2005-2006 Physics Olympiad Preparation Program

– University of Toronto –

Solutions. Set 6: AC Circuits, Electronics and General

Problem 1.

A linear conductor of length l and mass m is attached to identical vertically suspended springs with the spring constant k (fig.1). A uniform magnetic field $\mathbf{B}$ is perpendicular to the plane of the circuit. A capacitor with the capacitance C is initially charged, and the voltage across the capacitor is V_0. After the switch S is closed, the conductor starts oscillating harmonically with period that is significantly greater than the time of the discharging of the capacitor.
Find the amplitude of the oscillation.

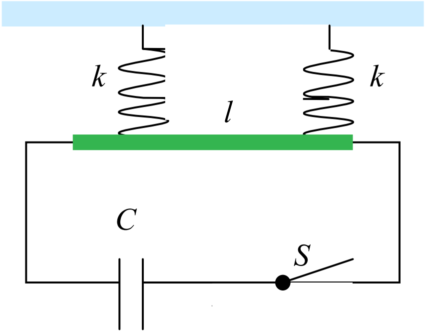

Fig.1.

Solution

The solution is based on the statement that the time of discharging of the capacitor Δt is very small compared with the period of oscillation.
Just after the switch is closed, a short pulse of discharge current appears in the circuit. The instantaneous value of the magnetic force F_B on the segment of the linear conductor, suspended from two springs, produces a linear momentum dp of the segment that is given by:

$$F_B = IlB = \frac{dp}{dt}; \quad \Rightarrow \quad dp = lB(Idt) = lBdq. \quad (1.1)$$

The total change in the value of conductor's momentum can be found by integrating eq. (1.1):

$$\Delta p = lB\,\Delta q = lBCV_0 \qquad (1.2)$$

We can substitute the effect of the current pulse during the discharging of the capacitor with the mechanical shove of the segment attached to the springs. It causes the beginning of oscillation from the position of equilibrium.
The following process of oscillation depends only on the mechanical properties of the system, because the discharge current can be neglected after Δt has elapsed. For two identical springs, the angular frequency of oscillation is given by:

$$\omega = \sqrt{\frac{2k}{m}} \qquad (1.3)$$

For the harmonic motion in y-direction with amplitude A we have the following expressions for displacement y and velocity v_y:

$$y = A \cdot \sin(\omega t); \quad v_y = \omega A \cdot \cos(\omega t); \text{ and } v_{y,max} = \omega A \qquad (1.4)$$

The maximum velocity $v_{y,max}$ is observed at the position of equilibrium, where the conductor is pushed by the magnetic force. As the time interval Δt is small, we can neglect the displacement from the equilibrium during Δt. The relationship between the change in momentum of the conductor and its maximum velocity is given by:

$$\Delta p = m\, v_{y,max}.$$

Combining equation (1.2), (1.3), and (1.4), we can calculate the unknown amplitude A as follows:

$$lBCV_0 = m\omega A$$

$$\text{Finally,}\ A = \frac{lBCV_0}{m\omega} = \frac{lBCV_0}{\sqrt{2km}}$$

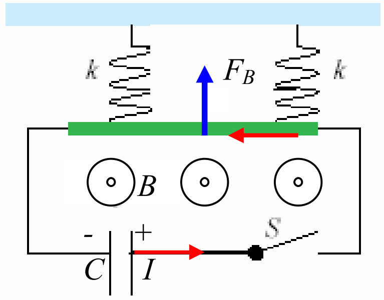

Fig.1.2.

Fig. 1.2 shows the possible directions of discharge current, magnetic field, and magnetic force at the very beginning of harmonic motion of the conductor. The uniform magnetic field is directed out of page. The force of gravity on the conductor is not shown.

The represented solution uses some uncertainties.
First, the discharging of an ideal capacitor continues for the infinite time. Therefore, the harmonic motion equation must include the magnetic force, which will always oppose the elastic force in springs. This will result in the different value for the frequency of oscillation. However, if the magnitude of the magnetic force is much less than the elastic force, we can neglect the first one. We have decided that it was possible for all times greater than Δt.
Second, we did not consider the masses and real elasticity of the other parts of the circuit that could influence the frequency of oscillation.
It is not appropriate to use the law of conservation of energy for a number of reasons: a) we do not know the resistance of the conductor(s); and b) we cannot calculate the energy loss due to radiation that always takes place when charged particles are involved in the accelerated motion as in our problem (the current decreases exponentially).

Problem 2.

Brief theory.

Energy emitted per second by one square meter of a hot object with the surface temperature T is given by Stefan-Boltzmann law:

$$S = \frac{E}{\Delta t \cdot A} = \frac{P}{A} = \sigma T^4 e \quad (2.1)$$

where S is called the intensity of radiation; P is the power of radiation; A is the area of the surface; T is the surface temperature in kelvins; $\sigma = 5.67 \cdot 10^{-8}$ W/(m$^2 \cdot$K^4), is a universal constant; and e is the emissivity. Its value varies from zero to unity. When an object is in thermal equilibrium with its surroundings, it radiates and absorbs energy at the same rate, and its temperature remains constant. An ideal absorber is called a **black body** and is defined as an object that absorbs all the energy incident on it, and for such an object, e = 1. For the purposes of estimation of some thermal parameters, many objects can be considered the black bodies with e = 1 in eq.(2.1).

All objects in our problem must be treated as black bodies.

A cloud of cosmic dust contains iron particles. The melting point for iron is $T = 1808$ K. Find the distance from the iron particle to the center of the Sun, at which the iron particle starts melting, using the following data: when the center of the Earth is at the distance of $1.50 \cdot 10^{11}$m from the center of the Sun, the intensity of solar radiation at the Earth's orbit is 1.37 kW/m^2.

Solution

Let us apply idealization for the distribution in space of sun radiation. We will consider that at the same distance from the sun the rate of energy flow through the unit area is the same. In other words, the sun emits energy uniformly in all directions. If X is the unknown critical distance from the sun to the iron particle, where the particle starts melting, then the rate of sun energy P_a absorbed by the particle of radius r is:

$$P_a = S_X \cdot \pi r^2$$

where S_X is the rate of flow of the solar energy through the unit surface area at a distance X from the sun, or the intensity of the solar radiation at distance X.

The particle will be heated until its temperature T matches the state of a "black body". Since this moment, the particle emits the same energy that absorbs during the same time interval. This is a state of thermodynamic equilibrium. In this state (see formula (2.1), for the particle it is true that:

$$P_{emission} = P_a$$

$$4\pi r^2 \sigma T^4 = S_X \cdot \pi r^2$$

$$4\sigma T^4 = S_X \qquad (2.2)$$

To find the relationship between S_X and unknown X, we can use information given for the Earth. Let S_E be the intensity of solar radiation at the Earth's orbit, and d be the distance between the Sun and the Earth. The law of conservation of energy (see fig 2.2) gives:

$$S_X \cdot 4\pi X^2 = S_E \cdot 4\pi d^2; \quad \text{and } S_X = S_E\, d^2 / X^2 \qquad (2.3)$$

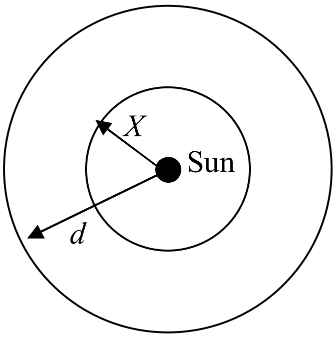

Fig.2.2

Combining eqs. (2.2) and (2.3), we obtain that

$$4\sigma T^4 = S_E\, d^2 / X^2$$

$$X = \frac{d}{2T^2}\sqrt{\frac{S_E}{\sigma}} = \frac{1.5 \cdot 10^{11}}{2 \cdot 1808^2}\sqrt{\frac{1370}{5.67 \cdot 10^{-8}}} \text{ m} = 3.57 \cdot 10^9 \text{ m} = 3.57 \cdot 10^6 \text{ km}$$

Problem 3.

A star of mass M and radius R emits from its "surface" a photon with frequency f_0. Find the gravitational shift for the frequency of the photon, $\Delta f / f_0$, very far from the star.

Solution

Qualitatively, the gravitational shift can be understood comparing the two frames of reference, or systems of measurements. The theory of General Relativity states that near the object with huge mass (the first frame of reference), there is observed a gravitational time dilation with respect to the second frame of reference far from sources of high gravity. While gravitational shift refers to what is seen, gravitational time dilation refers to what is deduced to be "really" happening once observational effects are taken into account. The period of oscillation measured far from the massive star gives greater value than the original period. As a frequency is a reciprocal to a period, for the frequency we obtain the opposite effect: the frequency of light obseved far from the star is less than the frequency of the light emitted by the star.

On the Earth we see the shifted frequency of Sun radiation. It is shifted to the lower frequencies and therefore to the greater wavelength. The phenomenon is called the ***gravitational redshift***.

It is possible to calculate with good certainty the gravitational redshift using the unmeasurable but convenient value of a mass of a photon. We can treat this value as a parameter obtained by the division of the photon's momentum p_{ph} by speed of light in free space c. Both are physically allowable quantities and can be measured. The energy of the photon is $E_{ph} = hf$, where h is Planck's constant. The photon's momentum is $p_{ph} = E_{ph} / c$. Therefore, the quantity with the dimension of a mass for the photon is expressed as: $\mu = hf / c^2$.

The potential energy of interaction between the star and the photon is given by:

$$U = -G\frac{M\mu}{r} = -G\frac{Mhf}{rc^2}$$

The law of conservation of energy for the photon can be written as:

$$U_S + hf_0 = U_\infty + hf \qquad (3.1)$$

where $U_S = -G\frac{Mhf}{Rc^2}$; and $U_\infty = 0$. From eq. (32.1) we can obtain that $\Delta f = f_0 - f = -U_S / h$.

$$\text{Finally: } \frac{\Delta f}{f} = \frac{GM}{Rc^2} \qquad (3.2)$$

The obtained relative value of redshift is the same for all wavelengths of light emitted by the star!

In addition, we can roughly estimate the gravitational redshift for the sunlight observed on the Earth. For this purpose, we should use the formula (3.1) with nonzero potential energy at the position of the observer, and neglect the gravitational field of the Earth and other planets. For the observer on the Earth the redshift is given by:

$$\frac{\Delta f}{f} = \frac{GM_S}{c^2}\left(\frac{1}{R_S} - \frac{1}{d}\right),$$

where M_S and R_S are the mass and the average conventional radius of the Sun; d is the distance between the Sun and the Earth. Calculations give $\Delta f / f \approx 2 \cdot 10^{-6}$.

Problem 4.

In the process of radioactive decay, a nucleus of ^{210}Po that is at rest emits an alpha particle (^4_2He) in the ground state (the quantum state with the lowest possible energy). Initially the alpha particle has a kinetic energy of 5.77 MeV.

Find:

(a) the other product of the decay;
(b) the initial speed of the nucleus of this product;
(c) what fraction of the total reaction energy (disintegration energy) does the kinetic energy of the product form;
(d) the heat emitted as a result of decay of 1.00 mg of ^{210}Po during the half-life time.

Solution

(a) The balance equation for the radioactive decay of the given isotope is following:

$$^{210}_{84}Po \quad \rightarrow \quad ^{4}_{2}He + ^{206}_{82}Pb \qquad (4.1)$$

(b) To choose whether to apply relativistic or non-relativistic formulae to kinetic energies of the products of reaction, it is quite enough to compare the initial kinetic energy of the α-particle and its rest energy. The rest energy of the α-particle is given by: $E_\alpha = m_\alpha c^2$. As the initial kinetic energy of the α-particle is given in electronvolts, we will use this unit to calculate the rest energy. The mass of α-particle can be found in physical tables in special units of atomic mass that is approximately equal to the number of nucleons in the nucleus. For α-particle this mass is 4.002603 a.m.u. For our estimation we can take m_α = 4.00 a.m.u. This value must be multiplied by the mass of a proton ($1.67 \cdot 10^{-27}$kg) to obtain the mass in kilograms.

$$E_\alpha = m_\alpha c^2 = 4.00 \cdot 1.67 \cdot 10^{-27} \cdot (3.00 \cdot 10^8)^2 = 6.023 \cdot 10^{-10}\text{J} = 3758 \text{ MeV}.$$

The ratio of the initial kinetic energy of the alpha particle to its rest energy is about $1.5 \cdot 10^{-3}$. This permits us to calculate all quantities with non-relativistic equations. As a nucleus of lead is much heavier, its velocity will be even less than that of the alpha particle.

For the single act of decay, shown in the eq. (4.1), we can apply the law of conservation of momentum and the law of conservation of energy.

Initially, the parent nucleus was motionless. This means that the magnitude of momentum p_α of the alpha particle equals the magnitude of momentum p_n of the product nucleus $^{206}_{82}Pb$.

$$v_n = \frac{m_\alpha}{m_n} v_\alpha = \frac{m_\alpha}{m_n}\sqrt{\frac{2K_\alpha}{m_\alpha}} = \frac{\sqrt{2K_\alpha m_\alpha}}{m_n} = \frac{\sqrt{2 \cdot 5.77 \cdot 10^6 \cdot 1.6 \cdot 10^{-19} \cdot 4.002603 \cdot 1.67 \cdot 10^{-27}}}{206 \cdot 1.67 \cdot 10^{-27}} =$$

$$= 3.25 \cdot 10^5 \text{ m/s},$$

where v is velocity; K is kinetic energy; subscript "α" refers to the quantity of the alpha particle; and subscript "n" refers to the quantity of the product nucleus.

(c) The total reaction energy $\Delta E = K_\alpha + K_n$. The relationship between the kinetic energy and momentum gives:

$$K_n = \frac{p_n^2}{2m_n} = \frac{p_\alpha^2}{2m_n} = K_\alpha \frac{m_\alpha}{m_n}$$

The unknown fraction of the total energy can be obtained as follows:

$$\frac{K_n}{\Delta E} = \frac{K_n}{K_n + K_\alpha} = \frac{K_\alpha m_\alpha / m_n}{K_\alpha m_\alpha / m_n + K_\alpha} = \frac{1}{1 + m_n / m_\alpha} = \frac{1}{1 + 51.5} = 1.90 \cdot 10^{-2} = 1.90\%$$

(d) The heat emitted during the given time interval, Q, is the kinetic energy of the products of reaction of all N nuclei undergoing the decay during this period. During the half-life time, one half of initial number of nuclei undergoes the decay. Therefore, $N = N_0 / 2$. The initial number of nuclei, N_0, can be found with given mass of the substance divided by the mass of one nucleus. The total energy of reaction during the half life is

$$Q = \frac{N_0}{2}\cdot\Delta E = \frac{N_0}{2}\cdot\left(K_n + K_\alpha\right) = \frac{N_0}{2}K_\alpha\cdot\left(\frac{m_\alpha}{m_n}+1\right) = \frac{M}{2\cdot(210\cdot 1.67\cdot 10^{-27})}K_\alpha\cdot\left(\frac{m_\alpha}{m_n}+1\right) =$$

$$= \frac{1\cdot 10^{-6}}{2\cdot(210\cdot 1.67\cdot 10^{-27})}\cdot 5.77\cdot 10^{6}\cdot 2.019 = 1.66\cdot 10^{25}\,eV = 1.66\cdot 10^{19}\,MeV = 2.66MJ$$

Problem 5 (A black box experiment).

A first-year university student did not like to use a heavy and large notebook for his records during the laboratories. He preferred to write experimental results without details on a little piece of paper, and relied on his memory. Once, he was not accurate with his data and did not make clear notes during the measurements of current and voltage in an experiment with electric circuit. The single source of information was a traditional little piece of paper with some sketches of experiments and measured values shown on fig.5.1. Actually, the student produced a problem with a so-called “black box”, the contents of which he now has to recall. The unknown electric circuit exists inside a “black box” with three terminals. Two terminals are grounded, i.e. connected to the earth, which potential is zero. The student remembers that he measured the current through the third, not grounded, terminal. All current scales are in amperes, and the voltage scales are in volts. Using the student’s notes, help him to find the contents of the black box.

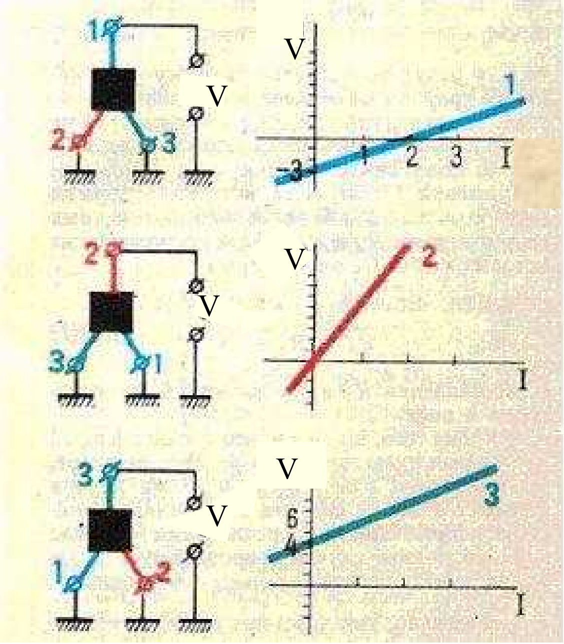

Fig.5.1

Solution

The first drawing shows, that the current between the joint terminals 2 and 3, and the first terminal is zero when the external applied voltage is -3 V. This is possible if the “black box” contains a power source either between terminals 1 and 2, or between terminals 1 and 3, or in both parts of the internal circuit.

The second figure shows that between terminals 1 and 2, and between terminals 3 and 2 may be only resistors, because current is zero when voltage is zero (Ohm’s law).

From the third drawing it is clear that the internal battery exists either between terminals 2 and 3, or between terminals 1 and 3. There may be batteries in both locations.

All the above can exist simultaneously when there is a battery connected in series with a resistor between terminals 1 and 3; and there are only resistors between terminals 1 and 2, and between terminals 2 and 3.

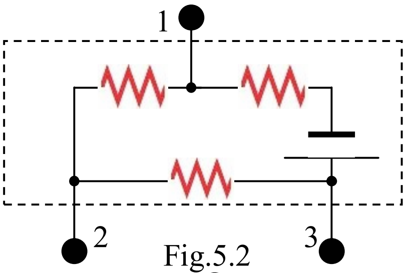

Fig.5.2

The possible connection of the parts of the circuit inside the “black box” is given on the fig. 5.2.

However, it is possible to find inside the “black box” a simpler circuit, which also matches the above conditions. It is shown on the fig. 5.3. Using the given scale of graphs from fig. 5.1, we can solve the problem quantitatively.

The first circuit from fig. 5.1 gives the following equations:

$$-3\text{V} = V_{battery}$$

$$R_1R_2/(R_1+R_2)\cdot 2\text{A} = 3\text{ V}$$

The slope of the second graph is equal to $R_2 = 6\ \Omega$.
Substituting R_2 by this value, we can obtain $R_1 = 2\ \Omega$.
This result can be also obtained with the third graph.

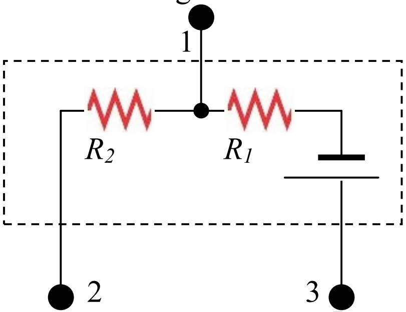

Fig. 5.3