2005-2006 Physics Olympiad Preparation Program

– University of Toronto –

Solutions. Set 5: Electricity and Magnetism

Problem 1

A little charged bead is inside the hollow frictionless sphere manufactured from the insulating material. Sphere has a diameter of 50 cm. The mass of the bead is 90 mg, its charge is 0.50 μC.
What charge must carry an object at the bottom of the sphere to keep hold the charged bead at the vertex of the sphere?

Solution

The charge Q to be placed at the bottom of the sphere must provide such electric force on the charged bead at the top of the sphere that is not less than the force of gravity on the bead at the top.

$$k\frac{qQ}{d^2} \geq mg \qquad (1.1)$$

where q is the charge of the bead; m is its mass; d is the diameter of the sphere; and $k = 8.99{\cdot}10^9 \text{Nm}^2/\text{C}^2$, is the Coulomb's constant.
This gives the solution for the value of the charge Q:

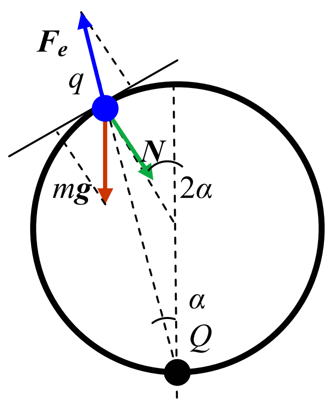


Fig. 1.1

$$Q \geq \frac{mgd^2}{kq} = 4.9\cdot 10^{-8}\,\text{C}$$

Now, we must find out whether the equilibrium is stable. Let us investigate the effect of the little displacement of the bead from the position of equilibrium with the help of fig.1.1. The equilibrium is stable if the net force is directed to the position of equilibrium. For the electric force $\boldsymbol{F_e}$, the force of gravity $\boldsymbol{mg}$, and the normal force of reaction $\boldsymbol{N}$ it means that the tangential component of the electric force must exceed the tangential component of the force of gravity, or:

$$k\frac{qQ}{d^2}\sin\alpha \geq mg\sin 2\alpha$$

As the displacement of the charged bead is little, $\sin\alpha = \alpha$, and $\sin 2\alpha = 2\alpha$.

$$k\frac{qQ\alpha}{d^2} \geq 2mg\alpha \qquad \Rightarrow \qquad Q \geq \frac{2mgd^2}{kq}$$

Finally, to hold the charged bead at the vertex of the sphere in the stable equilibrium, the object at the bottom of the sphere must carry the charge

$$Q \geq \frac{2mgd^2}{kq} = 2\cdot 4.9\cdot 10^{-8}C = 9.8\cdot 10^{-8}C$$

Problem 2

The diagram shows a circuit with given values for resistances R_1 and R_2 and electromotive forces of batteries ε_1 and ε_2. The internal resistances of the batteries can be neglected.

(a) Find the resistance R of the resistor when the maximum power is delivered to it.

(b) Find this maximum power.

Solution

I. The simplest solution is based on the Ohm's law and the method to find the maximum value with the help of the first derivative test. The connection of the batteries permits us to solve the problem without Kirchhoff's rules. For three parts of the circuit connected in parallel we can write the following equations:

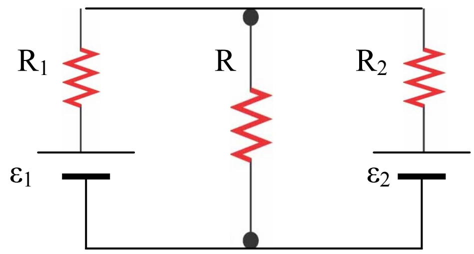


Fig.2.1

$$\varepsilon_1 - I_1R_1 = \varepsilon_2 - I_2R_2 = IR; \qquad I_1 + I_2 = I$$

The solution for the current I through the resistor R is given by:

$$I = \frac{\varepsilon_1 R_2 + \varepsilon_2 R_1}{RR_1 + RR_2 + R_1R_2}$$

The power delivered to the resistance R is $P = I^2R$, or

$$P = \frac{(\varepsilon_1 R_2 + \varepsilon_2 R_1)^2 R}{[R_1R_2 + R(R_1 + R_2)]^2} \qquad (2.1)$$

(a) To find the resistance that provides the maximum power, we take the first derivative in respect to R and equate it to zero. The result for the resistance R is given by:

$$R = \frac{R_1R_2}{R_1 + R_2} \qquad (2.2)$$

(b) The obtained value is a maximum because for zero resistance the power is zero, and for the very great resistance current becomes negligible, and the power also approaches zero.
Substituting the calculated resistance (2.2) in the equation (2.1), we can obtain the power:

$$P_{\max} = \frac{(\varepsilon_1 R_2 + \varepsilon_2 R_1)^2}{4R_1R_2(R_1 + R_2)} \qquad (2.3)$$

II. The alternate solution uses the other approach involving the calculation of the equivalent electromotive force and equivalent resistance. The given circuit can be transformed into the equivalent one shown on the fig.2.2.

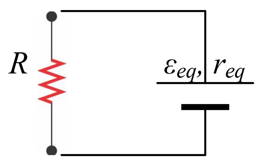


Fig.2.2

As the real batteries have no internal resistance, the equivalent resistance for the equivalent battery can be obtained as follows:

$$r_{eq} = \frac{R_1R_2}{R_1 + R_2} \qquad (2.4)$$

(a) It is known that the maximum power is delivered to a resistor with the resistance equal to that of the battery. Therefore, the unknown resistance is equal to the equivalent resistance, and the result coincides with the formula (2.2).
If we shorten the circuit eliminating the resistor R, the current can be found as:

$$I = \frac{\varepsilon_{eq}}{r_{eq}} = \frac{\varepsilon_1}{R_1} + \frac{\varepsilon_2}{R_2} \qquad \Rightarrow \qquad \varepsilon_{eq} = \frac{\varepsilon_1 R_2 + \varepsilon_2 R_1}{R_1 + R_2} \quad (2.5)$$

(b) To calculate the maximum power, we can use the results (2.4), (2.5), and the Ohm's law for the equivalent circuit:

$$P_{\max} = I^2 R = \left(\frac{\varepsilon_{eq}}{r_{eq} + R}\right)^2 \cdot R = \frac{(\varepsilon_1 R_2 + \varepsilon_2 R_1)^2}{4 R_1 R_2 (R_1 + R_2)} \qquad (2.6)$$

The results of two solutions are identical.

Problem 3

A cathode ray tube with the known applied voltage V is penetrated by the uniform magnetic field B, directed along the axis of the tube. There is a small diffuse spot on the screen of the operating tube. Varying B it is possible to obtain the focused electron trace on the screen when the magnetic field is equal to B_0, $2B_0$, $3B_0$, etc.
(a) Explain this phenomenon and the value of B_0.
(b) How is it possible to find the ratio of the electron charge and electron mass basing on the results of the described experiment?

Solution

(a) After accelerating in the electric field of the tube, all electrons appear in the magnetic field with identical components of velocities in the direction of the axis of the tube:

$$v_{\parallel} = \sqrt{\frac{2eV}{m}} \qquad (3.1)$$

The motion of electrons along the axis of the tube is the uniform motion. In addition the electrons have initially negligible random components in the perpendicular direction. In the magnetic field these components are changed by the magnetic force that plays the role of a centripetal force:

$$\frac{mv_{\perp}^2}{R} = evB \qquad (3.2)$$

where $v_{\perp}$ is the component of the velocity of an electron perpendicular to the magnetic field and the axis of the tube. Equation (3.2) gives the value for the radius of rotation of electrons:

$$R = \frac{mv_{\perp}}{eB}$$

A period of rotation of an electron is given by:

$$T = \frac{2\pi R}{v_{\perp}} = \frac{2\pi m}{eB} \qquad (3.3)$$

The obtained period does not depend on the velocity of the electron. Therefore, all electrons perform one revolution around the axis of the tube in the same time regardless of their "perpendicular" velocities. Thus, the electron moves inside the tube along the helix path, and the trajectories of all electrons intersect after each full revolution. The points of intersection can give a spot on the screen of the tube if the distance L traveled by the electrons to the screen is divisible by the displacement of the electron along the axis during one revolution:

$$L = n v_{\|} T = n\sqrt{\frac{2eV}{m}} \cdot \frac{2\pi m}{eB} = \frac{n}{B} \cdot 2\pi \cdot \sqrt{\frac{2mV}{e}} \qquad (3.4)$$

When $B = B_0$, $n = 1$; when $B = 2B_0$, $n = 2$, and so on.

(b) If we adjust all parameters for $n = 1$, the charge to mass ratio for the electron can be obtained from the formula (3.4):

$$\frac{e}{m} = \frac{8\pi^2 V}{L^2 B_0^2}$$

where L, V, and B_0 can be measured directly.

Problem 4

Brief theory.

When an electric current in a loop or coil is non-constant, it creates the variable magnetic field that, in turn, creates a changing flux through the coil. The changing flux induces the electromotive force (emf) of induction in the coil. This phenomenon is called a self-induction. As the origin of the self-induction is the changing current in the coil, the emf (ε) of self-induction is a function of the rate of change of the current, and is given by:

$$\varepsilon = -L\frac{\Delta I}{\Delta t} \qquad (4.1)$$

where L is an inductance of the coil. The value of inductance depends on the geometry of the coil, number of its turns, and the core material. For a single loop L depends on its shape and size. The unit of inductance in the SI is 1 H (henry) = 1 V·s / A

A current-carrying circular loop of radius R = 50 mm manufactured from a very thin wire has an inductance of L = 0.26 μH and is inserted into the uniform magnetic field B = 0.50 mT. The plane of the loop is perpendicular to the magnetic field lines. The conductor of the loop was cooled down to the superconductive state with zero resistance. After it, the magnetic field has been turned off.

Find the current in the loop.

Solution

The superconductive state of a material means zero electric resistance. Therefore, the current in such material can flow for the infinite time. The electric current is a source of magnetic field. According to the Faraday law of induction and Lenz's law the current induced as a result of the change in the magnetic flux opposes the cause of its own appearance.

After the external magnetic field is turned off, the electromotive force must become zero, because the resistance is zero (current is nonzero):

$$\varepsilon = R \cdot I = 0.$$

The electromotive force in this case consists of two components:

ε_{ind} – the electromotive force of induction due to the changing external magnetic field; and

ε_{si} – the electromotive force of self-induction due to the change in the induced current and its magnetic flux through the loop:

$$\varepsilon = \varepsilon_{ind} + \varepsilon_{si} \qquad (4.2)$$

From the equation (4.2) it is evident that two electromotive forces must be directed oppositely. For two *emf* we have:

$$\varepsilon_{ind} = -\frac{\Delta\Phi_{external}}{\Delta t} = -\frac{\pi R^2 \Delta B_{external}}{\Delta t} \qquad \varepsilon_{si} = -L\frac{\Delta I}{\Delta t} \qquad (4.3)$$

Substituting the equations (4.3) in the equation (4.2), we can obtain:

$$-\pi R^2(0 - B) = L\,(I - 0)$$

$$I = \frac{\pi R^2 B}{L}$$

Calculation gives the following:

$$I = \frac{3.14 \cdot 25 \cdot 10^{-4} \cdot 5 \cdot 10^{-4}}{2.6 \cdot 10^{-7}} = 15A$$

5. Drive experiment.

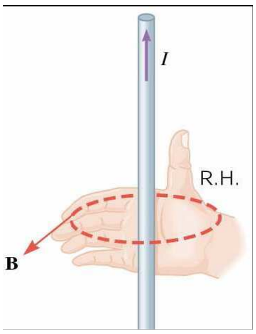
On a drive, you decide to demonstrate your family how powerful Physics is. You have a piece of wire and a compass.

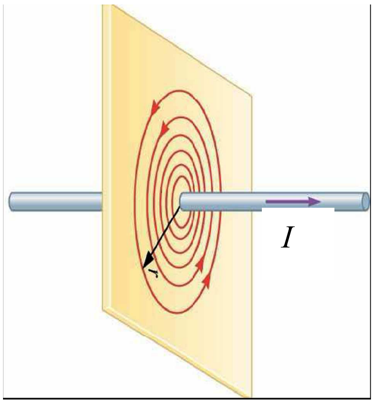
(a) How to determine the electric charges on the terminals of the automobile storage battery using these tools and a bulb from the automobile kit?

(b) After the first success, you continue to demonstrate the power of Physics by determining the horizontal component of the magnetic field of the Earth using the same equipment. Explain the method in detail.

Solution

(a) Set up a circuit consisting of the automobile storage battery and a bulb. One of the wires must be linear. A compass needle being under the linear wire turns and shows the direction of current according to the right screw law. Direction of current shows the direction from the positively charged terminal to the negatively charged terminal.





(b) To obtain the value of the horizontal component of the magnetic field of the earth, we need some additional equipment: an ammeter and a protractor. Besides we must know the formula to calculate the magnetic field of the linear conductor carrying the current. In general, we need to calculate a tangent of an angle between the direction of the magnetic field of the earth in the absence of the current carrying conductor and the direction of the resultant field of the earth and the conductor (fig.5.1). This is possible if initially we find the direction of the magnetic fields separately and then arrange the linear conductor in the way that permits us to apply the geometric method shown on the drawing.

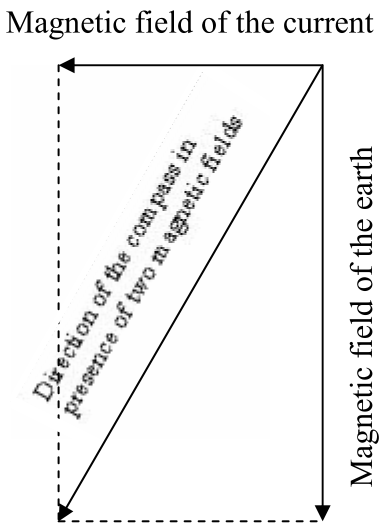


Fig.5.1