

**2004-2005 Physics Olympiad Preparation Program**  
 – University of Toronto –

**2004 POPTOR Weekend**  
 will be held from May 13 to May 15

**Solution Set 6: AC Circuits, Particles, and General**

**Problem 1**

A circuit consists of two identical capacitors with capacitance  $C$  and a coil with inductance  $L$ . Initially the switches were opened and the first capacitor was charged with the charge  $q_0$ . Another capacitor was not charged. After the switches have been simultaneously closed, the system becomes an oscillatory circuit, or  $LC$  circuit, where charge and current harmonically oscillate in time.

Find all parameters of oscillations of charges and current in this circuit.

Propose a mechanical analogy for this system.

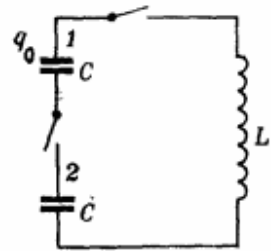


Fig. 1.1

**Solution**

Fig.1.1 shows a kind of so-called  $LC$ -circuit. After the switches are closed, the first capacitor starts to discharge providing current through the coil (inductor), and the second capacitor starts to charge. Due to the phenomenon of self-induction, the current in the coil cannot jump immediately to its maximum and cannot immediately drop to zero. Current, charge in the capacitors, voltage, and energy in the circuit oscillate. Current oscillates with frequency  $\omega$ , and its value obeys the sinusoidal law:

$$I = I_0 \sin \omega t$$

To find the amplitude  $I_0$  of oscillation we can apply the law of conservation of energy. At any instant the total energy is a sum of energy of magnetic field in the coil and energy of electric field in two capacitors. If  $L$  is the inductance of the coil,  $q_1$  and  $q_2$  are the charges in the capacitors, then the total energy is given by:

$$U_{total} = \frac{q_0^2}{2C} = \frac{LI^2}{2} + \frac{q_1^2}{2C} + \frac{q_2^2}{2C}$$

Let us put all terms with charges to one side and take into account that  $q_1 + q_2 = q_0$ :

$$\frac{LI^2}{2} = \frac{1}{2C} [q_0^2 - (q_1^2 + q_2^2)]$$

(1.1)

$$I^2 = \frac{1}{LC} [q_0^2 - (q_1 + q_2)^2 + 2q_1q_2] = \frac{2}{LC} q_1q_2$$

The product of two charges is maximum when  $q_1 = q_2 = q_0/2$  (1.2). This result can be obtained from equation (1.1) if we substitute one of the charges as  $q_2 = q_0 - q_1$ , and equate the derivative

of  $I^2$  with respect to  $q_1$  to zero. Thus, the amplitude value for current corresponds to the condition (1.2), and

$$|I_0| = q_0 \sqrt{\frac{1}{2LC}}$$

For a simple  $LC$ -circuit with one capacitor with capacitance  $C_0$  the frequency

$$\omega = \frac{1}{\sqrt{LC_0}}$$

In our case two capacitors with capacitance  $C$  in series are equivalent to the capacitor with capacitance  $C_0 = C/2$ . That is why, the frequency of oscillation of current in the given circuit is:

$$\omega = \sqrt{\frac{2}{LC}}$$

Current in the circuit oscillates as:

$$I = q_0 \sqrt{\frac{1}{2LC}} \sin\left(\sqrt{\frac{2}{LC}} \cdot t\right) \quad (1.3)$$

To obtain the law for oscillation of the charge, let us substitute  $I$  in the formula (1.1) with its expression from (1.3) and the value of charge  $q_2$  in (1.1) with  $q_0 - q_1$ :

$$\frac{q_0^2}{2LC} \sin^2\left(\sqrt{\frac{2}{LC}} t\right) = \frac{2}{LC} q_1 q_0 - \frac{2}{LC} q_1^2$$

Solutions of this equation are:

$$q_1 = \frac{q_0}{2} \left[ 1 \pm \cos\left(\sqrt{\frac{2}{LC}} t\right) \right]$$

The same result can be obtained for  $q_2$ . That is why we can consider that the charge in each capacitor oscillates with the same frequency  $\omega$  according to the law:

$$q_1 = \frac{q_0}{2} \left[ 1 + \cos\left(\sqrt{\frac{2}{LC}} t\right) \right],$$

$$q_2 = q_0 - q_1 = \frac{q_0}{2} \left[ 1 - \cos\left(\sqrt{\frac{2}{LC}} t\right) \right]$$

A mechanical system with the similar properties is shown in fig.1.2.

At the initial moment (a) one of the springs is compressed, another one has its natural length. At the state of equilibrium (b) both of springs are compressed to the same extent (spring constants are equal; no friction). Inertia of the mass plays the similar role as inductance of the coil.



Fig.1.2 (a)



Fig.1.2 (b)

### Problem 2

The electric field vector in the electromagnetic wave of frequency  $\omega = 2 \times 10^{16} \text{ s}^{-1}$ , amplitude-modulated with frequency  $\Omega = 2 \times 10^{15} \text{ s}^{-1}$ , changes in time according to the equation:

$$E = E_0 [1 + \cos(\Omega t)] \cos(\omega t)$$

where  $E_0$  is a constant.

The wave knocks out electrons from the atoms of hydrogen with the ionization energy of  $W = 13.5 \text{ eV}$ . Determine the energy of these electrons.

### Solution

The amplitude modulation of the wave can be imagined as a production of a number of monochromatic waves with modified frequencies. To find the frequencies we must try to transform the given equation to a sum of cosinusoidal or sinusoidal waves.

$$\begin{aligned} E &= E_0 [1 + \cos(\Omega t)] \cos(\omega t) = E_0 \cos(\omega t) + E_0 \cos(\Omega t) \cos(\omega t) = \\ &= E_0 \cos(\omega t) + \frac{1}{2} E_0 \cos[(\omega - \Omega)t] + \frac{1}{2} E_0 \cos[(\omega + \Omega)t] \end{aligned} \quad (2.1)$$

Now we can operate with three monochromatic waves with frequencies  $\omega - \Omega$ ,  $\omega$ , and  $\omega + \Omega$ .

If we switch to the quantum description of radiation, the corresponding photons will have energies  $W_1 = 2\pi\hbar(\omega - \Omega) = \hbar(\omega - \Omega)$ ;  $W_2 = \hbar\omega$ ;  $W_3 = \hbar(\omega + \Omega)$ , where  $h$  and  $\hbar$  are two expressions for the Planck's constant;  $h = 6.63 \times 10^{-34} \text{ Js}$ ;  $\hbar = h / 2\pi = 1.05 \times 10^{-34} \text{ Js} = 6.59 \times 10^{-16} \text{ eV}\cdot\text{s}$ . These values must be calculated and compared with the ionization energy of an atom. If energy of a photon is less than the ionization energy of the atom, the photon cannot ionize the atom and liberate an electron.

We will calculate energy in electronvolts to simplify the comparison. After the calculation we obtain the following:

$$W_1 = 1.19 \text{ eV}; W_2 = 13.2 \text{ eV}; W_3 = 14.5 \text{ eV}.$$

First two waves cannot ionize the hydrogen atom. The third wave can.

The kinetic energy of the liberated electron is:

$$K = W_3 - W = 14.5 \text{ eV} - 13.5 \text{ eV} = 1 \text{ eV}$$

As it is seen the unit of energy must be chosen upon the convenience. If we express the same energy in Joules,  $K = 1.6 \times 10^{-19} \text{ J}$ , that is much less convenient than previously calculated result. In the Particle Physics the unit, **electronvolt** is usually accepted.

### Problem 3

In 1923 Arthur Holly Compton in Chicago discovered one of the most famous phenomena that proved the particle-wave dualism for all natural objects. It was found that the scattering of photons from electrons could be explained by treating photons as point-like particles with energy  $hf$  ( $h = 6.63 \times 10^{-34} \text{ J}$  is Planck's constant;  $f$  is a photon's frequency) and momentum  $hf/c$  ( $c$  is speed of light in free space). Such scattering may be considered a collision, in which the energy and momentum of the isolated system of the photon and the electron are conserved. The Compton shift equation gives the change in the photon's wavelength after scattering on any particle, and the new direction of photon after the scattering:

$$\lambda' - \lambda_0 = \frac{h}{mc}(1 - \cos \theta),$$

where  $\lambda'$  is the changed wavelength of a photon;  $\lambda_0$  is the initial photon's wavelength;  $m$  is the mass of the particle;  $\theta$  is the angle between the changed and the initial direction of the motion of a photon (angle of scattering). The coefficient  $\frac{h}{mc}$  is called the Compton wavelength of the particle. For electron it is equal to 0.00243 nm.

Find the wavelength of the x-rays before scattering on the motionless electrons if the maximum kinetic energy of the electrons after the Compton scattering is 0.19 MeV.

### Solution

In general, the momentum vector diagram for the photon scattering from motionless electron (Compton effect) is shown in the fig. 3.1:  $\mathbf{p}'_{electron}$  is momentum of the electron after the scattering;  $\mathbf{p}_{photon}$  is momentum of the photon before the scattering;  $\mathbf{p}'_{photon}$  is momentum of the scattered photon.

In addition to the law of conservation of momentum, we can use the law of conservation of energy taking into account that energy of photon is proportional to the photon's frequency  $\nu$  and inversely proportional to the wavelength  $\lambda_{photon}$ :

$$E_{photon} = h\nu = \frac{hc}{\lambda_{photon}},$$

where  $h$  is the Plank's constant.

The law of conservation of energy for the system of electron and photon gives the following:

$$K = h\nu - h\nu' = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) \quad (3.1)$$

We can combine it with the given equation for the Compton wavelength shift:

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos \theta) \quad (3.2)$$

The angle of scattering  $\theta$  is not given, but we can determine it from the condition of the maximum kinetic energy of the electron after scattering. The maximum electron kinetic energy corresponds to the maximum difference of the frequencies in eq.3.1. For a photon  $h\nu = pc$ . Thus, the maximum kinetic energy of electron corresponds to the maximum difference of the momentum of the initial and scattered photon. Fig.3.1 helps to understand that this occurs when  $\theta = 180^\circ$ , and  $\cos \theta = -1$ . Thus, eq.3.2 may be replaced by the new one:

$$\lambda' - \lambda = \frac{2h}{mc} \quad (3.3)$$

Let us find  $\lambda'$  from eq.3.3 and substitute it in eq.3.1 for maximum kinetic energy of electron:

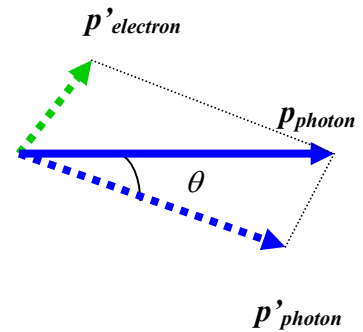


Fig.3.1

$$\frac{K_{\max}}{hc} = \frac{1}{\lambda} - \frac{1}{\lambda + \frac{2h}{mc}} = \frac{2h}{mc\lambda\left(\lambda + \frac{2h}{mc}\right)}$$

The quadratic equation for the unknown wavelength is:

$$\lambda^2 + 2\frac{h}{mc}\lambda - \frac{2h^2}{mK_{\max}} = 0 \quad (3.4)$$

From the two possible solutions for the eq.3.4 we should choose only the positive one:

$$\lambda = \frac{h}{mc} \left( \sqrt{1 + \frac{2mc^2}{K_{\max}}} - 1 \right) = \lambda_C \left( \sqrt{1 + \frac{2mc^2}{K_{\max}}} - 1 \right) \quad (3.5)$$

where  $\lambda_C = 0.00243\text{nm} = 2.43 \times 10^{-12}\text{m}$  is the Compton wavelength of electron.

To simplify calculations, we can find the value of the rest energy of electron under the root of eq.3.5 in electronvolts:  $mc^2 = 0.511\text{ MeV}$  (is given in all physics handbooks and textbooks).

$$\lambda = 2.43 \times 10^{-12} \text{ m} \left( \sqrt{1 + \frac{2 \times 0.511}{0.19}} - 1 \right) = 3.71 \times 10^{-12} \text{ m} = 3.71 \text{ pm}$$

#### Problem 4

In 1923, in his doctoral dissertation, Louis Victor de Broglie postulated that, “because photons have wave and particle characteristics, perhaps all forms of matter have wave as well as particle properties”. De Broglie suggested that material particles of momentum  $p$  should also have wave properties and a wavelength of

$$\lambda = \frac{h}{p} \quad (\text{the de Broglie wavelength}),$$

where  $h$  is Planck’s constant. The frequency of a particle is  $f = E / h$ . The further numerical experiments with electron diffraction proved the de Broglie hypothesis.

A beam of electrons accelerated by the potential difference  $V = 25\text{ V}$  is normally incident on a two-slit barrier with the slit separation of  $d = 50\ \mu\text{m}$ .

Find the distance between two adjacent maximums of the diffraction pattern on the screen which is separated from the barrier by  $l = 100\text{ cm}$ .

#### Solution

The diffraction (or interference) pattern in the problem consists of the dark and bright fringes on the photographic plate. Dark fringes correspond to the maximum intensities in this case.

To solve the problem we will use the equation for the distance  $\Delta x$  between two consecutive maximums on the screen for given distance  $l$ , slit separation  $d$ , and wavelength  $\lambda$  considering it the de Broglie wavelength of electron:

$$\Delta x = \frac{\lambda l}{d} \quad (4.1)$$

According to the de Broglie theory  $\lambda = \frac{h}{p}$  (4.2), where  $p$  is the momentum of electron.

The kinetic energy of electrons can be calculated as:  $K = e\Delta V = 25 \text{ eV}$ . The energy is much less than the rest energy of electron (0.511 MeV), that is why the electrons in our problem are nonrelativistic. The relationship between momentum and kinetic energy is:

$$K = \frac{p^2}{2m} \quad (4.3)$$

Combining eqs.4.1, 4.2, and 4.3 we can compose a formula to calculate  $\Delta x$  :

$$\Delta x = \frac{h}{p} \frac{l}{d} = \frac{h}{\sqrt{2mK}} \frac{l}{d} = \frac{h}{\sqrt{2me\Delta V}} \frac{l}{d} \quad (4.4)$$

Calculation of eq.4.4 can be also simplified with the use of the value of the rest mass and the Planck's constant in electronvolts:

$$h = 4.14 \times 10^{-15} \text{ eV s}$$

$$\Delta x = \frac{hc}{\sqrt{2mc^2K}} \frac{l}{d} = \frac{4.14 \times 10^{-15} \times 3 \times 10^8}{\sqrt{2 \times 0.511 \times 10^6 \times 25}} \frac{1}{50 \times 10^{-6}} = 4.91 \times 10^{-6} \text{ m} = 4.91 \mu\text{m}$$

### Problem 5

The carbon dating is a method to date organic samples using the present ratio of two carbon isotopes:  $^{14}\text{C}$  to  $^{12}\text{C}$  in the carbon dioxide molecules. In living organisms this ratio is about  $1.3 \times 10^{-12}$ . After the organism dies, it no longer absorbs  $^{14}\text{C}$  from the atmosphere, and so the ratio of  $^{14}\text{C}$  to  $^{12}\text{C}$  decreases as the result of the beta decay of  $^{14}\text{C}$ , which has a half-life of 5730 years. There are two different formulae for the same radioactive decay law:

$$N = N_0 e^{-\lambda t} \quad \text{or} \quad N = N_0 \cdot \left(\frac{1}{2}\right)^{\frac{t}{T}}$$

where  $N$  is the number of undecayed radioactive nuclei in the time interval  $t$ ;  $N_0$  is the initial number of nuclei;  $\lambda = \ln(2)/T$  is a decay constant;  $T$  is called a half-life, it is the time it takes half of a given number of radioactive nuclei to decay.

The decay rate  $R$  is called an activity. It is a derivative of a number of nuclei with respect to time:

$$R = \left| \frac{dN}{dt} \right| = N_0 \lambda e^{-\lambda t} = N \lambda = R_0 e^{-\lambda t}$$

$R_0$  is the decay rate at  $t = 0$ .

The technique of radioactive carbon dating has been successfully used to measure the ages of many organic relics up to 25 000 years old.

A piece of charcoal of mass 25.0 g is found in the ruins of an ancient city. The sample shows a  $^{14}\text{C}$  activity of 250 decays/min.

How long has the tree from which this charcoal came been dead?

### Solution

First, let us calculate the decay constant for  $^{14}\text{C}$ :

$$\lambda = \frac{0.693}{T} = \frac{0.693}{(5730 \text{ yr})(365 \text{ days})(24 \text{ hr})(3600 \text{ s})} = 3.83 \times 10^{-12} \text{ s}^{-1}$$

The number of  $^{14}\text{C}$  nuclei can be calculated in two steps. First, the number of  $^{12}\text{C}$  nuclei in 25g of carbon is

$$N = \frac{6.02 \times 10^{23} \text{ nuclei/mol} \times 25 \text{ g}}{12 \text{ g/mol}} = 1.25 \times 10^{24} \text{ nuclei}$$

Assuming that the ratio of  $^{14}\text{C}$  to  $^{12}\text{C}$  is  $1.30 \times 10^{-12}$ , we see that the number of  $^{14}\text{C}$  nuclei in 25.0 g is

$$N_0 = (1.30 \times 10^{-12})(1.25 \times 10^{24}) = 1.63 \times 10^{12} \text{ nuclei}$$

Hence, the initial activity of the sample is

$$R_0 = \lambda N_0 = (3.83 \times 10^{-12} \text{ s}^{-1})(1.63 \times 10^{12}) = 6.24 \text{ decays/s} = 374 \text{ decays/min}$$

We can now calculate the age of the charcoal using the relationship between the activity at any time and the initial activity (see introduction to the problem):

$$e^{-\lambda t} = \frac{R}{R_0} \quad (4.1)$$

Because  $R$  is given, and  $R_0$  was found, we can calculate  $t$  by taking the natural logarithm of both sides of eq.4.1:

$$-\lambda t = \ln\left(\frac{R}{R_0}\right) = \ln\left(\frac{250}{374}\right) = -0.403$$

$$t = \frac{0.403}{\lambda} = 1.05 \times 10^{11} \text{ s} = 3.33 \times 10^3 \text{ years}$$