2005-2006 Physics Olympiad Preparation Program

– University of Toronto –

Solutions. Set 4: Optics and Waves

Problem 1

A bat, moving at 5.00 m/s, is chasing a flying insect. If the bat emits a 40.0 kHz chirp and receives back an echo at 40.4 kHz, at what speed is the insect moving toward or away from the bat? Take the speed of sound in air to be v = 340 m/s.

Solution

We should use the Doppler shift formula for the moving source ***and*** the moving observer:

$$f' = \left(\frac{v + v_O}{v - v_S}\right) f \qquad (1.1)$$

where f is the original frequency of sound produced by a source;
f' is a frequency of sound received by the observer;
v is a speed of sound in the medium (air in our problem);
v_O is the speed of the observer in the motionless frame of reference; and
v_S is the speed of the source in respect to the motionless frame of reference.

In expression (1.1), the signs for the values substituted for v_O and v_S depend on the direction of the velocity. A positive value is used for motion of the observer or the source ***toward*** the other, and a negative sign for motion of one ***away from*** the other.

In our problem, it is convenient to separate two processes: 1) the bat is a source of sound, and the insect is the observer; and 2) the insect reflects the sound wave and becomes a source, while the bat now is the observer.

For the first part of solution, the frequency of sound detected by the insect is given by:

$$f_{ins} = \left(\frac{v + v_{ins}}{v - v_{bat}}\right) f$$

Let us decide that the bat is chasing the insect, and the sign for its velocity must be always positive. The sign of the velocity of the insect is unknown. Thus,

$$f_{ins} = \left(\frac{340 + v_{ins}}{340 - 5.00}\right) 40kHz$$

For the echo frequency, we can obtain the following:

$$40.4 = \left(\frac{v + v_{bat}}{v - v_{ins}}\right) f_{ins} = \left(\frac{340 + 5.00}{340 - v_{ins}}\right)\left(\frac{340 + v_{ins}}{340 - 5.00}\right) 40.0$$

Solving the equation, we can find that v_{ins} = - 3.31 m/s. It means that the insect is flying away from the bat! This is, of course, natural for the insect in such circumstances!
Therefore, the bat is gaining on its prey at (5.00 – 3.31) m/s = 1.69 m/s.

Problem 2

A surface of a glass plate is covered with a thin layer of water. Light with the wavelength $\lambda = 0.680$ μm strikes the surface at the angle 30.0° with respect to the normal to the surface. Due to the evaporating of the water layer, the intensity of the reflected light changes periodically. The time interval between the appearances of the maximum of intensity is equal to 15.0 min. Indices of refraction of glass and water are respectively 1.50 and 1.34.
Find the rate of decrease of the water layer thickness.

Solution

To understand the appearance of the interference pattern, it is quite enough to draw a diagram for two rays 1 and 2 (fig.2). The angle of incidence is deliberately shown greater than 30° that makes the diagram clearer.
The condition of constructive interference on the surface of the water layer is given by:

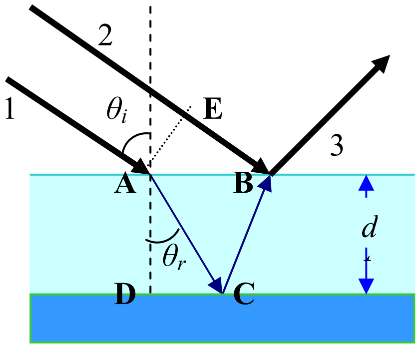


Fig.2

$$m\lambda = (AC + CB)\cdot n_W - EB =$$
$$= 2\cdot AC\cdot n_W - AB\cdot\sin\theta_i = \qquad m = 1, 2, 3,\ldots(2.1)$$
$$= 2\frac{d}{\cos\theta_r}n_W - 2d\tan\theta_r\sin\theta_i$$

Formula (2.1) contains the following notation:
$m\lambda$ is the optical path difference between the rays 2 and 3 (ray 3 is the result of refraction at point **A**, reflection at point **C,** and refraction at point **B** of the ray 1);
d is the instantaneous thickness of the water layer;
n_W is index of refraction of water;
θ_i is the angle of incidence of the rays 1 and 2 on the surface of the water;
θ_r is the angle of refraction of the ray 1.
The relationship between the angles of incidence and refraction is known as the Snell's law:

$$\sin\theta_i = n_W\sin\theta_r \quad (2.2)$$

Combining equations (2.1) and (2.2), we can obtain an expression for the path difference that matches the condition of constructive interference:

$$m\lambda = 2d\sqrt{n_W^2 - \sin^2\theta_i} \qquad (2.3)$$

In time interval Δt, the condition of constructive interference will look like:

$$(m-1)\lambda = 2(d-\Delta d)\sqrt{n_W^2 - \sin^2\theta_i} \qquad (2.4)$$

After subtracting equation (2.4) from equation (2.3), we obtain:

$$\lambda = 2\Delta d\sqrt{n_W^2 - \sin^2\theta_i}\,, \qquad \text{and } \Delta d = \frac{\lambda}{2\sqrt{n_W^2 - \sin^2\theta_i}}$$

Now, we can calculate the rate of change of the thickness of the layer as follows:

$$\frac{\Delta d}{\Delta t} = \frac{\lambda}{2\Delta t\sqrt{n_W^2 - \sin^2\theta_i}} = \frac{6.8\cdot 10^{-7}}{2\cdot 15\cdot 60\cdot\sqrt{(1.34)^2 - (0.5)^2}} = 3.04\cdot 10^{-10}\frac{m}{s} = 1.01\frac{\mu m}{h}$$

Problem 3.

The diffraction grating has 200 grooves per millimeter. Non-coherent light beam consists of waves with the wavelengths in the range from $\lambda - \varDelta\lambda$ to $\lambda + \varDelta\lambda$ and average wavelength $\lambda = 550$ nm. The light strikes the diffraction grating along the normal to the surface.

Find the maximum value of $\varDelta\lambda$, for which the adjacent spectra obtained with the diffraction grating do no overlap.

Solution

Let us calculate $\varDelta\lambda$, for which the adjacent spectra just start to overlap. It happens when

$$m(\lambda - \varDelta\lambda) = (m\text{-}1)(\lambda + \varDelta\lambda)$$

$$2m - 1 = \frac{\lambda}{\Delta\lambda} \qquad (3.1)$$

This condition must take place for the farthest from the middle line of spectra. In this case, it is automatically valid for all other lines. To find the greatest value for *m*, we can use the general condition for the observing maximum intensity in the diffraction grating pattern:

$$d \sin\theta = m\lambda$$

For our problem and maximum value of *m*, this condition is following:

$$d = m\,(\lambda + \varDelta\lambda),$$

because the maximum value for the sine function is one, and we must consider the lines with greater wavelength, as they are diffracted to the greater angle. For biggest *m* we obtain the following:

$$m = \frac{d}{\lambda + \Delta\lambda} \qquad (3.2)$$

The value of $\varDelta\lambda$ is unknown. However, *m* must be an integer number. To calculate the unknown in numbers, we have to find the value for *d*.

$$d = (1 / 200)\text{ mm} = 5 \cdot 10^{-6}\text{ m} \qquad (3.3)$$

The maximum integer number according to (3.2) and (3.3) can be only 9, so, $m \leq 9$ depending upon the value of $\varDelta\lambda$. $\varDelta\lambda$ can be obtained from (3.1) with $m = 9$.

$$\Delta\lambda = \frac{\lambda}{2 \cdot 9 - 1} = 32.4\text{ nm} \quad (3.4)$$

Now, let us check whether the line of the 8th order with the greatest wavelength and the line of the 9th order with the least wavelength do not overlap:

$$8\cdot (\lambda + \Delta\lambda\,) \leq 9\cdot (\lambda - \Delta\lambda) \qquad (3.5)$$

If the calculation is performed with the rounded numbers and three significant digits in each value involved, the found $\varDelta\lambda$ =32.4 nm does not match the condition (3.5) as rounding actually increases this number over the permitted threshold. However, the exact number for $\varDelta\lambda$ gives the equation in the formula (3.5).

If the result of calculation is expected to be used in a real experiment, we should take into account that the mean wavelength is given with the precision of units of nanometers. In this case, the obtained value for $\varDelta\lambda$ must be taken with the same precision without rounding. In other words, we must consider the integer number of nanometers for $\varDelta\lambda$. This decision brings us to the correct result in the inequality (3.5). For the practical purposes, $\varDelta\lambda = 32$ nm.

Problem 4 (The first version was corrected, see Comments)

While an artificial diffraction grating is used to study a spectrum of a light beam, electromagnetic waves with wavelengths much less than the wavelength of the visible light are used to study the natural diffraction grid, namely a crystalline structure. Electromagnetic waves in this range are called the x-rays.

Fig. 4.1 shows a two dimensional description of the reflection of an x-ray beam from two parallel crystalline planes separated by a distance *d*. The beam reflected from the lower plane travels farther than the beam reflected from the upper plane. The effective path difference is $2d\sin\theta$. The condition for constructive interference is given by Bragg's law:

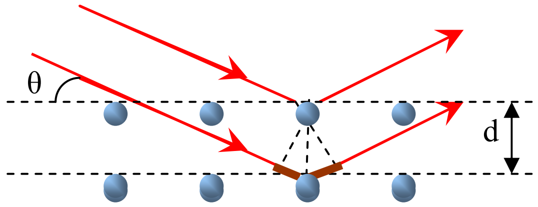


Fig.4.1

$$2d\sin\theta = m\lambda,\ m = 1, 2, 3, \ldots$$

If the wavelength and diffraction angle are measured, the Bragg equation can be used to calculate the spacing between atomic planes.

In our problem, the crystalline of NaCl with density D = 2.16 g/cm^3 is exposed to x-rays with the glancing angle $\theta = 60.0^{\circ}$. The reflected rays form the second order maximum.

(a) Find the wavelength of x-rays used in the experiment.
(b) Explain why the visible light cannot be used to obtain the diffraction pattern on crystals.

Solution

For the second-order maximum, the Bragg equation gives the following wavelength for the x-rays:

$$2d\sin\theta = 2\lambda$$

$$\lambda = d\sin 60^{\circ} = \frac{d\sqrt{3}}{2}$$

To find *d* we must recall that the crystalline lattice of the table salt, or NaCl, has a cubic form. If V_1 is a volume per one atom inside the cubic lattice, the distance between the neighbor atoms *d* can be found as: $d = \sqrt[3]{V_1}$. The result for *d* can be obtained basing on the given density:

$$D = \frac{M}{V} = \frac{m_1}{V_1} = \frac{\mu/2}{N_A V_1}; \quad \Rightarrow \quad V_1 = \frac{\mu}{2DN_A}$$

where *M* is the mass of a salt sample; *V* is its volume; *m* is "the average mass" of one atom; μ is the molar mass of NaCl ; and N_A is the Avogadro number.

(a) Finally: $\lambda = \frac{\sqrt{3}}{2}\sqrt[3]{\frac{\mu}{2DN_A}} = \frac{\sqrt{3}}{2}\sqrt[3]{\frac{58.5}{2\cdot 2.16\cdot 6.02\cdot 10^{23}}} = 2.44\cdot 10^{-8}\,cm = 2.44\cdot 10^{-10}\,m$

(b) The value of the wavelength is of the same order of magnitude as the period of the crystalline lattice (value *d*). Visible light has the wavelengths near $5.50\cdot 10^{-7}$m that is about one thousand times greater than in the described experiment with x-rays. Therefore, the visible range of electromagnetic waves cannot be used to obtain the diffraction pattern on a crystal.

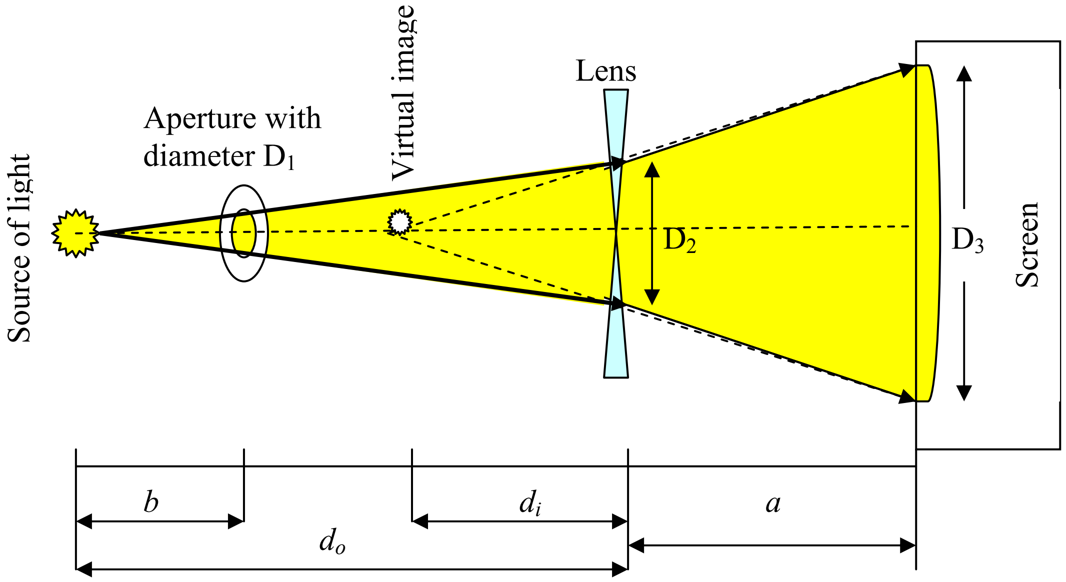
Problem 5 (experimental)

Using an available diverging lens (e.g. of the glasses), measure the focal distance of the lens.
Propose as many methods of this measurement as you can.
Explain in detail every method, submit it with the list of equipment, and sketch an experimental setup.
It is important to submit your solution with a guide in such a way that the members of POPTOR team will be able to follow your instructions and perform similar measurements to find the focal distance of their glasses with diverging lenses.
No error analysis is needed. However, the comparison of your result with the known value (for glasses) or the comparison of the results of different methods will be rewarded.

Solution

The diverging lens cannot produce a real image. That is why we cannot measure its focal distance directly.

One of the possible and simple methods is shown on the diagram. You need a little source of light, an aperture (a piece of cardboard with a circular hole), a diverging lens, a screen, and a ruler.



We do not observe the virtual image of the source of light. However, we can use the laws of geometric optics and the virtual image to derive the expression for the focal distance of the lens. After the source of light, the aperture with known diameter D_1, the lens, and the screen are set up, the diameter of the illuminated spot on the screen D_3, and distances d_o, b, and a are measured, we can start solving the problem represented on the diagram.

From similar triangles, we can obtain that:

$$\frac{D_2}{d_o} = \frac{D_1}{b} \quad \Rightarrow D_2 = \frac{d_o D_1}{b}$$

$$\frac{D_2}{d_i} = \frac{D_3}{a + d_i} \quad \Rightarrow D_2 = \frac{d_i D_3}{a + d_i}$$

$$\frac{d_o D_1}{b} = \frac{d_i D_3}{a + d_i} \qquad \Rightarrow a d_o D_1 + d_o d_i D_1 = d_i b D_3$$

$$d_i (b D_3 - d_o D_1) = a d_o D_1 \qquad \Rightarrow d_i = \frac{a d_o D_1}{b D_3 - d_o D_1}$$

d_i is the distance from the lens to the virtual image. In the formula for the diverging lens this value is negative.

According to the formula for the lens:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$

The obtained focal distance f must be also obtained as a negative value.