2005-2006 Physics Olympiad Preparation Program

- University of Toronto –
Solutions. Set 2: Mechanics

Problem 1

A heavy disk with radius R is rolling down hanging on two non-stretched strings wound around the disk. The free ends of the strings are attached to the disk. The strings are always tense during the motion. At some instant, the angular velocity of the disk is ω, and the angle between the strings is α (fig.1.1).
Find the velocity of the disk at this moment.

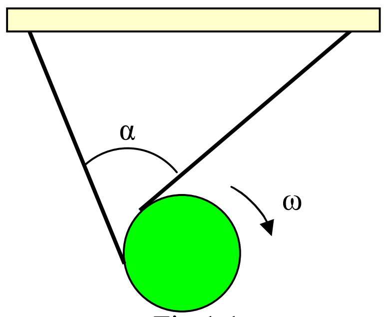

Fig.1.1

Solution

The question is about the instantaneous linear velocity of the center of mass of the disk.
One of the possible solutions can be based on the concept of the instantaneous axis of rotation. An example of such axis is shown in fig. 1.2. For the disk, rolling to the right with the velocity $\mathbf{v}_{CM}$ of its center of mass, the instantaneous axis coincides with the line of contact between the disk and the surface at some instant. Fig. 1.2 shows point P of this axis.

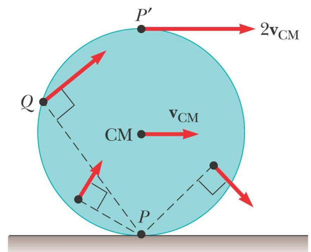

Fig. 1.2

In general, the motion of a rolling object is a combination of the translation of its center of mass and rotation of the object around the axis through the center of mass.
To simplify the problem, we can consider the motion of a rolling disk as a pure rotation round the instantaneous axis with the angular speed ω given by: $\omega = v_{CM} / R$, where R is the radius of the disk. This idea permits us easily calculate the velocity of any point, which is instantly at some distance d from the axis through point P. For instance, the speed of the point Q is $v_Q = \omega \cdot d_{PQ}$. The velocity vector is perpendicular to the line PQ. The last property is a key to find the instantaneous axis of rotation in any case.
In our problem we have to find such an axis, for which the vector of velocity for any point on the disk is perpendicular to the line joining the point with this axis.

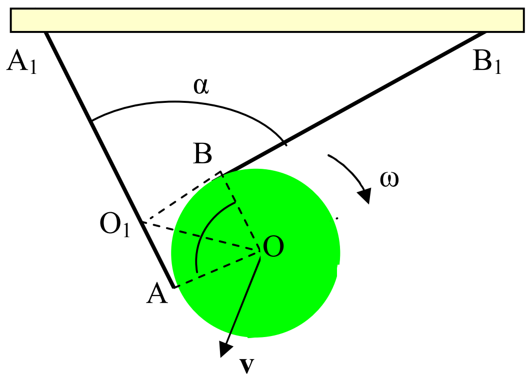

Fig.1.3

Fig.1.3 helps to solve the problem. As the strings are non-stretched, the velocity of point B is perpendicular to the line BB_1, and the velocity of the point A is perpendicular to the line AA_1 (the velocities do not have components along the strings). Thus, the intersection of these lines, O_1, belongs to the instantaneous axis at the moment when the angle between the strings is α.

The angle between two radii OA and OB is also α. The distance from the center of mass to the instantaneous axis is $OO_1 = R / \cos(\alpha/2)$.

Therefore, the center of mass has a linear velocity of $v = \dfrac{\omega R}{\cos\left(\dfrac{\alpha}{2}\right)}$

Problem 2.

A spacecraft with mass m_0 is moving with constant velocity v_0 in free space. To change its direction the maneuvering jet engine is turned on, which starts to eject fuel at a constant velocity $\boldsymbol{u}$ relative to the spacecraft. During the maneuver, vectors $\boldsymbol{v_0}$ and $\boldsymbol{u}$ are always mutually perpendicular. Finally, the mass of the spacecraft becomes m.

Find the angle between the initial and final direction of motion of the spacecraft.

Solution 1

According to the law of conservation of a linear momentum:

$$\Delta\boldsymbol{p}_{\text{fuel}} + \Delta\boldsymbol{p}_{\text{sc}} = 0,$$

where $\boldsymbol{p}_{\text{fuel}}$ is the momentum of the exhausted fuel; and $\boldsymbol{p}_{\text{sc}}$ is the momentum of the spacecraft.
It is better to use the differential, or very little, changes of momentum to consider some values constant during the process: $d\boldsymbol{p}_{\text{fuel}} + d\boldsymbol{p}_{\text{sc}} = 0$ (2.1).

The vector diagram on the fig.2.1 shows the change in the momentum of the spacecraft during very little time interval dt. During this time interval the mass of the spacecraft has an average instantaneous value m, and the decrease in this mass is equal to the mass of fuel exhausted: $-dm = dm_{fuel}$ (2.2). It is also accepted to consider a very little vector $m d\boldsymbol{v}$ perpendicular to the vector $m\boldsymbol{v}_0$ because of a very small value of the angle $d\alpha$.

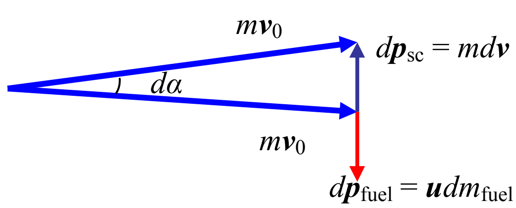

Fig.2.1

Using the diagram (fig.2.1) let us rewrite the eq.2.1:

$$\boldsymbol{u} dm_{\text{fuel}} + m d\boldsymbol{v} = 0$$

Taking into account the opposite direction of the vectors $\boldsymbol{u}$ and $d\boldsymbol{v}$ and the relationship (2.2) for the masses we obtain the following:

$$u\, dm = -\, m\, dv$$

For a small angle $d\alpha$ it is true that $dv = v_0 \cdot d\alpha$. Finally: $u\, dm = -\, m\, v_0\, d\alpha$, or

$$\frac{dm}{d\alpha} = -m\frac{v_0}{u} \qquad (2.3)$$

From the course of Grade 12 Calculus it is known that when the change in a function is proportional to the function, this function is an exponential one, and the solution of the eq.(2.3) is given by:

$$m = m_0 e^{-\frac{v_0}{u}\alpha}$$

Solving this equation for unknown α we obtain:

$$\alpha = \frac{u}{v_0}\ln\left(\frac{m_0}{m}\right) \qquad (2.4)$$

Solution 2

This solution is based on the eq.(2.1), the third Newton's law, and the definition of force, as a rate of change of a linear momentum:

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad (2.5)$$

The second Newton's law is not a definition of force. It is valid only for the objects with constant mass. In our case as well as in any problem on the rocket propulsion the perfect expression for the force is eq. (2.5).

We will also use the same vector diagram as in the first solution.

The force exerted on the spacecraft is opposite to the force exerted on the fuel, which is given by the eq.(2.5):

$$\vec{F} = -\frac{\vec{u}\cdot dm_{fuel}}{dt} = \frac{\vec{u}\cdot dm}{dt}$$

Its magnitude is equal to:

$$F = -\frac{u\cdot dm}{dt} \qquad (2.6)$$

because dm/dt is a negative value; it is a rate of decrease in the mass of the spacecraft.

This force is perpendicular to the velocity of the spacecraft and thus is a centripetal force. One of the expressions for the centripetal force is following:

$$F = v_0\omega = v_0 d\alpha /dt \qquad (2.7)$$

Eqs. (2.6) and (2.7) give:

$$-\frac{u\cdot dm}{dt} = \frac{mv_0\cdot d\alpha}{dt}$$

After integrating it by time, we obtain the equation (2.3), from the first solution.

Problem 3.

A horizontal plate with mass M lies on two rotating cylinders with equal angular speed of rotation, but opposite directions of rotation. The distance between the axes of cylinders is l. The coefficient of kinetic friction between the plate and the material of the cylinders is μ.

(a) Prove that the plate initially at the equilibrium position will start the harmonic oscillation if it is slightly displaced from the equilibrium in horizontal direction.

(b) Find the frequency of the oscillations.

(c) What is the result of the same displacement if the angular velocities have opposite directions?

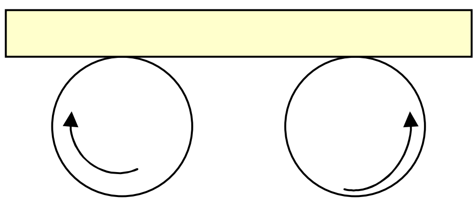

Fig. 3.1

Solution

a) The main feature of the harmonic motion (oscillation) is following: the force exerted on the object is directly proportional to the object's displacement, and the vector of force is oppositely directed to the vector of displacement.

Fig.3.2 shows the forces on the plate after it was shifted to the right from the position of equilibrium at the middle point between the axes of two cylinders. The value of displacement is x.

In this position of the plate, two normal forces N_1 and N_2 are not equal, and therefore the forces of friction between the plate and each of cylinders are also different. The horizontal component of the net force is a source of the horizontal acceleration.

Normal forces can be found from the following system of equations:

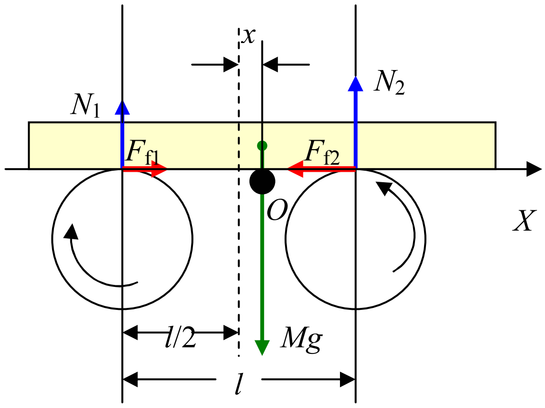

Fig.3.2

$N_1 + N_2 - Mg = 0$ (no vertical acceleration);

$N_1\left(\frac{l}{2}+x\right) - N_2\left(\frac{l}{2}-x\right) = 0$ (zero torque).

The solution is following: $N_1 = Mg\left(\frac{1}{2}-\frac{x}{l}\right)$; $N_2 = Mg\left(\frac{1}{2}+\frac{x}{l}\right)$

The net force in the horizontal direction is given by: $F = F_{f1} - F_{f2} = \mu(N_1 - N_2) = -\frac{2\mu Mg}{l}x$

The net force is proportional to the displacement and has the opposite direction. This force causes harmonic oscillation.

b) The frequency of oscillation can be found from the equation of motion of the plate:

$$Ma = -\frac{2\mu Mg}{l}x \qquad or \qquad a + \frac{2\mu g}{l}x = 0$$

This equation gives the well known solution for the harmonic oscillation with angular frequency:

$$\omega = \sqrt{\frac{2\mu g}{l}}$$

c) If the direction of rotation of both cylinders is changed to the opposite one, the net force will also change its direction and will push the plate in the direction of displacement. Instead of oscillation the motion will become uniformly accelerated.

Problem 4.

A solid cylinder with mass M and radius R rotates around its axis of symmetry with initial angular speed ω_0. Then this rotating cylinder is carefully put with its lateral side on the horizontal plane. The coefficient of kinetic friction between the cylinder and the plane is μ.
Find:
1) the time interval in which the cylinder is rotating with slipping;
2) the total work of the force of friction.

Solution

1) The equation of motion for the rotation of a rigid body is given by:

$$I\frac{\Delta\omega}{\Delta t} = \tau \qquad (4.1)$$

where I is a moment of inertia of a rigid body; ω is its angular speed; and τ is the torque of a force that causes the change of the angular speed. In our problem a solid cylinder or disk has the moment of inertia $I = MR^2/2$. The torque of the force of friction is equal to μMgR. The eq.(4.1) can be rewritten as:

$$\frac{MR^2}{2}\frac{\omega_f - \omega_0}{\Delta t} = -\mu MgR \quad (4.2)$$

Negative sign appears because the torque is directed to decrease the angular speed. Solution of the eq.(4.2) gives:

$$\omega_0 - \omega_f = \frac{2\mu g\Delta t}{R} \quad (4.3)$$

Force of friction decelerates rotation, and at some moment the cylinder starts to roll. There appears the linear motion of the center of mass with constant speed $v = \omega_f R$. From the definition of force as a rate of change of a linear momentum, we can obtain:

$$M\frac{\Delta v}{\Delta t} = \mu Mg \quad \text{or} \quad v = \mu g\Delta t,$$ because initially the linear speed was zero.

The relationship between the linear and the angular speed gives:

$$\omega_f = \frac{\mu g\Delta t}{R} \quad (4.4)$$

Combining eqs(4.3) and (4.4) we can calculate the unknown time interval:

$$\Delta t = \frac{\omega_0 R}{3\mu g} \quad \text{and} \quad \omega_f = \frac{\omega_0}{3} \quad (4.5)$$

2) The work of the force of friction is equal to the change of kinetic energy of the cylinder:

$$W = \frac{I\omega_f^2}{2} + \frac{Mv^2}{2} - \frac{I\omega_0^2}{2} = \frac{MR^2}{2}\frac{(\omega_0/3)^2}{2} + \frac{MR^2(\omega_0/3)^2}{2} - \frac{MR^2}{2}\frac{\omega_0^2}{2} = -\frac{MR^2\omega_0^2}{6}$$

Problem 5 (experimental)

The Hooke's law in the case of elongation of a uniform wire can be written as:

$$\frac{F}{A} = Y\frac{\Delta L}{L_i}$$

where F is an applied force; A is a cross-sectional aria of the wire; Y is_ the Young's modulus of the material of the wire; ΔL is the increase in the length under the force applied; L_i is the initial length of the wire. Sometimes, you can hear that the Young's modulus is stress over strain, because F/A is called the stress, and $\Delta L/L_i$ is called the strain.

Experiment:

Stretch the wire in horizontal direction and firmly attach the ends to some stable support. It can be a vertical wall or ceiling or something else.

Attach the object of a known mass to the center of the wire and carefully release it.

Measure the displacement of the object in the vertical direction. You must be sure that the distance between the ends of the wire has not changed.

Calculations:

With given L_i, Y, diameter of the wire d, and mass of the object m calculate theoretically the same value of displacement.

Compare the two values and briefly discuss the difference.

Solution

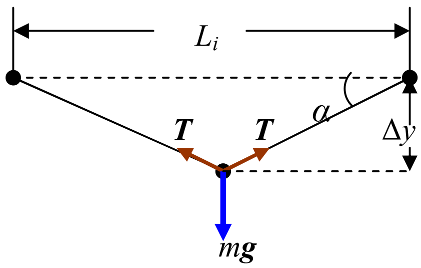

Fig.5.1 shows the quantities, which are used to calculate the vertical displacement of an object on the string. The condition of static equilibrium gives:

$$mg = 2T\sin\alpha = 2T\frac{\Delta y}{\sqrt{\Delta y^2 + \frac{L_i^2}{4}}} \qquad (5.1)$$

Force of tension

$$T = YA\frac{\Delta l_1}{L_i/2} = Y\frac{\pi d^2}{4}\frac{\sqrt{\Delta y^2 + \frac{L_i^2}{4}} - \frac{L_i}{2}}{\frac{L_i}{2}} \qquad (5.2)$$

Solving together equations (5.1) and (5.2) we obtain:

$$\Delta y = L_i\sqrt[3]{\frac{mg}{2\pi d^2 Y}}$$