2007-2008 Physics Olympiad Preparation Program
University of Toronto

Solutions to Problem Set 1: General
Last revised: November 07, 2007

1. This problem seems to require just a few simple manipulations of the projectile motion formulae. However, one should be careful with the analysis, since the existence and the number of the solutions depend on the particular value of v_0.

(i) The ball should be launched above the horizontal line, $\alpha_0 \geq 0$, otherwise the "one-jump task" cannot be completed.

With the (x, y) coordinate axes as indicated in Figure 1, the equations of motion read

$$x = -R + (v_0 \cos\alpha_0)t, \quad (1)$$

$$y = (v_0 \sin\alpha_0)t - \frac{g}{2}t^2. \quad (2)$$

From (1) we express t, and substitute it into (2) to obtain the equation of the trajectory

$$y = (x+R)\tan\alpha_0 - \frac{g}{2}\frac{(x+R)^2}{v_0^2\cos^2\alpha_0}. \quad (3)$$

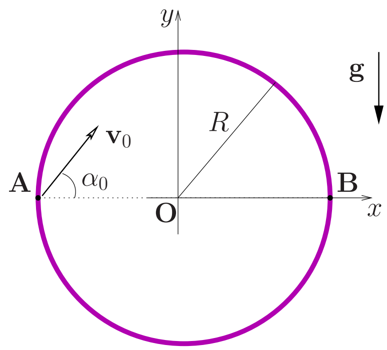

Figure 1: Illustration to the problem 1(i).

Point B is on the trajectory, therefore its coordinates obey (3); by substituting $(x_B = R,\ y_B = 0)$ into (3) we obtain

$$\sin 2\alpha_0 = \frac{2gR}{v_0^2}. \quad (4)$$

Seemingly this is the answer, since now the angle α_0 can be expressed via the $\arcsin$ of a function of v_0, namely

$$\alpha_0 = \frac{1}{2}\arcsin\left(\frac{2gR}{v_0^2}\right). \quad (5)$$

Notes:

(a) In our case $\arcsin : (0,1] \to (0,\pi/2]$, therefore equation (5) gives angles in the $\alpha_0 \in (0,\pi/4]$ interval only.

(b) The time it takes to complete the motion on the x and y axes is the same, hence $2R/v_0 \cos\alpha_0 = 2v_0 \sin\alpha_0/g$ and we may arrive at (4) in a shorter way.

However, the number of solutions may depend on the particular values of v_0. A discussion is therefore in order.

Indeed, a quick glance at (4) tells us that since $0 \le |\sin 2\alpha| \le 1$ we necessarily need $0 \le 2gR/v_0^2 \le 1$. Therefore, we have solution(s) only if $v_0 \ge \sqrt{2gR}$.

If $v_0 = \sqrt{2gR}$ then $\alpha_0 = \pi/4$, and this solution is unique.

If $v_0 > \sqrt{2gR}$ then a classic result from the projectile motion theory tells us that for a given v_0 there are always *two* angles that provide the same horizontal projectile range, namely $\alpha_0 = \gamma$ (given by (5)) and $\alpha_0 = \pi/2 - \gamma$. Note that $0 < \gamma < \pi/4$, and indeed $\sin(2\gamma) = \sin(\pi - 2\gamma)$. Therefore, for any given v_0 (with $v_0 > \sqrt{2gR}$) point B can be reached in one jump by launching the ball under two different angles, as described above. However, this is not the entire story yet!

One of the solutions has $\alpha_0 < \pi/4$ and is given by (5); the angle is monotonically decreasing as v_0 increases.

For the solution with $\alpha_0 > \pi/4$ we have

$$\alpha_0 = \frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{2gR}{v_0^2}\right), \tag{6}$$

and now α_0 increases as v_0 increases. However, at some point we have to stop increasing v_0 and hence α_0, since otherwise the ball will hit the "dome" if we aim too high, and there is no one-jump solution for some of the $\alpha_0 > \pi/4$. Therefore, for sufficiently large v_0 we end up with only one solution, with $\alpha_0 < \pi/4$ as given by (5).

Let us now find the values of v_0 that pose the risk of hitting the sphere. One way of doing it is to demand that the equation of the trajectory (3) and the equation of the circle $x^2 + y^2 = R^2$ have no common solutions, other than the points A and B. Since the trajectory is symmetric relative to the Oy axis, one may consider just the case $-R < x < 0$ and $0 < y < R$ (upper left quadrant).

Nevertheless, the rather involved algebra & trigonometry may be avoided by noticing that in any case, if the ball is to hit the dome, the highest point of the " impact trajectory" would be above the top of the dome (this method works if the trajectory does not cross the circle twice in the left-upper quadrant, as depicted with dotted line in the upper half of Figure 2; we shall show that such weird trajectory does not occur in our case). Hence, to avoid hitting the dome it is sufficient to demand that the maximum height (y_{max}) the ball may achieve is smaller than R.

From the symmetry of the trajectory we see that y_{max} occurs at $x = 0$, which gives

$$y_{max} = R\tan\alpha_0 - \frac{g}{2}\frac{R^2}{v_0^2\cos^2\alpha_0} < R. \tag{7}$$

The above inequality implies

$$\tan\alpha_0 < 1 + \frac{gR}{v_0^2}\frac{1}{2\cos^2\alpha_0}. \tag{8}$$

From (4) one can express gR/v_0^2 via $\sin 2\alpha_0$, and use a bit of trigonometry to arrive at

$$1 < \tan\alpha_0 < 2, \tag{9}$$

where we added the left-hand side inequality to emphasize that the analysis is carried out for $\alpha_0 > \pi/4$. This result states that the maximum angle one can aim at (and still get from A to B in one jump) should be smaller than $\alpha_{0max} = \arctan 2 \approx 0.35\pi \approx 63.5°$. From (4) we find the velocity corresponding to this angle, namely $v_0 < \sqrt{5}/2\sqrt{2gR}$. The reader may also check that for $\alpha_0 = \pi/4$, the ball only reaches the height $y_{max} = R/2$.

Summary:

- we have no solution if $v_0 < \sqrt{2gR}$
- we have one solution (or two degenerated solutions if you prefer to say so) if $v_0 = \sqrt{2gR}$, namely $\alpha_0 = \pi/4$.
- we have two solutions for α_0 (given by (5) and (6)) as long as $\sqrt{2gR} < v_0 < \sqrt{5}/2\sqrt{2gR}$.
- for $v_0 \geq \sqrt{5}/2\sqrt{2gR}$ we only have one solution, with $\alpha_0 < \pi/4$, as given by (5).

There is one loose end to tie up, namely to prove that the dotted weird trajectory depicted in the upper half of Figure 2 is not actually possible. Let us show that in our case the parabola is always below the circle arc, provided

$$\frac{\pi}{4} < \alpha_0 < \arctan 2, \tag{10}$$

and the corresponding condition on v_0,

$$\sqrt{2gR} < v_0 < \sqrt{5}/2\sqrt{2gR}. \tag{11}$$

We have to prove that for any $-R < x < 0$ the following inequality holds

$$y^2_{trajectory} < R^2 - x^2, \tag{12}$$

subject to the constraints (11) and (10). Here $y_{trajectory}$ is given by (3). After some algebraic manipulations we arrive at the equivalent inequality

$$F(z) \equiv z - 2 + z\left[\tan\alpha_0 - \frac{\xi}{4}z(1+\tan^2\alpha_0)\right]^2 < 0, \tag{13}$$

where $z \equiv (x+R)/R$, with $0 < z < 1$, and $\xi \equiv 2gR/v_0^2$, with $4/5 < \xi < 1$. In the square bracket we use the trigonometric relation $1 + \tan^2\alpha_0 = 2\tan\alpha_0/\sin 2\alpha_0$ and account for (4) to arrive at

$$(z-2)\left[1 + \frac{z(z-2)\tan^2\alpha_0}{4}\right] < 0. \tag{14}$$

We notice that the quadratic function of z in the square bracket is always positive for $1 < \tan\alpha_0 < 2$ (indeed, the discriminant of the equation $[\dots] = 0$ is $\Delta = \tan^2\alpha_0(\tan^2\alpha_0/4 - 1) < 0$, and hence the $[\dots]$ is one-signed for all z, in this case positive). For $0 < z < 1$ the factor $(z-2) < 0$ and therefore the inequality (14) is fulfilled over the domain of interest. Our proof is now complete.

(ii) From equation (5) we see that $\alpha_0 = 0$ only if $2gR/v_0^2 = 0$. But since $gR \neq 0$, the only way to get 0 is to demand $v_0 \to \infty$. Therefore, one needs $v_0 = \infty$ in order to reach B in one jump. But this is not a physically sensible solution since v_0 in any case has to be less that the speed of light. Alternatively, and mathematically more correct, you may say that the problem has no solution. Both explanations will receive full mark (but we expect *some* explanation).

(iii) There is one obvious solution, in which the horizontally launched ball hits point C, and bounces upward to hit point B, as suggested by the dashed trajectory in Figure 2. The motion of the "shadows" of the ball on the x– and y–axis takes the same time. We have $x = v_0\tau = R$, and

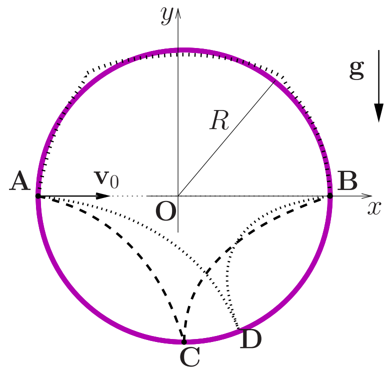

Figure 2: Illustration for the problem # 1, part 2.

$y = -g\tau^2/2 = -R$, where τ is the time the ball falls from A to C. Expressing τ from the y equation and substituting into the x equation we obtain

$$v_0 = \sqrt{\frac{gR}{2}} = \frac{1}{2}\sqrt{2gR}, \tag{15}$$

where we forced out the $1/2$ factor on purpose, so that one can relate the answer to the well-known result for the speed of free fall over a height R.

It is necessary to argue that there cannot be any other solution, on some odd trajectory of the sort depicted with dotted line in the bottom half of Figure 2. A rigorous proof exists, but we provide here only a heuristic explanation (although it is not bulletproof!). Suppose v_0's value is given by (15); then by increasing the speed a little bit, the symmetry of the trajectory is broken, and the ball doesn't hit B anymore (the ball overshoots the y_B level, or may bounce back, to the left of the point D). The same reasoning applies if one takes smaller values of v_0 (undershooting).

2. The finite–infinite pairs of resistance values in the first set of measurements lead us to the conclusion that the box contains diode(s). Indeed, when the "+" of the ohmmeter is applied to the anode, the diode opens, and since the diode is ideal the forward conduction resistance is 0; on the contrary, if "$-$" of the ohmmeter is applied to the the anode, the diode does not allow any significant current to pass, therefore the equivalent resistance is $\Delta V/I \to \infty$. In our case the anode of the diodes seems to be connected towards the terminals A and B, as shown in Figure 3. Based on the first set of measurements we take $R_{ac} = 8\,\mathrm{k}\Omega$ and $R_{bc} = 5.5\,\mathrm{k}\Omega$. The proposed scheme also satisfies the "double" ∞ measurement from Set 1, since then the diodes are connected head-to-head. At first glance the measurements from the Set 2 seem to agree with the circuit in Figure 3. Indeed, the resistance between the pair AC–B is just R_{bc} or ∞, depending on the polarity, and the same applies to the BC–A measurement (either ∞ or R_{ac}).

However, the very last measurement in Set 2 claims the finite resistance of the AB–C pair to be $4.8\,\mathrm{k}\Omega$, while based on our circuit guess we expect the same resistance to be $R_{AB-C} = R_{ab}R_{bc}/(R_{ab}+R_{bc}) = 3.26 \approx 3.3\,\mathrm{k}\Omega$. There are two possibilities: either the table has a typo (not in this case), or the simple scheme shown in Figure 3 is not entirely correct.

At this point one may notice that the resistors R_{bc} and R_{ac} may be composite, in the sense that they may be made up of a few resistors connected in series *and* parallel. Notice that the $4.8\,\mathrm{k}\Omega$ measurement is a higher resistance than what we expected based on the initial guess, which hints

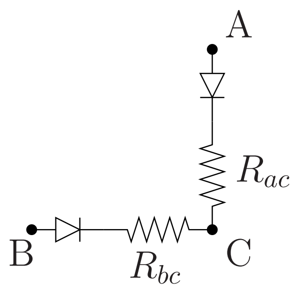

Figure 3: Black box – initial guess

to the equivalent resistance being some serial/parallel, rather than purely parallel connection of the resistors (the parallel connection gives too low of an equivalent resistance).

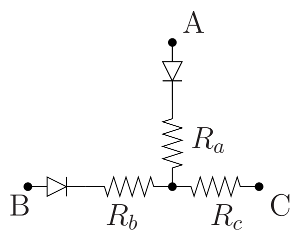

Figure 4: Black box – second (and final) guess

The scheme of a circuit that might agree with the measurements is presented in Figure 4. According to what we know so far

$$
\begin{aligned}
R_b + R_c &= R_{bc}, &(16)\\
R_a + R_c &= R_{ac}, &(17)\\
\frac{R_a R_b}{R_a + R_b} + R_c &= 4.8\text{k}\Omega. &(18)
\end{aligned}
$$

We express R_a, R_b from the first two equations as functions of R_c to arrive at the equation

$$R_c^2 - 9.6R_c + 20.8 = 0, \tag{19}$$

which has two roots, both positive, $R_{c-} = 3.3\,\text{k}\Omega$ and $R_{c+} = 6.3\,\text{k}\Omega$. However, the following inequalities should hold: $0 < R_a < R_{ac},\ 0 < R_b < R_{bc}$ and $R_c < 5.5\,\text{k}\Omega$). Therefore, we only retain $R_c = 3.3\,\text{k}\Omega$, since the second root is larger than $5.5\,\text{k}\Omega$.

Finally, we conclude that the black box contains the circuit shown in Figure 4, with two diodes connected as depicted, and with the resistors having the values:

$$R_a = 4.7\,\text{k}\Omega, \quad R_b = 2.2\,\text{k}\Omega, \quad R_c = 3.3\,\text{k}\Omega.$$

One can easily check that both sets of measurements are now satisfied (appropriate rounding to 2 significant digits needs to be performed).

Notes:

(a) The solution is not unique! Indeed, apart from the trivial non-uniqueness of "making up" each resistor using a series or parallel connection, the "star" (a.k.a. "Y") connection of the R_a, R_b, R_c can in any case be replaced by the so-called Δ connection, as depicted in Figure 5. The formula for the "Delta–wYe" transformation (as the engineers call it) can be easily obtained by expressing the equivalent resistance between any two pairs of terminals (1-2, 2-3, 1-3 in Figure 5) using either the Y or the Δ scheme. For example, $r_1 + r_3 = R_{31}(R_{23} + R_{12})/(R_{12} + R_{23} + R_{31})$, and the remaining two cyclic permutations.

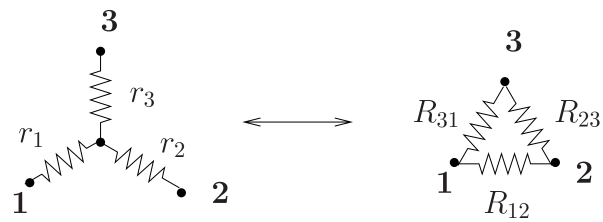

Figure 5: Black box - part 3

(b) More on the engineering side of the problem. The resistors used in this problem have real-life values, since 2.2, 3.3, 4.7 (kΩ) are part of the IEC 60063 standard E6 series (20%): $(10, 15, 22, 33, 47, 68) \times 0.1$. Note that $2.2 + 20\% \approx 3.3 - 20\%,\ 3.3 + 20\% \approx 4.7 - 20\%$, and similar relationships for the rest of the values in the E6 series; in practice this means that if you have a box with resistors belonging to this 20%-tolerance series, you have a chance to find whatever intermediate value of resistance you wish.

3. Let us consider, at an arbitrary location on the wire, a very short segment $\Delta\ell$ as shown in Figure 6. The Ohm's law for the $\Delta\ell$ segment reads

$$R = \rho \frac{\Delta\ell}{S}, \tag{20}$$

therefore, the potential difference across the segment is

$$\Delta U = I_0 R = I_0 \rho \frac{\Delta\ell}{S}. \tag{21}$$

Note that ρ was assumed constant across the $\Delta\ell$ stretch (this is a valid approximation if $\rho' \Delta\ell \ll \rho$, where ρ' is the derivative of ρ with respect to the coordinate along the wire; in the limit $\Delta\ell \to 0$ this is even "more true", since ρ' stays finite, in general).

From the electrostatics we know that

$$|\vec{E}_\ell| = \lim_{\Delta\ell \to 0} \frac{\Delta U}{\Delta\ell} = \frac{\rho I_0}{S}, \tag{22}$$

where $\vec{E}_\ell$ is the intensity of the electric field at the location we picked our $\Delta\ell$. Therefore, the equation (22) gives the dependence of E on ρ, at a given location along the wire.

Let ΔQ be the volume charge in the wire segment $\Delta\ell$. Then in the limit $\Delta\ell \to 0$, the field to the left and to the right of the segment is given by

$$E_L = E_\ell - \frac{\Delta Q}{2\varepsilon_0 S}, \qquad E_R = E_\ell + \frac{\Delta Q}{2\varepsilon_0 S}, \tag{23}$$

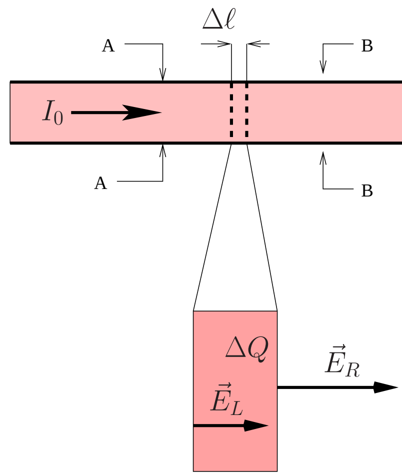

Figure 6: Illustration to problem # 3

where we used the formula for the electric field near an (infinite) planar charge distribution of surface density $\Delta Q/S$ (the $\Delta\ell$ segment is not drawnto scale; in fact we have $\Delta\ell \ll \sqrt{S}$, so that we may think of the charge distribution as being an infinite plane tested in its close vicinity). Therefore, the change in the magnitude of the electric field $|\vec{E}|$ inside the wire as we pass through a charged region is

$$\Delta E = E_R - E_L = \frac{\Delta Q}{\varepsilon_0 S} = \Delta\rho \frac{I_0}{S}, \tag{24}$$

where for the last equality we took into account (22), now written for E_L and E_R. Of course, the same result could have been obtained by applying directly the Gauss's theorem for the volume element of length $\Delta\ell$ and cross section S.

Equation (24) relates the charge that accumulates in a given volume with changes in ρ, namely

$$\Delta Q = \varepsilon_0 I_0 \Delta\rho. \tag{25}$$

It can be seen easily that over a finite length (say, the segment AB) the total charge is just the sum of individual elementary ΔQ's, hence

$$Q_{\rm AB} = \varepsilon_0 I_0 (\rho_B - \rho_A). \tag{26}$$

In our case $Q_{\rm AB} > 0$, since the problem stated $\rho_B > \rho_A$.

4. In the first case the light passes through the lens twice. Therefore, the setup lens + mirror is equivalent to a system consisting of a lens "twice as powerful", in the sense that the optical power is $D_1 = 2D_0$, or in terms of equivalent focal length $1/F_1 = 2/F_0$. The light source was at distance $F_0 = 2F_1$, which is twice the focal length of the equivalent system. Therefore, the image coincides with the source itself, $l_1 = F_0 = 10\,\mathrm{cm}$.

When the lens is half-immersed in water the optical power of the compound system is $D_2 = 2D_0 + 2D$, where D is the optical power of the water "lens". In terms of focal distances we have $1/F_2 = 2/F_0 + 2D$. But from the condition of the problem we have $l_2 = 2F_2$. Hence,

$$\frac{2}{l_2} = \frac{2}{F_0} + 2D \to D = \frac{F_0 - l_2}{l_2 F_0} < 0. \tag{27}$$

In the third case we have $D_3 = 2D_0 + 4D$ and $l_3 = 2F_3$. Therefore,

$$\frac{2}{l_3} = \frac{2}{F_0} + 4\frac{F_0 - l_2}{l_2 F_0}, \tag{28}$$

which leads to

$$l_3 = \frac{l_2 F_0}{2F_0 - l_2} = 30\,\mathrm{cm}. \tag{29}$$

5. Assuming the gas is ideal (we don't have many other choices anyway) one has the equation of state

$$p = \frac{m}{V}\frac{R}{\mu}T, \tag{30}$$

where V is the volume of a given region of the atmosphere, and m and T– the mass and the temperature of the gas in there (here, R is the universal gas constant). The region is large enough to be able to talk about (macroscopic) pressure, and small enough to be able to assume a uniform T. Since $m/V = \rho$, with ρ being the density, (30) can be re-written in a truly local form involving only the intensive properties of the gas,

$$p = \rho\frac{R}{\mu}T. \tag{31}$$

We see that if ρ is constant then $p = p(T)$, or vice-versa, $T = T(p)$. The question is now: how does the pressure vary in this planetary atmosphere? At this point we realize that the pressure is nothing but some force divided by the surface area; in our case it is the weight of the gas above a certain location that determines the pressure. At the base of a vertical column of cross-section area S we have

$$p = \frac{mg}{S} = \frac{(\rho S h)g}{S} = \rho g h, \tag{32}$$

where we assumed g to be uniform in the column. The task is now to find g. From Newton's law, the weight W of a given mass m on the planet is

$$W = G\frac{Mm}{a^2}, \tag{33}$$

where a is the distance of the mass m to the center of the planet. But $W = mg$, and therefore

$$g = G\frac{M}{a^2}. \tag{34}$$

Close enough to the surface of the planet we assume as before that g is uniform (which is an OK approximation for any decent sized planet with shallow atmosphere); since $h \ll r$ the variation of g is negligible over the region where atmosphere resides, and we may replace in (34) a by r (the radius of the planet).

We now combine (32), (31), and (34), to obtain

$$T = \frac{\mu g h}{R} = \frac{\mu h}{R}\frac{GM}{r^2}. \tag{35}$$

Note: If instead of $\rho = \mathrm{const.}$, we would have assumed T to be constant (and let ρ vary), then the pressure would have decreased exponentially with height: $p = p_s \exp(-z/H)$, where $H = RT/(\mu g)$ is the height scale, and z the geometric height, measured from the surface upward.