2006-2007 Physics Olympiad Preparation Program

– University of Toronto –

Solutions. Set 4: Optics and Waves

Problem 1

Light, or electromagnetic wave, carry momentum and hence exerts pressure on the surface of incidence. Depending on the absorption factor of the surface, the pressure may be greater or less. If we treat the electromagnetic radiation as a beam of photons, the pressure depends on whether the collision between the photon and the surface is a perfectly elastic or non-elastic. The pressure, produced by molecules of a gas on the walls of the container, has a similar nature, as a pressure of photons on the surface. The intensity I of a light beam is an average power passing through the unit area, perpendicular to the beam. The light pressure P is related to the intensity as: $P = k I / c$, where, k is the absorption factor; and c is the speed of light. $k = 2$ for the totally reflective surface normal to the beam of photons.

(a) Determine the value of k for the totally absorbing surface (black body).

Lasers have been used to suspend spherical glass beads in the Earth's gravitational field. The laser is the source of photons with one frequency (coherent radiation). The momentum p of the photon and its energy E are related as: $p = E / c$.

(b) A black bead has a mass m and a density ρ. Determine the radiation intensity needed to support the bead.

(c) If the beam has a radius r, what is the power required for this laser?

Solution

(a) Similar to the perfectly elastic collision of a small particle with a massive wall, the absolute value of change of the linear momentum of the particle due to the collision is equal to 2x(normal, with respect to the wall surface, component of the momentum of the particle before the collision). After the perfectly inelastic collision, the particle sticks to the wall. Therefore, its linear momentum is completely lost. It means that the absolute value of change of the linear momentum of the particle in this case is equal to the initial momentum, or is two times less than that in the elastic collision. The pressure on the wall is a result of an average force exerted on the wall by the incident particles. The force is a rate of change of the linear momentum. Hence, we must conclude that for the totally absorbing surface (inelastic collisions of photons with the surface), $k = 1$.

(b) Think of light going up and being absorbed by the bead which presents a face area πr_b^2. Actually, the area of interaction of the bead surface with the beam of photons is $2\pi r_b^2$. But the angle of incidence for the parallel beam of photons is different at different points on the surface of the hemisphere of the bead. The black color of the bead means the total absorption of all incident photons. We are interested only in one component of force, directed vertically upwards. Therefore, we can consider the pressure to be a result of acting of this component of force on the cross-sectional area πr_b^2. As pressure and intensity are related via $P = I / c$, the magnitude of the vertical force is given by:

$$F_\ell = \frac{I\pi r_b^2}{c} = mg = \rho\frac{4}{3}\pi r_b^3 g \quad \text{and} \quad I = \frac{4\rho g c}{3}\left(\frac{3m}{4\pi\rho}\right)^{1/3} = \boxed{8.32\times 10^7\ \mathrm{W/m^2}}$$

(c) Upon the definition of intensity, the power is (intensity) x (area of the laser beam):

$$\text{Power} = IA = \left(8.32\times 10^7\ \mathrm{W/m^2}\right)\pi\left(2.00\times 10^{-3}\ \mathrm{m}\right)^2 = \boxed{1.05\ \mathrm{kW}}$$

Problem 2

Light passes from air into flint glass with index of refraction n.

(a) What angle of incidence must the light have if the component of its velocity perpendicular to the interface is to remain constant?

(b) Can the component of velocity parallel to the interface remain constant during refraction?

Solution

Before answering the questions of the problem, it is useful to recall the law of refraction (Snell's law) and definitions of some quantities involved in the equations for refraction.
Upon the definition of the index of refraction n for a medium, the velocity of propagation of light in this medium v relates to the velocity of light in vacuum c as: $v = c/n$. For air, we consider the index of refraction $n_{air} = 1$.
The Snell's law is given by the equation (see Figure 2.1):

$$\sin\theta_1 = n\sin\theta_2 \qquad (1)$$

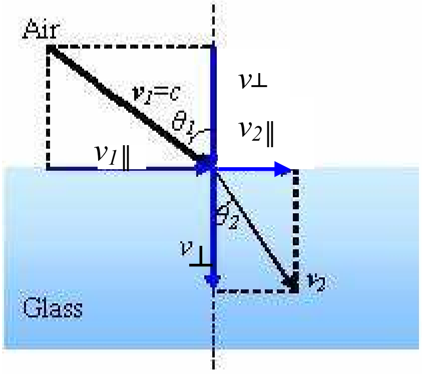


Figure 2.1

(a) For the normal component of velocity to be constant,

$$v_1\cos\theta_1 = v_2\cos\theta_2$$

or $$c\cdot\cos\theta_1 = \left(\frac{c}{n}\right)\cos\theta_2 \qquad (2)$$

We multiply Equations (1) and (2), obtaining: $\sin\theta_1\cos\theta_1 = \sin\theta_2\cos\theta_2$

$$\sin 2\theta_1 = \sin 2\theta_2 .$$

The solution $\theta_1 = \theta_2 = 0$ does not satisfy Equation (1) and must be rejected. The physical solution is $2\theta_1 = 180° - 2\theta_2$ or $\theta_2 = 90.0° - \theta_1$. Then Equation (1) becomes:

$$\sin\theta_1 = n\cos\theta_1, \text{ or } \tan\theta_1 = n$$

which yields $\theta_1 = \tan^{-1} n$.

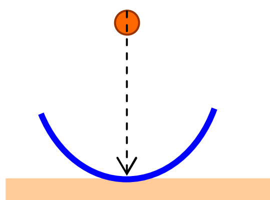
(b) Light entering the glass slows down and makes a smaller angle with the normal. Both effects reduce the velocity component parallel to the surface of the glass, so that component cannot remain constant, or will remain constant only in the trivial case $\boxed{\theta_1 = \theta_2 = 0}$.

Problem 3

A ball is dropped at t = 0 from rest 3.00 m directly above the vertex of a concave mirror that has a radius of curvature of 1.00 m and lies in a horizontal plane.
(a) Describe the motion of the ball's image in the mirror.
(b) At what time do the ball and its image coincide?

Solution

(a) The image starts from a point whose height above the mirror vertex is given by



$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}=\frac{2}{R} \qquad \frac{1}{3.00\text{ m}}+\frac{1}{q}=\frac{1}{0.500\text{ m}} .$$

Therefore initially, the distance from the image to the vertex is $q = 0.600$ m
As the ball falls, p decreases and q increases. Ball and image pass when $q_1 = p_1$.
When this is true,

$$\frac{1}{p_1}+\frac{1}{p_1}=\frac{1}{0.500\text{ m}}=\frac{2}{p_1} \quad \text{or} \quad p_1 = 1.00\text{ m} .$$

As the ball passes the focal point, the image switches from infinitely far above the mirror to infinitely far below the mirror. As the ball approaches the mirror from above, the virtual image approaches the mirror from below, reaching it together when $p_2 = q_2 = 0$.

(b) The falling ball passes its real image when it has fallen

$$3.00\text{ m} - 1.00\text{ m} = 2.00\text{ m} = \frac{1}{2}gt^2, \text{ or when } t = \sqrt{\frac{2(2.00\text{ m})}{9.80\text{ m/s}^2}} = \boxed{0.639\text{ s}}$$

The ball reaches its virtual image when it has traversed

$$3.00\text{ m} - 0 = 3.00\text{ m} = \frac{1}{2}gt^2, \text{ or at } t = \sqrt{\frac{2(3.00\text{ m})}{9.80\text{ m/s}^2}} = \boxed{0.782\text{ s}} .$$

Problem 4

A photoelectric effect is a process of emanation of an electron from the surface of a metal after the absorption of a photon with energy enough to make an electron free with some initial kinetic energy. The minimum energy needed to release the electron with zero kinetic energy is called the work function of the material of the surface. Electrons in the photoelectric effect are called the photoelectrons.

A light source emitting radiation at 7.00×10^{14} Hz is incapable of ejecting photoelectrons from a certain metal. In an attempt to use this source to eject photoelectrons from the metal, the source is given a velocity toward the metal.

(a) Explain how this procedure produces photoelectrons.

(b) When the speed of the light source is equal to 0.280c, photoelectrons just begin to be ejected from the metal. What is the work function of the metal?

(c) When the speed of the light source is increased to 0.900c, determine the maximum kinetic energy of the photoelectrons.

Solution

The Einstein's formula for the photoelectric effect gives the following relationship among the frequency of a photon f, the work function of the metal φ, and the maximum kinetic energy of the ejected electron $K_{\max}$:

$$hf = \varphi + K_{\max} \qquad (1)$$

where h is the Planck's constant (h = 6.63x10^{-34} J·s). If $f < f_{\text{cut}}$, where f_{cut} is called the cutoff frequency, the photoelectric effect is impossible. So, $f_{cut} = \varphi / h$.

(a) When the photon source moves toward the metal, the incident photons are Doppler shifted to higher frequencies that exceed f_{cut} and make the photoelectric effect possible.

(b) If v = 0.280c, the frequency $f' = f_{cut}$.

$$f' = f\sqrt{\frac{1+v/c}{1-v/c}} = (7.00\text{x}10^{14})\ (1.33) = 9.33\text{x}10^{14}\text{ Hz.}$$

Therefore, φ = (6.63x10^{-34} Js) · (9.33x10^{14} Hz) = 6.18x10^{-19} J = 3.87 eV .

(c) At v = 0.900 c f' = 3.05x10^{15} Hz; and

$K_{\max} = hf' - \varphi$ = (6.63x10^{-34} Js) · (3.05x10^{15} Hz) / (1.60x10^{-19} J/eV) – 3.87 eV = 8.78 eV

Problem 5 (experimental)

For this experiment you will need a water tank, a timer, a ruler, thoroughness and patience.

1) Use a method of dimensions (or units) to determine the relationship among the speed c of propagation of waves in the water tank, the acceleration due to gravity g, and the depth H of water in the tank. This method gives a formula without exact dimensionless constants that may also exist in the function c (g, H). Hint: there are no other physical quantities, except g and H, in the formula. Show your work and the obtained formula.
2) Perform a number of experiments with waves varying the depth of water in the tank and measuring quantities necessary to calculate the speed of propagation of the waves. Hint: Try to produce the waves with approximately same frequency.
   Organize results of your measurements in the form of a table.
3) Determine the value(s) of constant(s) in the formula based on the performed experiments.
4) Show the final formula for the function c (g, H).

Solution

1) The method of dimensions is based on the derivation of the dimension of the unknown function using the proper combination of dimensions of the given quantities. In our problem, the dimensions may be combined in the following way:

$$\frac{m}{s} = \sqrt{\left(\frac{m}{s^2}\right)m}$$

For the function we can conclude that

$c = A\sqrt{gH}$, where A is a dimensionless constant, or just a number. (1)

This solution is not the single possible but the simplest one.

2) The next step is to examine the validity of the proposed relationship. This may be simplified by switching to the logarithmic scale:

$$\ln c = \ln A + \tfrac{1}{2}\ln(gH) \qquad (2)$$

The table for measured values may look like:

#	H	gH	$\ln(gH)$	c	$\ln c$
1					
2					
3					

3) Using the data from the Table, we should plot results as {ln$c$} vs. {ln(gH)}.
If the initial assumption (1) was correct, the plots must show a linear function with a slope of ½ and ln$A$ as the axis {ln$c$}intercept.

4) The correct value of A is 1.