2007-2008 Physics Olympiad Preparation Program
University of Toronto

Solutions to problem set 2: Mechanics
Last revised: December 06, 2007

1. It turns out this problem is deceivingly simple. A 100% analytic solution is in principle possible, although it proves to be fairly tedious to carry out. To make our life easier we will employ some computational tools here and there, as we develop our solution.

We first notice that for $v_0 \to 0$ all the fragments end up in the bottom hemisphere (hence $\eta = 1$), while for $v_0 \to \infty$ the gravity has little time to act and therefore $\eta \to 1/2$.

Let us now increase v_0 starting from very small values. The first collision with the upper hemisphere occurs for the fragments launched at $\alpha_0 = \pi/4$ with $v_0 = \sqrt{gR}$. Indeed, under these conditions the points on the horizontal diameter are being hit. For $v_0 < \sqrt{gR}$ all the fragments eventually end up in the bottom hemisphere, and so $\eta = 1$. Note also that the top of the sphere is reached for $v_0 = \sqrt{2gR}$, and an "educated guess" suggests that for $v_0 > \sqrt{2gR}$ all of the high-angle trajectories will hit the upper hemisphere. Of course, we assume there are no relativistic effects and that the fragments do not collide with each other in-flight.

A sketch of $\eta(v_0)$ based on the above asymptotic reasoning is shown in Figure 1.

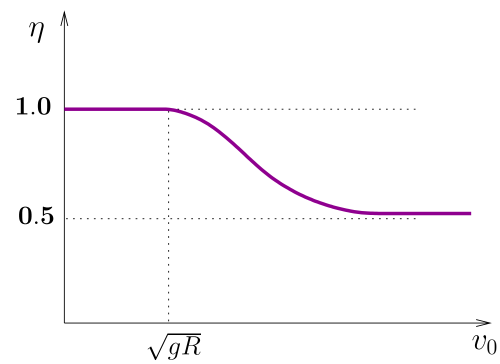

Figure 1: A sketch of $\eta(v_0)$ based on asymptotics.

The non-trivial stuff occurs for $\sqrt{gR} < v_0 < \sqrt{2gR}$. It is clear that given the isotropic initial distribution of the fragments, the fraction η is proportional to the solid angle formed by the trajectories of the fragments that do not touch the upper hemisphere. By definition, the solid angle of a cone with apex angle 2θ, is the area of a spherical cap on a unit sphere: $\Omega = 2\pi\,(1 - \cos\theta)$. Note that the solid angle of the whole sphere is 4π. Our goal is therefore to determine, for a given v_0, which is the set of angles α_0 for which the trajectory touches (or intersects) the sphere in the upper hemisphere. The associated *total* solid angle Ω_{upper} can then by determined, and the fraction η can then be found as the difference $\eta = 1 - \Omega_{upper}/4\pi$.

By choosing the reference frame with the origin in the center of the circle, the equation of motion reads

$$y = x\tan\alpha_0 - \frac{1}{2}\frac{g}{v_0^2}(1 + \tan^2\alpha_0). \tag{1}$$

The trajectory of any given fragment is two-dimensional, and (1) is true in any vertical section of the sphere; it is therefore sufficient to look at the 2-D picture (and find α_0), and then move to the 3-D case by using the solid angle. The equation of the dome (circle, in 2-D) is

$$y^2 = R^2 - x^2. \tag{2}$$

From the symmetry of the problem it is clear that we may consider just the upper-right quadrant, $0 \leq x, y \leq R$. For trouble-free mathematics, let us non-dimensionalize the equation of the trajectory and of the circle, by introducing $X \equiv x/R$ and $Y \equiv y/R$. Furthermore, for conciseness let us denote $t \equiv \tan\alpha_0$. With these notations (1) and (2) become

$$Y = Xt - \frac{\beta}{2}(1+t^2)X^2, \tag{3}$$

$$Y^2 = 1 - X^2, \tag{4}$$

where

$$\beta = \frac{gR}{v_0^2} \tag{5}$$

is the parameter that accounts for the initial speed of the fragments (as we have seen in the preliminary discussion above). Note that $\beta \to 0$ as $v_0 \to \infty$.

With a "brute force" approach, the problem may be solved, eventually, by expressing Y from (3), substituting in (4), and looking for the set of t's that allows (at a given β) for solutions (X_0, Y_0) of the resulting $F(X) = 0$ equation, such that $X_0 \in [0, 1]$ *and* $Y_0 \in [0, 1]$. The fourth-order polynomial equation $F(X) = 0$ may have four real-valued roots, but from the physics of the problem we only look for the one or maybe two roots that give (X_0, Y_0) in the upper-right quadrant (the other two will be in the lower-left and lower-right quadrants).

While in principle such a solution is possible (there are formulae that give you the roots of a fourth-order polynomial equation, but they are not pretty[1]), we should try to get it in a simpler way, based in part on the physics of the problem.

For further use, here is the $F(X)$ we are talking about:

$$F(X) = \frac{\beta^2}{4}(1+t^2)^2X^4 - \beta t(1+t^2)X^3 + (1+t^2)X^2 - 1. \tag{6}$$

Clearly, we do not really need the whole set of solutions, but only the extreme values of t for which our problem has solution. In physical terms, for a fixed β (i.e. fixed v_0) we are looking for the smallest and the largest values of $\tan\alpha_0$ (i.e., α_0) for which the trajectory touches/intersects the unit circle in the first quarter. Note that this method relies on the interval of α_0's being compact (i.e., not fragmented).

To develop some intuition on what is going on let us have a look at Figure 2, where a set of trajectories has been plotted (19 different $\tan\alpha_0$, with α_0 uniformly distributed in the $(0, \pi/2)$ interval) for a few values of β. Note that except for the $\pi/4$ angle, the trajectories are drawn in pairs. Their launch angles are of the type $\gamma, \pi/2 - \gamma$, where $\gamma < \pi/4$, and therefore their projectile ranges are the same; we shall call the trajectory with $\alpha_0 = \pi/2 - \gamma$ the "conjugated" trajectory.

[1]For some mathematical details on quadratic and quartic polynomial equations Wikipedia had at the time of writing some useful reading: see http://en.wikipedia.org/wiki/Quadratic_equation, and http://en.wikipedia.org/wiki/Quartic_equation

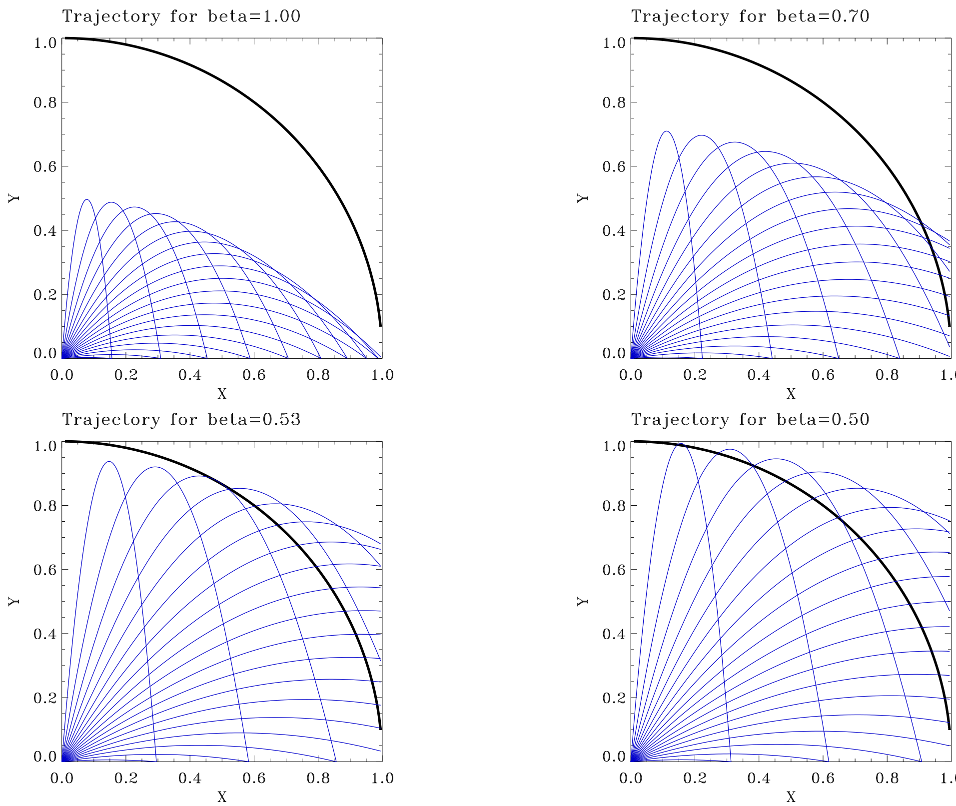

Figure 2: Illustration to problem # 1. Each plot corresponds to a given velocity, for several values of $\alpha_0 \in (0, \pi/2)$. The values of β are $1.0,\ 0.7,\ 0.53,\ 0.5$, as shown in the plot headers.

As argued above, for $\beta > 1$ the fragments do not touch the upper hemisphere, and for $\beta < 0.5$ all the high-angle trajectories ($\alpha_0 > \pi/4$) do intersect the circle. However, for $\beta < 0.5$ there are low-angle trajectories ($\alpha_0 < \pi/4$) that do not touch the upper hemisphere.

It seems fairly clear that for a given $\beta \in (0.5, 1)$ all the trajectories with $\alpha_0 < \alpha_{min}(\beta)$ end up in the lower hemisphere. Here α_{min} is found by demanding the corresponding low-angle trajectory to barely make it to the $(1, 0)$ point on the horizontal diameter, that is $\sin 2\alpha_{min} = \beta$. We get

$$\alpha_{min} = \frac{1}{2} \arcsin \beta \leq \frac{\pi}{4}. \tag{7}$$

As expected, (7) shows that for $\beta > 1$ we don't even have a chance to hit the upper hemisphere (since $\arcsin$ is not defined). Moreover, we see that even for very large v_0 we have $\eta > 1/2$, but $\eta \neq 1/2$ (since $\alpha_{min} > 0$, strictly). For further reference, we note that $\tan \alpha_{min}$ can be expressed as

$$\tan \alpha_{min} = \frac{1}{\beta} - \sqrt{\frac{1}{\beta^2} - 1}. \tag{8}$$

The $1/\beta + \sqrt{\ldots}$ solution has been rejected because the corresponding $\tan \alpha_0 > 1$, while here we look for $\alpha_{min} < \pi/4$.

The α_{min} low-angle and the "conjugated" high-angle trajectory are shown in Figure 3 (the thick dashed lines), for the case $\beta = 0.7$. We notice that the high-angle trajectory, for which

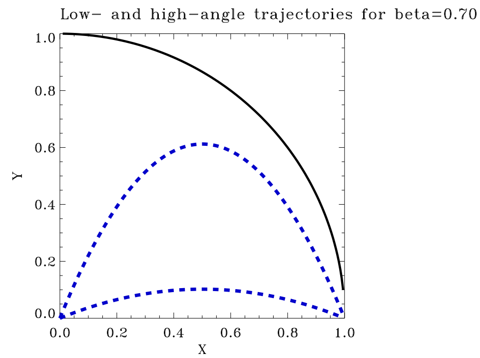

Figure 3: Illustration to problem # 1. The low-angle and its conjugate trajectory for $\beta = 0.7$.

$\alpha_{min}^{conj} = \pi/2 - \alpha_{min}$ *is* actually the α_{max} trajectory (at least for this β)! Indeed, for the given β any further increase of α_{min}^{conj} would have the fragment fall to the bottom hemisphere, while lower values would hit the sphere in the upper hemisphere; any $\alpha_0 \in (\alpha_{min}, \alpha_{max})$ ends up in the upper hemisphere. For further reference, the conjugate trajectory of α_{min} has

$$\tan \alpha_{min}^{conj} = \frac{1}{\beta} + \sqrt{\frac{1}{\beta^2} - 1} > 1. \tag{9}$$

However, this recipe is not the entire solution!

Indeed, a look at Figure 2, in particular to the plot $\beta = 0.53$ illustrates the issue: there are high-angle trajectories that touch (or intersect) the circle and have $\alpha_0 > \alpha_0^{conj}$ (for example, the 3-rd trajectory from the left on the $\beta = 0.53$ plot). Therefore, for some selected βs we have $\alpha_{max} > \max(\alpha_{min}^{conj})$.

Nolens volens, we have to appeal to some more math. Notice that for all $\alpha \in [\alpha_{min}, \alpha_{max}]$ we do seem to have hits with the upper half of the circle (this is what is meant by “compactness” of the solution interval for α). A proof of this conjecture is not provided here, although the statement seems to be true.

We obtain α_{max} by taking the limit of the high-angle trajectory barely touching the sphere. The situation resembles the behaviour of the 3-rd trajectory from the left on the $\beta = 0.53$ plot in Figure 2. The solution X_0 is then a double root for $F(X) = 0$, and therefore we may write

$$F(X) = (X - X_0)^2 G(X), \tag{10}$$

with $G(X)$ being a quadratic polynomial[2]. Notice also that the derivative of F at X_0 is zero, which means that the tangents to the trajectory and the circle coincide at the point (X_0, Y_0). In the general case X_0 is not the coordinate of the top of the trajectory X_{top}, but to the right of it ($X_{top} < X_0 < 1$). To convince yourself that this is true, notice that the tangent to the top of the trajectory is always parallel to the x-axis, while the tangent to the circle is in general slanted (except when $X_0 = 0$). One last, but important remark: X_0 has to be a local extremum, for our “double root” method to work as envisioned (actually a maximum for the $F(X)$ defined in (6)). Therefore, the second

[2]If you are not familiar with the concept of “double root”, think of the simple case of $(x-1)^2 = 0$, and move the parabola in the vertical, by substracting or adding a small ϵ to the left hand side, to see how the number of solutions in real numbers changes from two distinct roots, to 1+1 (double root) and to none.

derivative of $F(X)$ should be negative when evaluated at X_0. In fact $F''(X_0) = 2G(X_0)$. Note that $F'(X_0)$ and $F''(X_0)$ may vanish at the same point, in which case X_0 is a saddle point, not an extremum. Such as situation (almost) occurs with the 4-th trajectory from the left on the $\beta = 0.53$ plot (by increasing α_0 a bit the 4-th trajectory touches the circle from the outside, at X_0, and intersects the circle at some point $X^* < X_0$. Note however, that for the same β there are even higher-angle trajectories that intersect the circle, therefore in order to determine α_{max} with the double-root approach we have to look for solutions where the trajectory touches the circle from the inside!

So far, we figured out that the $\beta < 0.5$ and $\beta > 1$ cases are trivial. For $\beta < 1$ but close to 1 (for example, $\beta = 0.7$) we found that the low-angle α_{min} and the conjugated high-angle trajectory provide the extreme α_0's. However, for $\beta \to 0.5$ we noticed supplementary solutions, which cannot be obtained via the "conjugate trajectory" approach.

We are now looking to find the $t_{max} \equiv \tan \alpha_{0\,max}$ for which there is a double root X_0 and $F''(X_0) < 0$ (we assume β as being given). Note that $G(x) = aX^2 + bX + c$ is determined by three parameters. By expanding (10) and comparison with (6) we obtain five equations, for the coefficients of the powers of X. After expressing a, b and c the extra two equations are going to give us t_{max} and X_0. It was essential to have some supplementary information on the roots, so that we could solve $F(X) = 0$ without appealing to the general formula. What is left is just a bunch of algebraic manipulations. However, they are not of a pretty kind, which is surprising given the apparent simplicity of the problem!

Let $G(X) = aX^2 + bX + c$ and substitute it in (10). By identifying the coefficients of the powers of X in (6) and (10) we obtain

$$a = \frac{\beta^2}{4}(1+t^2)^2, \tag{11}$$

$$b - 2X_0 = -\beta t(1+t^2), \tag{12}$$

$$c - 2bX_0 + ax_0^2 = 1 + t^2, \tag{13}$$

$$bX_0^2 - 2cX_0 = 0, \tag{14}$$

$$cX_0^2 = -1. \tag{15}$$

We have five equations, with five unknowns (a, b, c, X_0 and t). From (11), (14) and (15) we get

$$a = \frac{\beta^2}{4}(1+t^2)^2, \quad b = -\frac{2}{X_0^3}, \quad c = -\frac{1}{X_0^2}. \tag{16}$$

while from (12) and (13) we may now obtain X_0 and t. We have

$$\frac{3}{X_0^2} + \frac{\beta^2}{4}(1+t^2)^2 X_0^2 = 1 + t^2, \tag{17}$$

$$\frac{1}{X_0^2} + \frac{\beta^2}{4}(1+t^2)^2 X_0^2 = \frac{\beta t}{2}(1+t^2)X_0, \tag{18}$$

or, after denoting $T = 1 + t^2$ and $U = X_0^2$

$$\frac{\beta^2}{4}T^2U^2 - TU + 3 = 0, \tag{19}$$

$$\frac{\beta^2}{4}T^2U^2 - \frac{\beta}{2}TU\sqrt{T-1}\sqrt{U} + 1 = 0. \tag{20}$$

From (19) we obtain

$$(TU)_{\pm} = \frac{2}{\beta^2}(1 \pm \sqrt{1-3\beta^2}) = \frac{2V}{\beta^2}, \tag{21}$$

which tells us that we expect solutions only when $1 - 3\beta^2 \geq 0$, that is for $\beta \leq 1/\sqrt{3} \approx 0.577$. Here we denoted $V = (1 \pm \sqrt{1-3\beta^2}) > 0$. We shall return to the choice of V_+ or V_- shortly. By substituting TU in (20) and doing some algebraic manipulations we obtain

$$X_0^2 = U = \frac{2V}{\beta^2} - \left(\frac{V}{\beta} + \frac{\beta}{V}\right)^2, \quad 1 + \tan^2\alpha_0 = T = \left[1 - \frac{V}{2}\left(1 + \frac{\beta^2}{V^2}\right)^2\right]^{-1}. \tag{22}$$

To get an idea of the behaviour of the "+" and "−" branches of the solution, in Figure 4 we have plotted $V_{\pm}$, $\sqrt{U_{\pm}}$ and $t_{\pm}$ (expressed from $T_{\pm}$). From the $t_{\pm}$ plot we see that the double-root method

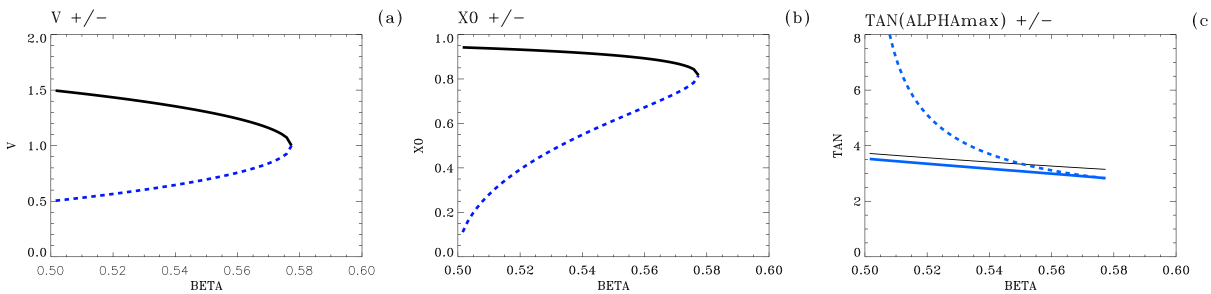

Figure 4: (a): $V(\beta)$ (V_- with dotted line, V_+ with solid line). (b): $X_0(\beta)$ (dashed line for the "−" branch, solid for "+"). (c): $t_{max}(\beta)$ corresponding to the the "+" (blue solid) and "−" (blue dashed) branches. The thin black line is $\tan\alpha_{min}^{conj}$.

provides the α_{max} for β's only up to $\beta \approx 0.55$. For $\beta \in (0.55, 1/\sqrt{3})$ the solution for the α_{max} is given by the "conjugate angle" method[3], and $\tan\alpha_{max}$ is given in this case by (9). As an example, we plotted the double-root solutions for $\beta = 0.52$ (Figure 5a). Notice that α_{max} is provided by the "−" branch, while the "+" branch provides the "saddle point" trajectory. Indeed, one may check that $G(X_0) < 0$ in the former case, and $G(X_0) > 0$ in the latter. The values of the $\tan\alpha_{\pm}$ were read off the Figure 4(c), for $\beta = 0.52$. We also plotted the trajectories for the $\beta = 0.55$ threshold, with $\tan\alpha_{max} \approx 3.3$, where the conjugate angle and the double root methods share the same trajectory of α_{max} (Figure 5b). We have now the entire solution:

- For $\beta > 1$ there are no collisions with the upper hemisphere, therefore $\eta = 1$.
- For $\beta \in (0.55, 1)$ the α_{max} is obtained by taking the trajectory conjugated to α_{min} ($\tan\alpha_{max}$ is given by (9); α_{min} is given by (7).
- For $\beta \in (0.5, 0.55)$ the α_{max} is obtained with the "double root" method (use the "−" branch), and its tangent may be expressed from (22); α_{min} is given by (7).

[3]For some trigonometric identities [e.g. $\tan(\arcsin(\beta))$] you may consult http://en.wikipedia.org/wiki/Arcsin, and http://mathworld.wolfram.com/InverseTrigonometricFunctions.html

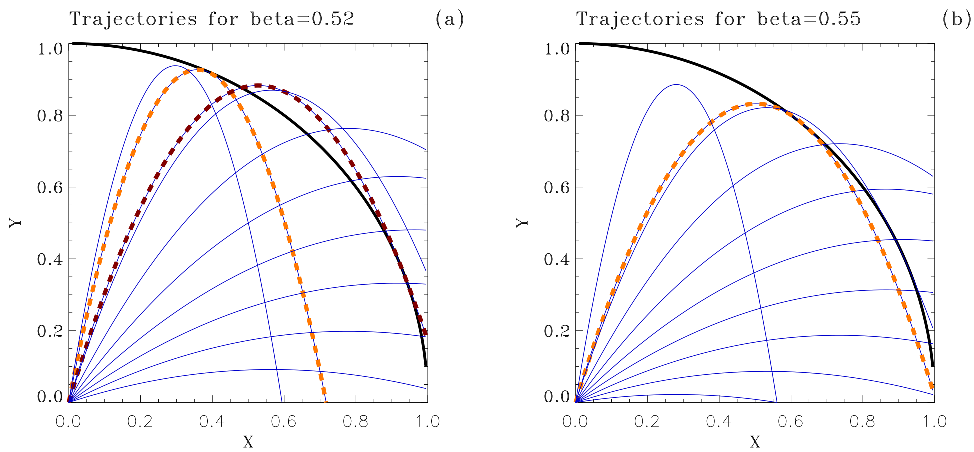

Figure 5: Illustration to problem # 1. (a): $\beta = 0.52$. The dotted trajectories correspond to $\tan\alpha_{max}$ for the "+" and "$-$" branches in the "double root" method. (b): $\beta \approx 0.55$. The dotted trajectory corresponds to $\tan\alpha_0 \approx 3.3$; for this (β, α_0) pair the "double root" and the "conjugate angle" methods give the same trajectory.

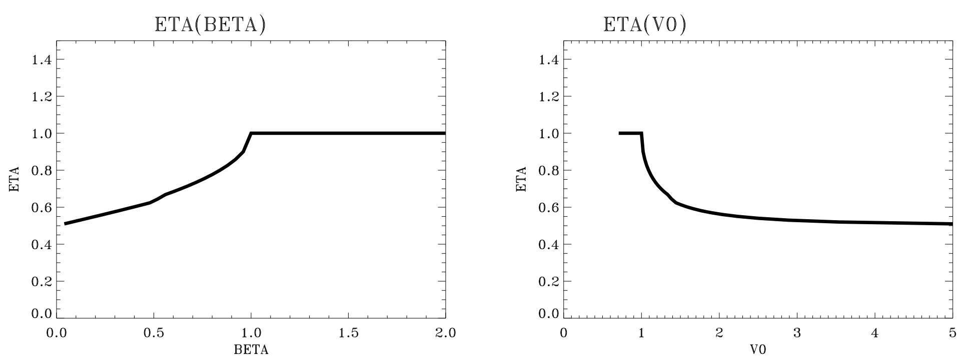

Figure 6: $\eta(\beta)$ (left plot) and $\eta(v_0)$ (right plot).

- For $0 < \beta \leq 0.5$ all the high-angle trajectories hit the dome; α_{min} is given by (7), and $\eta \to 1/2$ as $\beta \to 0$.

The interval $(\alpha_{min}, \alpha_{max})$ is compact (i.e. no gaps), therefore the associated solid angle is simply

$$\Omega_{upper} = 2\pi(\sin\alpha_{max} - \sin\alpha_{min}) \rightarrow \eta = 1 - (\sin\alpha_{max} - \sin\alpha_{min})/2. \qquad (23)$$

In the above we have used the trigonometric identity $\cos(\pi/2 - \gamma) = \sin\gamma,\ \forall\gamma$. The plot of $\eta(\beta)$ and $\eta(v_0)$ (with v_0 scaled by gR) is presented in Figure 6. Note that $\eta(v_0)$ differs from the asymptotic sketch, most notably in the behaviour of the second derivative, η''.

2. Let us take the coordinate system with the vertical Oy axis as shown in Figure 7. The top cylinder is A, the bottom one is B. After their release at $t_0 = 0$ both cylinders fall with the constant acceleration g. After the time interval $\tau = \sqrt{2h/g}$ the cylinders A and B are at locations of heights

h and 0, respectively (snapshot 1a in Figure 7) and both have the speed $v_0 = \sqrt{2gh}$. Cylinder B hits the table top perfectly elastic, and therefore bounces back with the same speed, but in the opposite direction (snapshot 1b).

The next interesting situation occurs when A and B collide for the first time (snapshots 2a and 2b). Since the cylinders are of equal masses, through a perfectly elastic collision they just swap their velocities, and part ways as depicted in 2b. The collision occurs at location y_1, where

$$y_1 = h - v_0\tau - \frac{g\tau^2}{2} = v_0\tau - \frac{g\tau^2}{2}. \tag{24}$$

In the above, τ is counted from the time of the snapshot 1b, and we expressed y_1 using the equations of motion for A and B, respectively. We find that $\tau = h/2v_0$; over this time interval the speed of A increases to $v_A = v_0 + g\tau = (5/4)v_0$, while the speed of B decreases to $v_B = v_0 - g\tau = (3/4)v_0$. This situation is depicted in snapshot 2a, and the first collision occurs at height $y_1 = (7/16)h$. Through the collision A and B swap velocities, as argued above, therefore $v_A(t_{1b}) = +(3/4)v_0$ and $v_B(t_{1b}) = -(5/4)v_0$, as shown in snapshot 2b.

In the snapshot 3a, A is still moving upward while B reaches the table (you may convince yourself that this is indeed the case, by comparing the time it takes A to reach the top of its trajectory; over the same time B would overshoot the table; hence B reaches the table before A reaches the top of its trajectory). In the snapshot 3b the B's velocity has changed direction, through the perfectly elastic collision with the ground. At this stage the B's location is at the table top level $h_B = 0$, and its speed is $v_B = \sqrt{2}v_0$ (e.g, you may use Galileo's formula), while A is at the location $h_A = (4\sqrt{2} - 5)h$ and its speed is $v_A = (2 - \sqrt{2})v_0$.

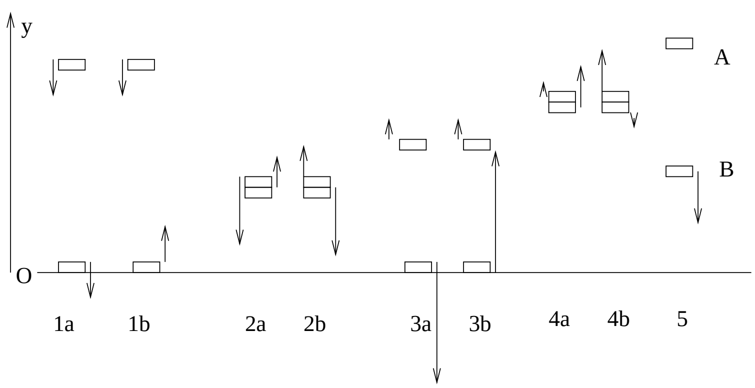

Figure 7: Illustration to problem # 2.

In snapshot 4a we see that B is catching up with A *before* A has a chance to reach the top of its own trajectory! The proof is a bit involved, although straightforward. One way of doing it is to find the location y_2 of the second collision

$$y_2 = h_A + v_A\tau - \frac{g\tau^2}{2} = v_B\tau - \frac{g\tau^2}{2}, \tag{25}$$

where τ is now counted from the time of the snapshot 3b. We find that

$$\tau = \frac{h}{v_0}\frac{3 - \sqrt{2}}{2}, \tag{26}$$

and is a simple matter to show that $\tau < v_A/g$, which means that the collision occurs before A reaches the top of its trajectory. The location of the collision is

$$y_2 = \frac{h}{16}(30\sqrt{2} - 27). \tag{27}$$

Once again, through the collision shown in snapshots 4a and 4b the cylinders swap velocities. Prior to their second collision the velocities of A and B were

$$v_A(t_{4a}) = +v_0\frac{5 - 3\sqrt{2}}{4}, \quad v_B(t_{4a}) = +v_0\frac{5\sqrt{2} - 3}{4}. \tag{28}$$

The velocity of A in the snapshot 4b is now $v_A(t_{4b}) = v_B(t_{4a}) = +v_0(5\sqrt{2} - 3)/4$. Hence A rises for an extra $\Delta y = v_A(t_{4b})^2/2g = h(5\sqrt{2} - 3)^2/16$.

Finally, the height A reaches after the second collision with B is $h_2 = y_2 + \Delta y = 2h$.

(b) The ugly calculations we had to carry out to get the height that A reaches after the second collision should perhaps put off any thoughts of inductively finding the upper limit on the height of the jump. Clearly, we need to look for a solution based on energy considerations. Indeed, the total mechanical energy of the system A+B is constant, and is equal to the energy at the very beginning, when the potential energy was the only form available in the system. The conservation equation is therefore

$$3gh = \frac{1}{2}v_A^2 + \frac{1}{2}v_B^2 + gh_A + gh_B. \tag{29}$$

The answer to part (b) is rather trivial for the case of two cylinders, since in the extreme case $v_B = 0$, $h_B = 0$ and with A at the top of its trajectory $v_A = 0$ we obtain $h_A = h_{max} = 3h$. However, this h_{max} cannot be reached, since A has reached this height by swapping the velocity with B, hence, there must be at least some kinetic energy (if not some potential energy, too) associated with the cylinder B (even if A was at rest when the velocity swapping occurred, so that B's kinetic energy is 0 after the swap, by the time A reaches the top of its trajectory, B has gained some kinetic energy on the expense of it's potential energy). Presumably tighter bounds on h_{max} can be found (although the problem only asks for this trivial bound).

The above result on h_{max} can obviously be generalized for $N \geq 3$. Indeed, with the same philosophy as above, we may set all the velocities in the system to 0, and consider the (unrealistic) case when $N - 1$ cylinders are at the table level, with only the top cylinder being at the top of its trajectory. Then

$$h_{max} \leq (1 + 2 + 3 + \cdots + N)h = \frac{N(N+1)}{2}h, \tag{30}$$

where the last equality is a known result from the arithmetics (to obtain it, write the middle sum in (30) from left to right and from right to left, add the two expressions side by side, and notice that now twice the sum equals $N(N + 1)$).

3. There are a few distinct stages in the motion of the cube, and we shall consider them separately.

(i) At the beginning the cube is still at rest, and the spring is getting stretched. This situation keeps on going until some time t_1 at which the elastic force in the spring ($|kx|$) reaches the maximum value of the static friction force ($\mu_0 mg$); after this moment the cube starts moving.

To find t_1, let us note that $kx_1 = \mu_0 mg$, where by the statement of the problem $x_1 = vt_1$. Therefore,

$$t_1 = \mu_0 \frac{mg}{kv} \approx 10\,\mathrm{s}. \tag{31}$$

For shortness, let us denote $k/m = \omega^2$. Then $t_1 = \mu_0 g/v\omega^2$. The elongation of the spring at time t_1 is

$$x_1 = vt_1 = \mu_0 \frac{g}{\omega^2}. \tag{32}$$

(ii) At and after the time t_1 the cube is moving. In the coordinate system moving with constant velocity $\vec{v}$ (as the free end of the spring) the velocity of the cube at time t_1 is $-\vec{v}$, and its coordinate is $x = 0$, while the spring is extended by $x_1 = vt_1$. The motion of the cube in the moving reference frame ('M' from now on) at t_1 is dictated by the net force due to the elastic force $F_e = kx_1$ and the kinetic friction force $F_f = \mu mg$. Note that $\vec{F}_f$ is oriented in a direction opposite to $\vec{v}$ (since in 'M' this is the direction of the cube velocity relative to the table).

Under the action of the elastic + constant forces the cube will effectuate a harmonic oscillation motion, with the coordinate being described (in the 'M' system) by

$$x_M = x_{M\,max} \sin(\omega t + \varphi) + x_0, \tag{33}$$

where $\omega = \sqrt{k/m}$, and x_0 is the equilibrium position, which can be found from $k(x_1 - x_0) - \mu mg = 0$, which leads to

$$x_0 = x_1 - \frac{\mu mg}{k} = x_1 - \mu \frac{g}{\omega^2} = \Delta\mu \frac{g}{\omega^2}, \tag{34}$$

where we denoted $\Delta\mu = \mu_0 - \mu$. The speed of the cube in the M-reference frame is

$$v_M = \dot{x}_M = x_{M\,max} \omega \cos(\omega t + \varphi). \tag{35}$$

We now find $x_{M\,max}$ and φ. By replacing in (33) and (35) t with t_1, $x_M = x(t_1) = 0$ and $\dot{x}(t_1) = -v$ we obtain

$$0 = x_{M\,max} \sin(\omega t_1 + \varphi) + \Delta\mu \frac{g}{\omega^2}, \tag{36}$$

$$-v = x_{M\,max} \omega \cos(\omega t_1 + \varphi). \tag{37}$$

The above equations may be rewritten as

$$x_{M\,max} \sin(\omega t_1 + \varphi) = -\Delta\mu \frac{g}{\omega^2}, \tag{38}$$

$$x_{M\,max} \cos(\omega t_1 + \varphi) = -\frac{v}{\omega}. \tag{39}$$

By taking the square of (38) and (39), and adding side-by-side we obtain

$$x^2_{M\,max} = \frac{(\Delta\mu)^2 g^2}{\omega^4} + \frac{v^2}{\omega^2} \approx 2\mathrm{m}^2 \ \Rightarrow x_{M\,max} \approx 1.4\mathrm{m}. \tag{40}$$

By dividing (38) and (39) we obtain

$$\tan(\omega t_1 + \varphi) = \Delta\mu \frac{g}{v\omega}, \tag{41}$$

which upon accounting for $\sin() < 0, \cos() < 0$ gives

$$\varphi = \pi + \arctan \frac{\Delta\mu g}{v\omega} - \omega t_1 \approx \frac{5\pi}{4}. \tag{42}$$

We now move back to the laboratory (resting) reference state. Since at t_1 the origins of the moving and resting reference frames coincide, we have

$$\begin{aligned} x &= x_M + v(t - t_1) = x_{M\,max} \sin(\omega t + \varphi) + v(t - t_1) + \Delta\mu \frac{g}{\omega^2} \\ &\approx \sqrt{2} \sin\left[(t - 10) + \frac{5\pi}{4}\right] + (t - 10) + 1 \text{ [m]}, \end{aligned} \tag{43}$$

$$\dot{x} = x_{M\,max}\omega \cos(\omega t + \varphi) + v \approx \sqrt{2} \cos\left[(t - 10) + \frac{5\pi}{4}\right] + 1 \text{ m/s}. \tag{44}$$

The above two equations describe the motion of the cube from $t = t_1$ until some time t_2, when the speed of cube relative to the table is 0, and the kinetic friction force is replaced by the static friction force. At $t = t_2$ we have

$$\dot{x} = x_{M\,max}\omega \cos(\omega t_2 + \varphi) + v = 0. \tag{45}$$

At this point we shall recall that the phase $\theta_1 = \omega t_1 + \varphi$ of $\cos$ at time t_1 was in the third quarter, and so the speed was positive. For later moments in time ($t > t_1$) the $\theta(t)$ increases, and so does $\cos\theta$, and therefore the speed increases. When the θ ends up in the second quadrant (or 6-th, if you prefer to say so) the $\cos\theta$ is negative; when it becomes equal to $\cos(\omega t_1 + \varphi)$ the speed of the cube is 0. Therefore, we may write

$$t - 2 = t_1 + \frac{2}{\omega}\left(\pi - \arctan \frac{\Delta\mu\, g}{v\omega}\right) \approx 15\,\text{s}, \quad x(t_2) \approx 7\,\text{m}. \tag{46}$$

On this occasion we note that the elastic force at time t_1 and t_2 are related by

$$F_f(t_1) - \mu mg = -(F_f(t_2) - \mu mg) \Rightarrow F_f(t_2) = 2\mu mg - F_f(t_1). \tag{47}$$

But $F_f(t_1) = \mu_0 mg$, therefore

$$F_f(t_2) = mg(2\mu - \mu_0) < \mu_0 mg. \tag{48}$$

Therefore, at t_2 the cube will stop and will not move until some time t_3, when $F_f(t_3)$ equals in magnitude the static friction force $\mu_0 mg$. The time interval $t_3 - t_2$ is

$$t_3 - t_2 = \frac{F_f(t_3) - F_f(t_2)}{kv} = \frac{2(\mu_0 - \mu)g}{v\omega^2} \approx 2\,\text{s}. \tag{49}$$

Afterward, the cycle repeats, so that $t_4 - t_3 = t_2 - t_1$, a.s.o. See Figure 8 for the plots of $x(t)$ and $\dot{x}(t)$.

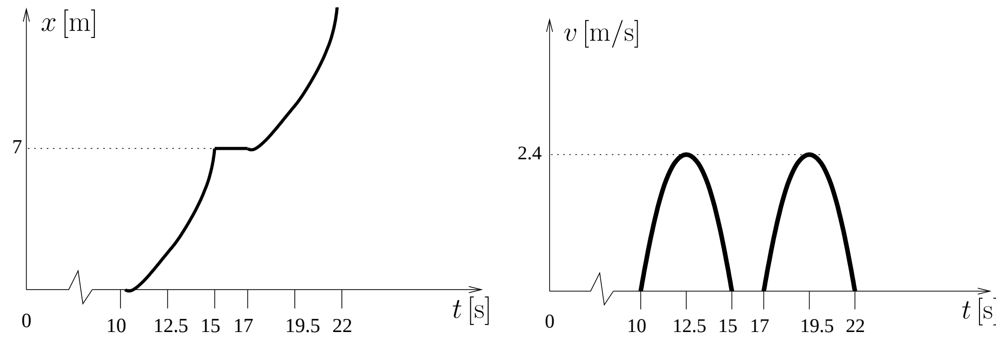

Figure 8: Illustration to problem # 3. The plots of coordinate (left) and velocity (right).

4. Assuming the time interval depends only on G and ρ, on dimensional grounds we may write $\tau = G^\alpha \rho^\beta$, with α, β being some coefficients to be determined. Since $[\tau] = T$, $[G] = M^{-1}L^3T^{-2}$ and $[\rho] = ML^{-3}$ one can easily find that $\alpha = \beta = -1/2$. Therefore

$$\tau = C\frac{1}{\sqrt{G\rho}}, \tag{50}$$

where C is just a numerical dimensionless coefficient. Assuming for now that $C \approx 1$ we get $\tau \approx 0.3 \times 10^{14}$sec ≈ 1 million years.

To refine this result, let us note that a given dust particle located at a distance R from the gravitational center of the cloud is attracted only by the mass contained by the sphere of radius R.

[This is a classic result, stemming from the "shell theorem", by which the net gravitational force on a point-like object inside a hollow spherical sphere is zero. It may be proved for example by taking a double integral (see Wikipedia, http://en.wikipedia.org/wiki/Shell_theorem). An elementary solution is also possible, which relies on the definition of the solid angle, the spherical geometry (and some like-triangles reasoning), and on the $1/r^2$ of the gravitational force dependence on the distance.]

Since the particles do not overtake each other during the accretion, the total mass acting gravitationally on our dust particle is constant, and therefore we may think of that mass as being concentrated in the center of the cloud. Our problem is then equivalent to finding the time of free fall to the center of gravitational attraction, starting from a distance R.

The fall of the particle can be pictured as the motion on a very elongated ellipsis, of big half-axis equal to $R/2$. Let us compare this motion with the motion on a circular orbit of radius R. The third Kepler's law tells us that

$$\frac{T_C^2}{T_E^2} = \frac{R^3}{(R/2)^3}, \tag{51}$$

where T_C (T_E) are the periods of the motion on the circular (elliptic) orbit. The period of motion on the circular orbit is easily expressed using the second Newton's law, as applied to the gravity

$$a_{cp}\frac{mv^2}{R} = G\frac{Mm}{R^2}, \tag{52}$$

where a_{cp} is the centripetal acceleration and

$$M = \rho\frac{4}{3}\pi R^3. \tag{53}$$

We then obtain

$$T_C = \frac{2\pi R}{v} = \sqrt{\frac{3\pi}{G\rho}}, \tag{54}$$

and hence

$$T_E = \frac{T_C}{2^{3/2}} = \sqrt{\frac{3\pi}{8G\rho}}. \tag{55}$$

Luckily, T_E does not depend on R. Then the time τ it takes the particle to fall on the attraction centre, which time is half of T_E, does not depend on R as well. We have

$$\tau = \sqrt{\frac{3\pi}{32G\rho}}. \tag{56}$$

Numerically, $\tau = 0.15 \times 10^{14}\mathrm{s} \approx 0.5$ million years.

5. Let us summarize the methodology we are going to use for ranking the methods.

In general, by saying that a measurement is "accurate" we understand that the value we get (in this case, the height h) is close to the "true" value. By "precise" we mean, in layman's words, that we can get a lot of significant digits in a trial and that by repeating the measurement we get about the same value. However, the measurement may be off from the true value, therefore high precision does not necessarily imply high accuracy. The popular "target" example on the meaning of precision and of accuracy is shown in Figure 9.

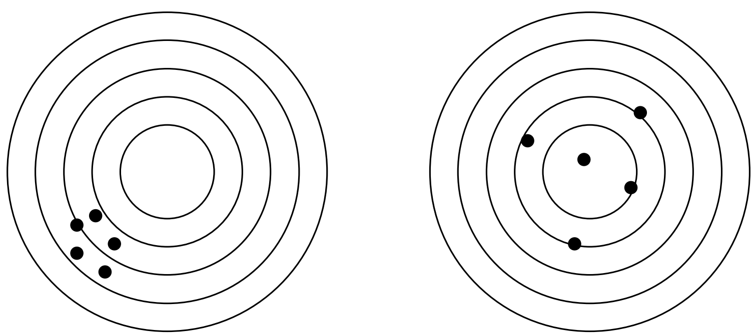

Figure 9: Illustration to problem # 5. Left plot: high precision, low accuracy; right plot: high accuracy, low precision.

Example of accuracy errors: inappropriate formula (e.g. not accounting for air resistance when we actually should), bias in the measurement (e.g. we measure the rope on some horizontal surface, but when the rope hangs it may stretch under its own weight, hence whatever meter marks we did on the rope, they are not accurate). Example of precision errors: using a coarse meter stick to measure small objects, human errors on starting/stopping the timer, etc.

The main criterion for ranking will be the *relative error* of a given measurement. Dr. Harrison's "Error analysis" web-document[4] has a succinct explanation on this matter. We shall systematically identify the largest contributors to the total relative error of the measurement with a given method,

[4]See http://faraday.physics.utoronto.ca/PVB/Harrison/ErrorAnalysis/

debate how does the accuracy errors compare to it, then analyze how the outcome may depend on the height of the building.

For further reference let us review the specifications of the main "players" we are dealing with.

A typical good-quality *aneroid barometer* (henceforth "barometer") has a mass of about 1kg, an accuracy of about $\pm 70\text{Pa}$ (that is, what you measure may be off by that much from the "true" value), and a graduation of the dial of 50Pa. Since we read this analog dial manually, we shall assume a reading error of about $\pm 20\text{Pa}$ or so. Basically this is the finest step in pressure that we may count on seeing (meaning that a given measurement may read $(xxxx40 \pm 20)\text{Pa}$ but not $(xxxx40 \pm 5)\text{Pa}$). The can-shaped barometer has a 200mm diameter, and 70mm thickness.

A typical climbing *rope* will have a mass of about 5kg/100m, about 1cm in diameter. We choose the "static rope" variety (N.B. for climbing you should use the "dynamic rope" variety!) meaning that we do not expect the rope to stretch. However, this is not entirely true, and one expects a 5% extension of the rope under a large load (typically, the load for testing is 10% of the minimal break strength (MBS); for a MBS of 10kN, we figure out that a mass of 100kg hanging freely stretches the rope by 5%). Since our barometer is much lighter, we expect the rope to stretch by 0.05% because of the barometer, and by about 0.15% or so due to the rope's weight (the weight of the rope acts gradually, hence we get less than 0.25% – what you'd expect if 5kg would hang at the end of the (massless) rope).

A regular *measuring tape* would be about 5m long, with a mark step of 1mm, and we assume an accuracy of about 0.1%. (1 mm in 1m). Reading error is about 1/3 of 1mm.

A *stopwatch* has a 0.01sec minimal reading. However, the precision of the time measurements is limited by our ability to decide when to stop/start the clock and to pass the command to the muscles, that is by our reaction time. Let's be optimistic about it, and assume about 0.3 sec as the precision error due to the above reasons.

We now proceed to discuss the experiments. The estimates of the errors presented below are just that: "estimates". They may increase or decrease by a factor of two (presumably) depending on your assumptions or experimental ability. Marks will be awarded for reasonable solutions, even if they differ in the final ranking. The most important thing here is to show your reasoning.

1) *Tie the barometer to a rope and lower it from the top of the building. Measure the length of the rope* $\rightarrow h$.

The biggest error is perhaps due to the wind blowing the rope (you won't get it vertical by just tying the barometer to the rope's end!). This is an accuracy error, I think, and it's pretty hard to cancel or take it into account, other than saying that perhaps it gives a $+10$cm error or so for the case A (40m building) and about $+1$m for the skyscraper (case B). The "$+$" for the error means that we expect the length of the rope to be systematically larger than the actual height of the building. Note that we cannot avoid this systematic error by adjusting our result to the above amounts (since these are just guesses, not calibration differences!). The relative error because of the wind is therefore about 0.3% for both A and B cases.

Suppose you measured and marked the rope while it was laying on a horizontal surface. Then there is an error coming from the stretching of the rope. A building that gives a reading of 40m on the rope is actually taller. The rope is stretched by about $0.2\% \times h$ (i.e. 8cm in case A, 80cm for case B), which is of the same order as the wind-induced error (although of opposite sign!). Again, we cannot get rid of this error altogether, since the percentage stretch values are just estimates given by manufacturer (presumably they are accurate within $\pm 20\%$ or so, therefore you may try to adjust your results somehow, and reduce this particular "stretch" error by a factor of 4 or so). For the *precision* of the measurements, given the length of rope, I'd bet on about ± 1cm for case A and perhaps ± 5cm for case B (it's not the reading error the biggest threat, but the repeatability of the measurement).

Summary: the accuracy relative error is about 0.5% for both A and B, with the precision being much better (say, 0.02%).

2) *Drop the barometer from the building's roof and clock the time it takes to reach the ground. Free fall theory* $\rightarrow h$.

In this case the main culprit will be the formula we use,

$$h = \frac{g}{2}\tau^2. \tag{57}$$

It assumes no air resistance, hence we end up with an accuracy error. How bad can it be? I presume in the case B (skyscraper) the barometer may reach its terminal velocity of "free" fall (80m/s or so), in which case we clearly have the wrong formula. By only taking one global measurement (the total fall time τ), and no measurements at intermediate positions, it is not possible to apply better fit formulae (for example, there is a first-year university level experiment which applies the free fall theory and the theory with weak friction to a dozen of intermediate heights measurements). For case A the velocity of the barometer should be about 25m/s, and the time of fall is about 3sec. In case B the time of fall should be about 10sec, but because of the air resistance I'd expect $\tau \approx 12\ldots 15$sec or so.

Variation of g with height does not bring in any sizeable error (see method # 3). For the error in measuring the fall time I'd estimate a ± 0.2sec error in case A, and perhaps a ± 1sec error in case B (by looking down from 400m it's hard to figure out when the barometer hit the ground).

Summary: precision error of about 5% in both A and B cases, and a hefty systematic error (i.e. accuracy) in case B (less in case A). The height is systematically overestimated, by let's say 15% in case A and as much as 50–100% in case B.

3) *Hang the barometer on a rope to form a pendulum. Measure the period of the oscillations at the ground and at the top of the building. The gravity field g varies with height, therefore the two periods will be different* $\rightarrow h$.

How much does g change with the height? First guess: not much, given our height range, which is much less than the radius of the Earth. At the ground level

$$g_0 = G\frac{M}{a^2}, \tag{58}$$

where M is the mass of the Earth, and $a \approx 6 \times 10^6$m is its radius. At some height $h \ll a$ we have

$$g = G\frac{M}{(a+h)^2} \approx G\frac{M}{a^2}\left(1 - \frac{2h}{a}\right), \tag{59}$$

where the approximation $(1+x)^n \approx 1 + nx$ for $nx \ll 1$ has been used (it follows from the Taylor series expansion of $(1+x)^n$, with n an arbitrary real number). We see that the relative change in g because of the change in height is

$$\left|\frac{\Delta g}{g_0}\right| \approx \frac{2h}{a}. \tag{60}$$

This gives us a change in g of about 0.001% in case A and of about 0.01% in case B! The question is now how does this "signal" ($\Delta g \approx 10^{-5} g$ in case A) compare with the errors in the measurements (the “noise”)? Can we actually detect the signal? From the formula for the period (T) of the small oscillations of the pendulum we may obtain g:

$$T = 2\pi\sqrt{\frac{\ell}{g}} \to g = (2\pi)^2\frac{\ell}{T^2}, \tag{61}$$

where ℓ is the length of the pendulum (let’s say $\ell = 1$m). The relative error in g due to errors in the T and ℓ estimates is

$$\frac{\delta g}{g} = \sqrt{\left(\frac{\delta\ell}{\ell}\right)^2 + \left(\frac{2\delta T}{T}\right)^2}. \tag{62}$$

Let us assume for now that ℓ does not change between the measurements (e. g. because of rope stretching or temperature variations). Note that (62) applies to both g_0 and g (with T replaced by T', and ℓ=const, as argued above). To error in time interval measurements is about ± 0.3sec, as argued before (to be picky, we assume an error of ± 0.2sec at the start and ± 0.2sec at the end of the time measurement, and combine these two errors in quadratures, to get the ± 0.3sec estimate). If one measures just one period of oscillation (in our case $T \approx$ 2sec), the $\pm 0.3\,\mathrm{sec}$ gives a huge relative error of 15%! However, by measuring the total time it takes for a few oscillations (say, 30) the relative error in T will be lower (assuming we don’t make mistakes in counting) because now the $\pm 0.3\,\mathrm{s}$ error applies to a larger time interval. In our case by measuring the time it takes for 30 oscillations, we end up with a relative error in T of just 0.5%!

The bottom line is that due to the error in T the estimate of g reads $g = g_{base}(1 \pm 0.005)$, where g_{base} is either g_0 or g measured at the top of the building. Note that the 0.5×10^{-2} is much larger than the “signal” (a 10^{-5} or 10^{-4} effect, depending on the case). In practice this means that g_0 and g are equal within the errors! No signal can be retrieved in this case!

Note that this conclusion does not apply to the method in general, but the the particular implementation we had here (barometer on the rope).

4) *If the sun is shining measure the length of the shadows of the building and of the barometer, measure the height of the barometer. Based on proportionality arguments one may now determine the height of the building.*

Let L and ℓ be the length of the shadows for the building and barometer, respectively, and D the diameter of the barometer (we shall set it standing on its side). The height of the building is then $h = DL/\ell$.

The main problem here is to determine the edge of the shadow of the building! The shadow does not have a clearly defined boundary. For case B the situation is pretty hopeless (I tried it with the CN tower, which is about 500m tall). Another issue is that the shadow moves, therefore you have to take the measurements in a matter of seconds. Anyway, just for the sake of completeness, let us write down some numerical estimates. One can perhaps determine the position of the shadow with a compound error of $\pm 1\,\mathrm{m}$ in case B, and $0.1\,\mathrm{m}$ in case A (the main contributor to the error comes from the boundary fuzziness). Since at $45\,^\circ$ latitude the shadow and the building have about the same size ($L \approx h$), this leads to a relative error of about 0.3% in L. The barometer’s shadow is about 20cm, and we may assume an error of about $1\,\mathrm{mm}$, that is about 0.5% relative error in ℓ. Assuming that the diameter of the barometer is known with a better precision than the length of the shadows, the total relative error in the height of the building ends up being of the order of 0.6% (apply a formula similar to (62)). The accuracy errors (are there any?) seem to be much smaller than the above precision error.

5) *Measure the pressure p at the ground and at the top of the building. The difference Δp allows one to estimate h.*

Let us assume for the beginning that the density of the air (ρ) is the same at the ground level and on the top of the building ($\rho \approx 1.2\,\mathrm{kg/m^3}$). Then the change in pressure and the height are related by

$$\Delta p = \rho g h, \tag{63}$$

which in case A gives $\Delta p \approx 500\,\mathrm{Pa}$, and $\Delta p \approx 5000\,\mathrm{Pa}$ in case B. As we argued in the specification section, the precision error in measuring the pressure with the aneroid barometer is about $\pm 20\,\mathrm{Pa}$. Note that the accuracy of the barometer is not of big concern here, since we look for the change in pressure, not for its absolute value! Since Δp is the difference of two pressure measurements (which are independent, from the point of view of reading error) the compound error is $\delta(\Delta p) = \sqrt{20^2 + 20^2} \approx 30\,\mathrm{Pa}$. The "signal" is larger by a factor of 16 (case A) and by a factor of 160 (case B), therefore the relative error in Δp (and therefore in h) seems to be about $\pm 6\%$ in case A and $\pm 0.6\%$ in case B.

Note however that the change in ρ with the height has been neglected, and the change in the barometer's behaviour with the pressure (!) and temperature (T) has been neglected as well. Let us assume for now that the change in T (presumably of about $1\,°\mathrm{C}$ in case A and perhaps $4\,°\mathrm{C}$ in case B) has a minimal impact. What about ρ's variation with the height? As we mentioned at the end of the last problem in the previous problem set, for an isothermal atmosphere the density of the air decreases exponentially with the distance from the surface. We have $\rho(z) = \rho_0 \exp(-z/H)$, where $H \approx 8\,\mathrm{km}$ is the density height scale in the Earth's atmosphere. Since $z/H \ll 1$ the exponential may be approximated by $1 - z/H$ (again, the first two terms in Taylor series expansion around 0). Therefore the relative change in ρ is $\Delta\rho/\rho \approx z/H$. In case A this is about 0.5%, while in case B this ends up being 5%! The moral: for the case B we have to revise the formula that links Δp with h, since the accuracy error (because of the wrong formula) is much larger than the estimated precision error! We do not follow the "improvement of formula" path here, because it will take us beyond the scope of the initial problem. Instead we recognize that in case B the result has a large accuracy error, and therefore the method # 5 in its simplest incarnation is perhaps worse than the other proposals.

The summary of our findings is presented in the Table 1, with the ranking displayed with Roman numerals (I for the best, V for the worst overall error). We see that the ranking does not depend on the measured height. (this seems to be a coincidence, not a general result!)

Method	1	2	3	4	5
Prec. error	0.02%	5%	n/a	0.6%	6%
Acc. error	0.5%	15%	n/a	$< 0.1\%$	0.5%
Rank	I	IV	V	II	III

Method	1	2	3	4	5
Prec. error	0.01%	5%	n/a	0.6%	0.6%
Acc. error	0.5%	50%	n/a	$< 0.1\%$	5%
Rank	I	IV	V	II	III

Table 1: Case A ranking (top) and case B (bottom).