

2006-2007 Physics Olympiad Preparation Program
- University of Toronto -
Solutions. Set 1: General

Problem 1

A small bead with mass M is attached to a very light string hung from a ceiling. The string can be torn by a force exceeding the value of $10 Mg$ where g is acceleration due to gravity. The bead on the string may oscillate harmonically with a period T_0 . A student decides to perform another experiment. He takes the attached bead aside from its equilibrium and pushes it in such a way that the bead performs a full revolution in a horizontal plane.

What is the time of one revolution of the bead that can be calculated with the data, given above?

Is this time a maximum or a minimum value of all possible?

Solution

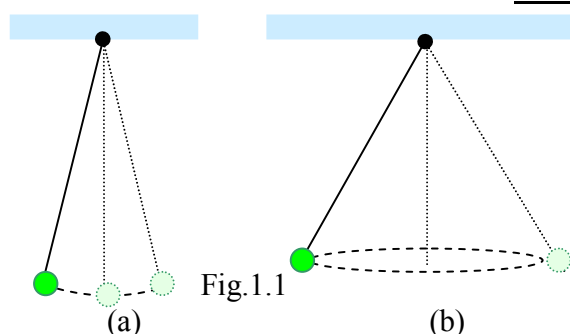


Fig.1.1

Fig.1.1 shows two possible experiments with the bead on the string:

(a) the bead is a simple pendulum that oscillates harmonically with the given period T_0 ; and

(b) the bead moves along a circular path in a horizontal plane.

To find the time of one revolution for the circular motion, we need the value of radius of rotation which depends on the length of the string. The length l may be found from a formula for the period of oscillation of the simple pendulum:

$$T_0 = 2\pi\sqrt{\frac{l}{g}} \Rightarrow l = \frac{T_0^2 g}{4\pi^2}$$

A free-body diagram on fig.1.2 gives the relationship between the centripetal force $M\omega^2 l \sin\alpha$, where ω is an angular speed of the bead, and the given value for the maximum force of tension $F_T = 10 Mg$:

$$M\omega^2 l \sin\alpha = F_T \sin\alpha \quad (1.1)$$

$$M\left(\frac{2\pi}{T}\right)^2 \frac{T_0^2 g}{4\pi^2} = 10Mg$$

$$T = \frac{T_0}{\sqrt{10}} \quad (1.2)$$

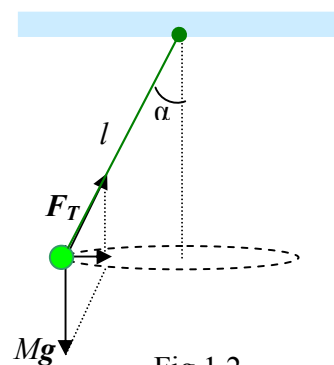


Fig.1.2

The obtained time of one revolution, or the period of rotation of the bead, given by eq.1.2, is the minimum value for the period. Eq.1.1 shows that an increase in the angular speed entails the increase of the force of tension. Therefore, the period is inversely proportional to the force of tension. The free-body diagram gives the general relationship between the angle α and the force of tension and the most general answer:

$$F_T = \frac{Mg}{\cos\alpha}, \quad T = T_0 \sqrt{\cos\alpha};$$

which permits to calculate the angle in our problem for the maximum possible force of tension: $\cos\alpha = 0.1$, and $\alpha = 84.3^\circ$. We also can conclude that for small angles α , $\cos\alpha \rightarrow 1$, and $T \rightarrow T_0$.

$$\text{Finally } T_0 \sqrt{\frac{Mg}{F_{T \max}}} \leq T \leq T_0$$

Problem 2

A sample of an ideal gas is in a vertical cylinder fitted with a piston. As 5.79 kJ of energy is transferred to the gas by heat to raise its temperature, the weight on the piston is adjusted so that the state of the gas changes from point A to point B along the semicircle, shown in fig. 2.1. Find the change in internal energy of the gas.

Solution

To find internal energy we must first calculate the work, done by the gas, and then apply the first law of Thermodynamics.

The work is always an area under the P versus V graph. In our thermal process, the area consists of area of the rectangle and an area of a semicircle

(the curve for P vs. V is a semicircle only for the chosen units).

The area of a true semicircle with radius r is $\frac{1}{2} \pi r^2$. However, we cannot easily apply this formula to the problem, as the dimensions of two axes of coordinates are different. For the circular shape of the P vs. V function it is possible to state that:

$$a (P - P_A)^2 + b (V_0 - V)^2 = \text{const}, \quad (2.1)$$

where a and b are constant factors who bring units of the terms to the identical one; $P_A = 300$ kPa; $V_0 = 3.6$ L; and const is proportional to a squared "radius" of the semicircle.

In general, we should convert units of pressure and volume into SI units. However, sometimes this is not necessary. In our problem, we need a product of pressure and volume to calculate work. If we take pressure in kilopascals and volume in litres, we obtain the same value of their product, as for the product of our data in pascals and cubic meters.

When $P = P_A$, $V = V_A = 1.2$ L; $b (V_0 - V_A)^2 = \text{const}$, and $b = \text{const} / (V_0 - V_A)^2$;

when $V = V_0$, $P = P_0 = 500$ kPa; $a (P_0 - P_A)^2 = \text{const}$, and $a = \text{const} / (P_0 - P_A)^2$.

A substitution of a and b in eq.2.1 by the obtained values and cancellation of the common factor const , give the new equation for the semicircle as:

$$\frac{(P - P_A)^2}{(P_0 - P_A)^2} + \frac{(V_0 - V)^2}{(V_0 - V_A)^2} = 1 \quad (2.2)$$

Equation 2.2 is a standard equation for an ellipse with the total area of

$$A = \pi \cdot (P_0 - P_A) \cdot (V_0 - V_A) = \pi (P_0 - P_A) \frac{V_B - V_A}{2} \quad (2.3)$$

Eq.2.2 is more general than eq.2.1, because it is applicable to the same process shown on the diagram with other units of pressure and volume.

Work W , done by the gas, is given by:

$$W = P_A (V_B - V_A) + A/2 = 2194 \text{ J.}$$

According to the 1st law of Thermodynamics, the change in the internal energy of the gas is:

$$\Delta U = Q - W = 3596 \text{ J} = 3.60 \text{ kJ}$$

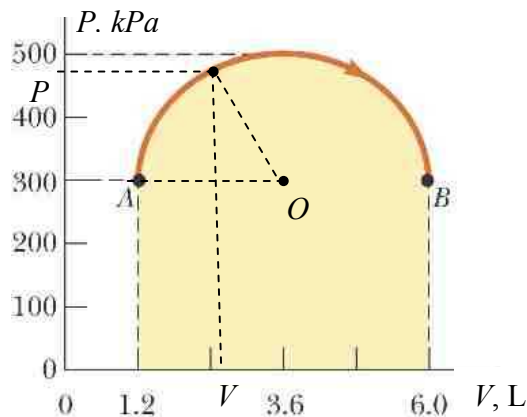


Fig.2.1

Problem 3.

A possible means for making an airplane invisible to radar is to coat the plane with an antireflective polymer. Radar waves have a wavelength of 3.00 cm and the index of refraction of the polymer is $n = 1.50$. How thick would you make the coating?

Solution

Treating the anti-reflectance coating like a camera-lens coating, one can obtain for its thickness t :

$$2t = \left(m + \frac{1}{2}\right) \frac{\lambda}{n}, \quad m = 0; \pm 1; \pm 2; \dots \quad (3.1)$$

Condition (3.1) is the condition for the destructive interference for the normally incident and reflected electromagnetic wave.

The minimum thickness of the coat refers to $m = 0$. This gives for the thickness:

$$t = \frac{\lambda}{4n} = \frac{3.00}{4(1.5)} \text{ cm} = 0.500 \text{ cm}.$$

This anti-reflectance coating could be easily countered by changing the wavelength of the radar to 1.50 cm now creating maximum reflection!

Problem 4

An oceanographer is studying how the ion concentration in sea water depends on depth. She does this by lowering into the water a pair of concentric metallic cylinders (Fig.4.1) at the end of a cable and taking data to determine the resistance between these electrodes as a function of depth. The water between the two cylinders forms a cylindrical shell of inner radius r_a , outer radius r_b , and length L much larger than r_b . The scientist applies a potential difference ΔV between the inner and outer surfaces, producing an outward radial current I . Let ρ represent the resistivity of the water.

- (a) Express the resistivity of the water in terms of the measured quantities L , r_a , r_b , ΔV , and I .
 (b) Estimate roughly the relationship between the resistivity of the water and the ion concentration in it.

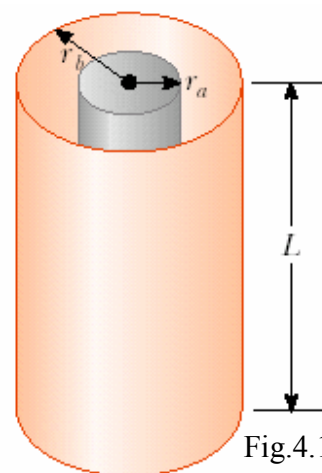


Fig.4.1

Solution

Solution of this problem needs integration, however, solutions with correct idea without integration will be also considered.

(a) Under the potential difference ions of water move between two cylinders in radial direction and experience collisions with molecules of the water. The motion of ions is electric current, and collisions produce the resistance. If we fill the space between the internal and external cylinders with very thin cylindrical shells, their area $A = 2\pi rL$ will continuously change due to the change in radius of the shell. Therefore, the resistance of each shell is different from the resistance of the adjacent shells. A thin cylindrical shell of radius r , thickness dr , and length L contributes resistance

$$dR = \frac{\rho}{A} dl = \frac{\rho}{2\pi rL} dr = \frac{\rho}{2\pi L} \frac{dr}{r}.$$

The resistance of the whole annulus is the series summation of the contributions of the thin shells. This is done by an operation of integration:

$$R = \frac{\rho}{2\pi L} \int_{r_a}^{r_b} \frac{dr}{r} = \frac{\rho}{2\pi L} \ln\left(\frac{r_b}{r_a}\right).$$

The Ohm's law permits to rewrite the equation as

$$\frac{\Delta V}{I} = \frac{\rho}{2\pi L} \ln\left(\frac{r_b}{r_a}\right)$$

Finally, we solve for $\rho = \frac{2\pi L \Delta V}{I \ln\left(\frac{r_b}{r_a}\right)}$.

(b) According to the definition, an electric current I is a charge passing through the cross section of a conductor per time unit: $I = \Delta q / \Delta t$. Because of the law of conservation of electric charge, current is independent of radius. If all ions in the water are charged identically, Δq is proportional to a product of the ion concentration and ion charge. The rough required relationship may be written as:

resistivity is inversely proportional to the ion concentration and the ion charge.

Problem 5

Student's determination of the half-life of ^{137}Ba in an experiment of the first-year Physics laboratory.

The radioactive barium isotope ^{137}Ba has a relatively short half-life and can be easily extracted from a solution containing its parent cesium (^{137}Cs). This barium isotope is commonly used in an undergraduate laboratory exercise for demonstrating the radioactive decay law. Undergraduate student using modest experimental equipment took the data, presented in fig.5.1. Along the vertical axis, the student plotted $\ln R$ that was a natural logarithm of a decay rate, or activity, R . R is the rate of change of the number of nuclei of a parent isotope due to the decay (a conversion into the other isotope). Determine the half-life for the decay of ^{137}Ba using the student's data.

Solution

The number N of nuclei with the half-life T remaining in time interval t is

$$N = N_0 2^{-\frac{t}{T}}, \quad (5.1)$$

where N_0 is the number of nuclei at an instant $t = 0$. Equation 5.1 is called the law of radioactive decay. The logarithmic form of eq.5.1 is following:

$$\ln N = \ln N_0 - \frac{t}{T} \ln 2 \quad (5.2)$$

To find a relationship between the number N and activity R let us write the definition of activity:

$$R = \left| \frac{dN}{dt} \right|$$

Applying the implicit differentiation to equation 5.2, we obtain

$$\frac{dN}{N} = -\frac{\ln 2}{T} dt \rightarrow \left| \frac{dN}{dt} \right| = \left| -N \frac{\ln 2}{T} \right| = N_0 2^{-\frac{t}{T}} \frac{\ln 2}{T} = R_0 2^{-\frac{t}{T}} \quad (5.3)$$

where $R_0 = N_0 \ln 2 / T$ is the activity at an instant of time $t = 0$. Taking the logarithm of equation 5.3, we obtain

$$\ln R = \ln R_0 - \frac{\ln 2}{T} t$$

which is the equation of a straight line with slope $-\ln 2 / T$.

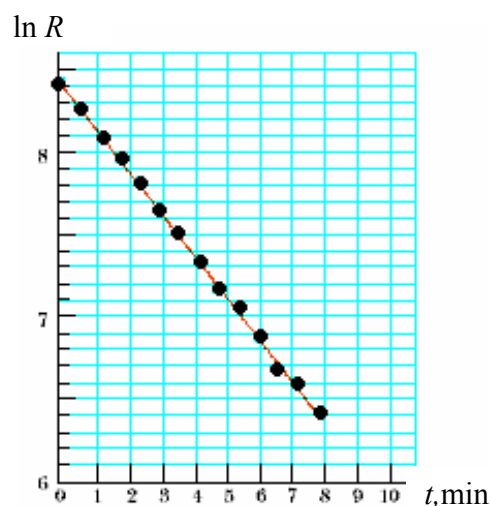


Fig.5.1

The logarithmic plot shown in Figure 5.1 is fitted by $\ln R = 8.4 - 0.25t$.

If t is measured in minutes, then slope is 0.25 per minute. The half-life is

$$T = \frac{\ln 2}{0.25} \text{ min} = 2.8 \text{ min}$$

The reported half-life of ^{137}Ba is 2.6 min. The difference reflects experimental uncertainties.