

2006-2007 Physics Olympiad Preparation Program
- University of Toronto -
Solutions. Set 2: Mechanics

Problem 1

Two similar prisms with the angle $\alpha = 30^\circ$ at the base have smooth surfaces and are initially placed one over the other and both on a frictionless horizontal desk as it is shown on the fig.1.1. The mass of the smaller prism is m , and its height is h_1 ; the mass of the bigger prism is $4m$ and its height is h_2 . After the smaller prism is released, it starts to slide down without friction. At some moment, the relative speed of the upper prism with respect to the lower one is v_{1rel} .

What is the speed of the bigger prism at this moment?

How long does it take the upper prism before it touches the desk?

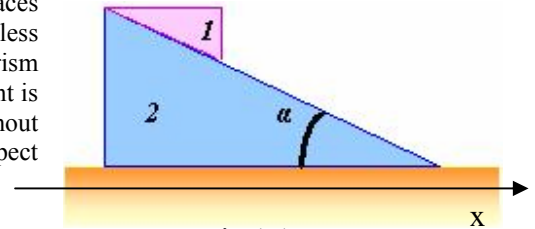


Fig.1.1

Solution

The system of two prisms is isolated in the x direction because of the absence of friction. There is no force in the x direction, and we can apply the law of conservation of linear momentum to the system. It is more convenient to solve the problem in the “laboratory” inertial frame of reference.

In this frame of reference, the horizontal component of velocity of the smaller prism at some instant of time is: $v_{1x} = v_{1rel} \cos \alpha - v_{2x}$. Therefore, the law of conservation of the x -component of the linear momentum is given by:

$$m_1(v_{1rel} \cos \alpha - v_{2x}) = m_2 v_{2x}$$

$$v_{2x} = \frac{v_{1rel} \cos \alpha}{1 + \frac{m_2}{m_1}} = \frac{\sqrt{3}}{10} v_{1rel}$$

Problem 2

A very light pulley is attached to a ceiling of an elevator moving upwards with a constant acceleration a_0 . Two blocks with masses m_1 and m_2 are connected by a string of negligible mass passing over a pulley.

Find (a) an acceleration of the block with m_1 with respect to the elevator; and

(b) a force exerted by the pulley on the ceiling of the elevator.

Solution

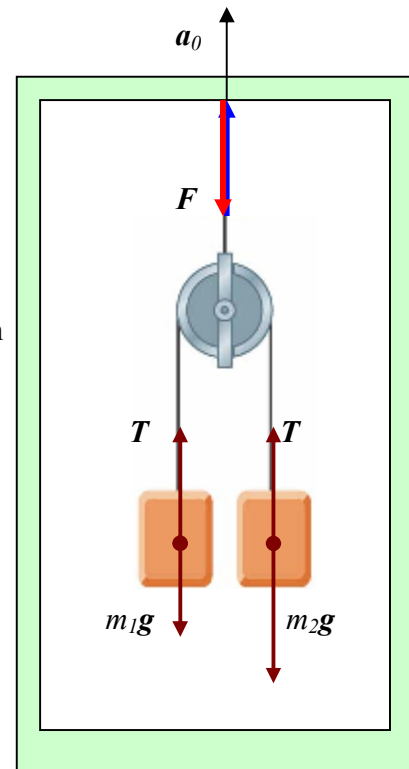
The direction of motion of the first block was not specified, therefore, we may choose it arbitrary upwards. A free-body diagram for masses, pulley and ceiling, as well as forces of action and reaction for the ceiling and the pulley is shown in Figure 2.1 with respect to the observer in an inertial “laboratory” frame of reference. In our version, the positive y -direction is upward. The force exerted by the pulley on the ceiling of the elevator is equal to the force, applied to a system of blocks to move it upwards with acceleration a_0 .

a is a magnitude of acceleration of each block with respect to the elevator. Force of tension T in the string is constant along the string, because the pulley is considered to have a negligible mass and therefore, no moment of inertia.

Combining all the above, we can write two equations of motion for two blocks as follows:

$$m_1(a + a_0) = T - m_1g \quad (2.1a)$$

$$m_2(a - a_0) = m_2g - T \quad (2.1b)$$



Addition of equations (2.1a) and (2.1b) gives for a :

$$a = \frac{(m_2 - m_1)(a_0 + g)}{m_1 + m_2} \quad \text{and for } F = 2T = \frac{4m_1m_2}{m_1 + m_2}(g + a_0)$$

To make the solution complete, we must write the obtained physical quantities as vectors:

$$\vec{a} = \frac{(m_2 - m_1)(\vec{a}_0 - \vec{g})}{m_1 + m_2}; \quad F = \frac{4m_1m_2}{m_1 + m_2}(\vec{g} - \vec{a}_0)$$

Problem 3

A period of harmonic oscillation of a simple pendulum of a length l is given by:

$$T_p = 2\pi\sqrt{\frac{l}{g}} \quad (3.1)$$

The period of harmonic oscillation of a mass m on a spring with a spring constant k is given by:

$$T_m = 2\pi\sqrt{\frac{m}{k}} \quad (3.2)$$

Basing on a similarity of harmonic processes, find the period of harmonic oscillation of a liquid in a glass tube shown in Figure 3.1. The left section of the tube is vertical; the right bend makes an angle of 30° with the vertical; and the cross-sectional area of the tube is everywhere same.

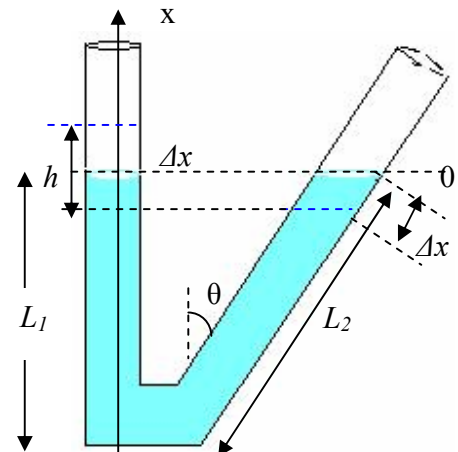


Fig.3.1

Solution

The main feature of the harmonic motion is following: the restoring force exerted on the oscillating object is oppositely directed and directly proportional to the displacement of the object from its equilibrium position.

In equilibrium position, liquid has same level in both sections. Let us choose the vertical segment of the tube to describe the displacement of the level along the chosen x direction and the corresponding restoring force. In equilibrium, $L_1 = L_2 \cos\theta$. If the level of liquid rises by the height of Δx in the vertical tube, the length of the tube filled with liquid to the right decreases by same value Δx . The difference in the levels $h = \Delta x + \Delta x \cos\theta = \Delta x (1 + \cos\theta)$ (3.1).

The restoring force is given by:

$$F_x = -\rho h A g = -[\rho A g (1 + \cos\theta)] \Delta x \quad (3.2)$$

where ρ is the density of the liquid; A is the cross-sectional area of the tube; and g is the acceleration due to gravity. Equation (3.2) matches the requirement for the restoring force in the harmonic motion.

To find a period basing on the similarity of the equations of motion for three objects: the simple pendulum, the mass on a spring, and the liquid in the bent tube, we have to write all three equations and compare the coefficients.

- 1) The simple pendulum demonstrates the motion along the arc, or circular motion. Therefore, its equation of motion is the equation for rotation and involves the moment of inertia of a point mass and the torque of the restoring force (α is the angular acceleration; $l \sin \theta \approx l \theta$ is an arm of the force of gravity; and θ is an angular displacement)

$$m l^2 \alpha - m g l \theta = 0; \quad \text{or } \alpha = \frac{g}{l} \theta \quad (3.3)$$

- 2) The mass on the spring moves under the Hook's law:

$$ma = kx; \text{ or } a = \frac{k}{m}x \quad (3.4)$$

3) The liquid in the bent tube:

$$ma = \rho Ag(1 + \cos\theta)x = \frac{m}{A(L_1 + L_2)} Ag(1 + \cos\theta)x = m \frac{g(1 + \cos\theta)}{L}x, \text{ or } a = \frac{g(1 + \cos\theta)}{L}x \quad (3.5)$$

where L is the length of the tube filled with the liquid (we ignore the lower segment of the tube). Comparing equations (3.3), (3.4), and (3.5), we can conclude that the period of harmonic oscillations of the liquid level in a bent tube has the following expression:

$$T = 2\pi \sqrt{\frac{L}{g(1 + \cos\theta)}}$$

Problem 4

On the North Pole of the Earth, an object is projected upward with an initial speed v_i which is less than the escape speed for the Earth.

(a) What altitude does the object reach?

(b) Does it change if the object is launched from Toronto? Why?

Check the **Science Links** on our front web page to find other necessary data for this problem.

Solution

(a) An object on either geographic pole is not involved into rotational motion around the Earth's axis. Therefore, it has no initial component of velocity associated with this rotation.

Gravitational potential energy is negative. The law of conservation of energy gives:

$$\frac{mv^2}{2} - G \frac{Mm}{R} = -G \frac{Mm}{(R+H)}, \text{ where } H \text{ is the unknown altitude.} \quad (4.1)$$

Taking into account that $G \frac{M}{R^2} = g$ on the earth's surface and dividing all terms of the equation by m , we can simplify the equation (4.1):

$$\frac{v^2}{2} - gR = -g \frac{R^2}{(R+H)} \text{ or } (R+H) = \frac{gR^2}{gR - \frac{v^2}{2}}. \quad (4.2)$$

Solving (4.2) for H , we obtain:

$$H = \frac{gR^2}{gR - \frac{v^2}{2}} - R = \frac{gR^2 - gR^2 + \frac{Rv^2}{2}}{gR - \frac{v^2}{2}} = \frac{R}{\frac{2gR}{v^2} - 1} \quad (4.3)$$

The escape speed of the Earth is $v_{esc} = \sqrt{2gR}$. Our solution (4.3) is valid only for speed less than the escape speed. When $v \rightarrow v_{esc}$, $H \rightarrow \infty$ which must be explained as the object will never return to the point on the Earth surface.

(b) The object launched from Toronto has a vertical (or normal) component of velocity v_{in} in the earth frame of reference and horizontal, or tangential, component v_{it} with which it rotates together with the earth surface around the earth's axis. v_{it} may be calculated from the following equation for

a circular motion of an object: $v = \omega r = \frac{2\pi}{T}(R \cos \theta)$ (4.4), where $T = 24$ hours; $R = 6.37 \times 10^6$ m;

and $\cos \theta = 0.73$ ($\theta = 43^\circ$ N is Toronto's latitude). So, $v_{it} = 338$ m/s = 1217 km/h.

But we cannot simply rewrite the equations (4.3) for Toronto with $v^2 = v_{it}^2 + v_{in}^2$ (4.5), because the object does not stop at the apex of its trajectory and equation (4.1) contains the kinetic energy term to the right hand side also. However, we can use equation (4.5) to calculate the vertical component of the launch velocity for the case when the object escapes the Earth. In this case, equation (4.1) has zero to the right hand side, because the object loses all its initial energy to escape from the Earth's gravitational field. For Toronto, the vertical, or normal, component v_{in} of the escape velocity is less than on the pole due to the natural contribution of the tangential component:

$$v_{in} = \sqrt{2gR - v_{it}^2}$$

Therefore, if we want a rocket to leave the earth's gravitation, it is preferable to launch the rocket from equator where the tangential component of the initial velocity is the greatest on the earth's surface, and direct the launch to the East. For Toronto v_{it} is about 3% of the escape speed.

To calculate the altitude of the object, launched from Toronto, we can apply the law of conservation of energy and the law of conservation of the angular momentum. This will be one of the solutions for the possible trajectories obtained with the Kepler's laws.

Qualitatively we can state that if for the escape velocity the vertical component decreases with moving of the launch point from the pole to South, this means that for the object launched from Toronto with the same vertical initial velocity as on the pole, the launch altitude H must be greater.

Problem 5 (experimental)

For this experiment you need a spool with threads as shown in Figure 5.1.

Place the spool on a horizontal surface, rough enough to prevent slipping of the spool while rolling along the surface. Permit the length of a free end of the thread to be about 20 cm.

- Pulling the thread make the spool to roll towards your hand.
- Pulling the thread make the spool to roll away from your hand.

Exercise 1.

Describe variables that can switch motion (a) to motion (b) and vice versa.

Exercise 2.

Propose a formula to calculate a linear acceleration of the axis of rotation of the spool along the horizontal surface without slipping.

Exercise 3.

Experimentally prove the correctness of your formula and give a complete description of the experiment and its results.

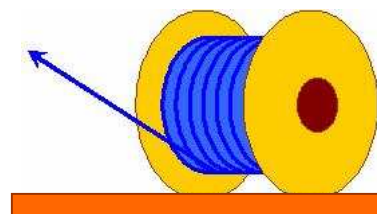


Fig.5.1

Solution

We must mention that this experiment, very simple at a first glance, is actually difficult to execute. The main two difficulties are: a) providing a constant force applied to the thread; and 2) measuring everything at the very beginning of motion, because later, the angle of the force is hard to maintain constant, and it changes as well as the thickness of the layer of threads on the spool!

In addition, the thread must be spooled uniformly with its free end close to the middle of the spool's length.

Exercise 1.

If we have only one spool with a thread, we can conclude that the only variable that changes the direction of motion is an angle θ (Fig. 5.2). The coefficient of friction is not given but it is constant, so it is not a variable for this specific problem.

If we have two geometrically identical spools with different amount of threads on them, it is also possible to have different directions of motion with the same angle θ . Therefore, for one spool in some time after beginning the experiment, same angle will also correspond to a different direction of motion because of the decrease of the thickness of the thread layer.

Exercise 2.

To find a linear acceleration a_x in a horizontal direction, let us write a system of two equations of motion, the linear and the rotational one:

$$ma_x = F\cos\theta - F_f \quad (5.1a)$$

$$I\alpha = F_f R - Fr \quad (5.1b)$$

where m is the instantaneous mass of the spool with thread; R and r are the radii shown in Figure 5.2; F_f is the force of friction; I is the moment of inertia of the spool (also instantaneous) which may be represented as

$I = bmR^2$ with b as a numerical constant; and α is an angular acceleration, related to the value of a_x as $a_x = \alpha R$.

Substituting all defined quantities into the system (5.1), the a_x may be obtained from:

$$a_x = \frac{F\left(\cos\theta - \frac{r}{R}\right)}{m(1+b)} \quad (5.2)$$

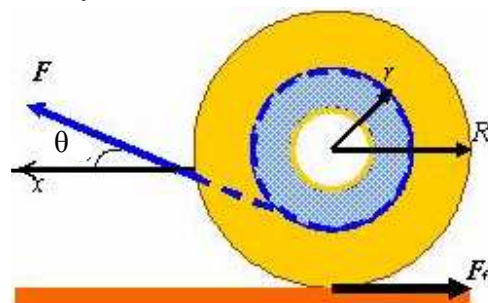


Fig.5.2

We should keep in mind that equation (5.2) gives the instantaneous value for a_x even for constant F and θ , because of the permanent changes in m and r .

The change in the direction of motion to the negative one (in our notation) happens when $\cos\theta$ becomes less than r/R . Therefore, the angle of the switch depends only on the ration r/R !

When θ starts to exceed 90° , a_x is still of the same “negative” direction. As it is really impossible to direct the thread in the way when $\theta > 270^\circ$, the spool will continue to roll in the negative x direction for any angle $\cos^{-1}(r/R) < \theta < 180^\circ$. The last conclusion uses the instantaneous value r , and therefore is valid together with the equation for instantaneous r as a function of time of observation. $I(t)$, $m(t)$ and $r(t)$ could be found and considered only with the information about the thread thickness and the internal radius of the spool, and are not discussed here.

The obtained result can be explained qualitatively comparing the torques of the force F and the force of friction. When the angle θ increases, the normal force decreases, and the force of friction also decreases. Its torque that provides the rolling of the spool in the positive x direction becomes less than the torque of the force F , and the direction of motion changes.