is

$$\varphi_{sol} = 3\Delta\varphi_{sol} - 2\Delta\varphi_{sol} = \Delta\varphi_{sol} = \alpha l$$

The direction of vector ***B*** in the light beam does not change in the magnetic field. That is why the angle of rotation of the plane of polarization in the magnetic field is

$$\varphi_{mag} = 5\beta B l$$

The total angle of rotation of the plane of polarization in our problem is:

$$\varphi = \varphi_{sol} + \varphi_{magn} = (5\beta B - \alpha) l$$

Experiment

Determine index of refraction of water using a thin stick, glass with water and a ruler. Describe your experiment in detail and submit the description with a drawing. Perform at least 5 measurements and calculate the mean and statistical (random) error of the series of experiments. Compare the result of your experiment with the tabulated value (n = 1.34), and calculate the absolute and percent difference. Make a conclusion on the precision of your method.

Solution

There is a number of experimental methods to obtain the index of refraction with this set of tools. One of them is explained with the help of fig.5.

Fill a glass with water put inside a thin stick and denote on the wall of the glass a position of the apparent point of contact between the wall and the stick.

Then pour out water and measure the depth of the point where they really contact. To construct an image of this point our eye needs at least two rays. Fig.6 shows the apparent and real incidence of these two rays from the points of contact on the water surface.

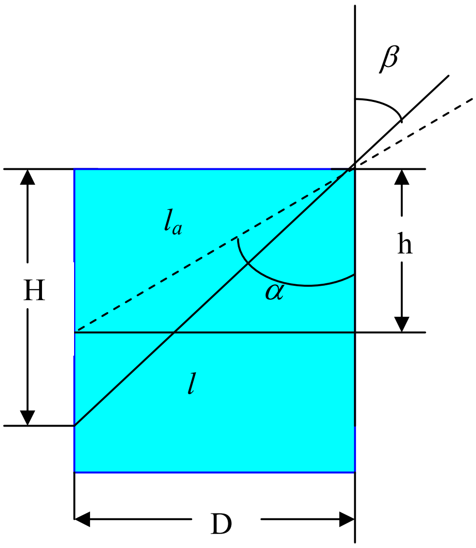


Fig.5

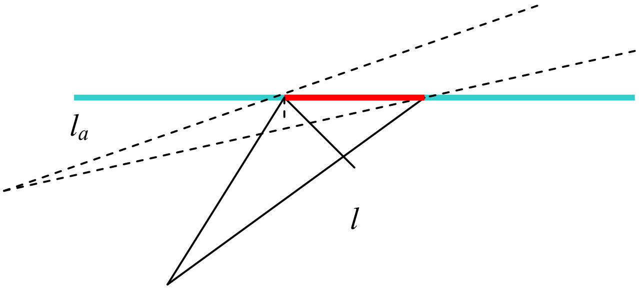


Fig.6

From this figure we can obtain the relationship:

$$\frac{l \cdot d\beta}{\cos\beta} = \frac{l_a d\alpha}{\cos\alpha}; \quad l = \frac{H}{\cos\beta}; \quad l_a = \frac{h}{\cos\alpha};$$

$$\sin\alpha = n\sin\beta$$

$$\cos\alpha d\alpha = n\cos\beta d\beta$$

$$\frac{d\alpha}{d\beta} = n\frac{\cos\beta}{\cos\alpha}$$

$$\frac{Hd\beta}{\cos^2\beta} = \frac{hd\alpha}{\cos^2\alpha}; \quad \frac{H}{h}\frac{\cos^2\alpha}{\cos^2\beta} = n\frac{\cos\beta}{\cos\alpha}; \qquad n = \frac{H}{h}\frac{\cos^3\alpha}{\cos^3\beta} = \frac{h^2}{H^2}\left(\frac{\sqrt{(D)^2 + H^2}}{\sqrt{(D)^2 + h^2}}\right)^3$$