

Solutions to problem set 4
Last revised: February 11, 2008

1. Let us assume the spider's eyes are located at point A (see Figure 1), at some height H above the North Pole of the globe. We shall first solve the complementary problem, asking the question: where should the fly be (on the globe) such that the spider *can* see it. The arc BDB' is visible to the

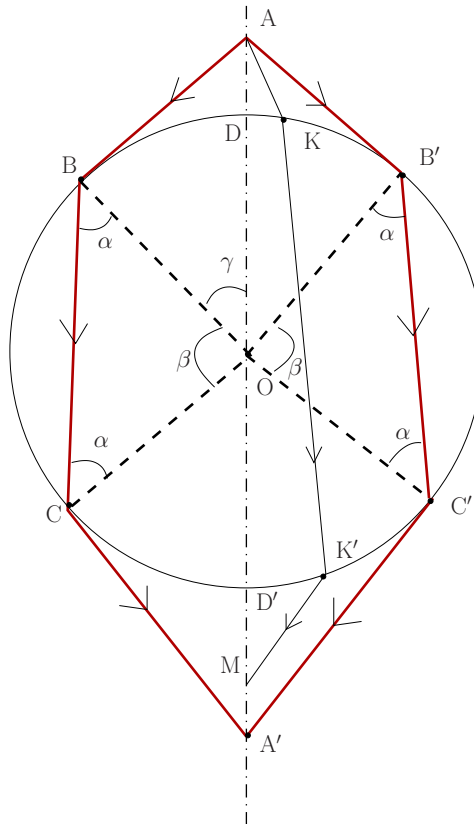


Figure 1: The stalking game

spider, where AB and AB' are tangents to the globe's surface. The ray AB will propagate through the globe following the line BC . The refraction angle α can be determined from the Snell's law,

$$\sin \alpha = \frac{1}{n}, \quad (1)$$

where the refraction index of the air was assumed to be unity. The ray ABC exits the globe and crosses the vertical diameter at point A' . It is clear that all the points on the spherical cap encompassed by the $CD'C'$ are visible to the spider (as an example, see the ray $AKK'M$).

Clearly, the spider can see only those portions of the globe's surface that are part of the spherical caps encompassed by the BDB' and $CD'C'$. In fact, the spider cannot see the band delimited by the "latitudes" BB' and CC' . The position of B is given by the angle γ , for which we have

$$\cos \gamma = \frac{R}{R + H}. \quad (2)$$

Notice that for $H \rightarrow 0$ we obtain $\gamma \approx 0$. From the geometry of the problem we see that $\beta = \pi - 2\alpha$. But $\sin \alpha = 1/n = \sqrt{2}/2 \rightarrow \alpha = \pi/4$, and hence $\beta \rightarrow \pi/2$. Therefore, in the limit $H/R \rightarrow 0$ the spider can see only the surface of the Southern hemisphere of the globe. The fly should therefore locate itself anywhere in the Northern hemisphere.

Note however, that since a real-life spider is not actually a material point, it wouldn't be wise for the fly to be too close to the spider's location.

2. Let us replace the continuous spherically symmetric medium with a discrete one (onion-like, with each layer having a constant n) – see Figure 2 for an illustration. The Snell's law applied to two neighbouring layers of radiuses R_1 and R_2 gives

$$n_1 \sin \alpha = n_2 \sin \beta. \quad (3)$$

In the triangle CAB we have

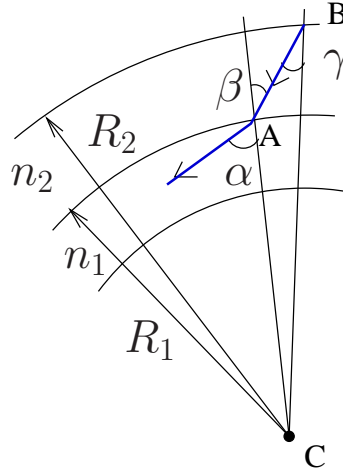


Figure 2: "Onion peeling".

$$\frac{\sin \gamma}{\sin \beta} = \frac{R_1}{R_2} \rightarrow \sin \beta = \sin \gamma \frac{R_2}{R_1}, \quad (4)$$

where we accounted for $\sin(\pi - \beta) = \sin \beta$. By substituting (4) into (3) one obtains

$$R_1 n_1 \sin \alpha = R_2 n_2 \sin \gamma, \quad (5)$$

This relationship is valid for any two adjacent layers, hence, in general $Rn(R) \sin \varphi(R) = \text{const}$ along the beam, where $\varphi(R)$ is the angle of incidence of the beam on the spherical layer of radius R . Using the expression for $n(R)$ we obtain

$$\frac{n_0}{R_0} R^2 \sin \varphi = \text{const}. \quad (6)$$

At the minimal distance from the symmetry center the angle between the radius and the beam is $\varphi = \pi/2$. Therefore,

$$\frac{n_0}{R_0} R_1^2 \sin 30^\circ = \frac{n_0}{R_0} R_{min}^2. \quad (7)$$

We obtain $R_{min} = R_1 \sqrt{\sin 30^\circ} = 0.566 \text{ m}$. Note that $R_{min} > R_0$.

3. Let us consider a region in the dust “cloud”, as depicted in Figure 3. On the path of the light rays we choose a short distance, small enough that across $\delta\ell$ the dust particles do not overshadow each other. Then the light is absorbed due to the cross section ΔS of all the dust particles in the layer. For a unit cross section of the layer we have the *absorbing* cross section

$$\Delta S = N \delta\ell \pi r^2 = \frac{m}{(4/3)\pi r^3 \rho} \delta\ell \pi r^2 = \frac{3m\delta\ell}{4r\rho}, \quad (8)$$

where N is the number of dust particles in the unit volume and ρ is the density of the material that makes up the dust. One could write (8) for both cases and divide side-by-side to obtain

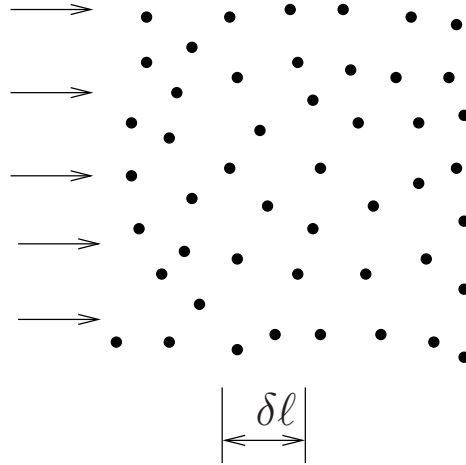


Figure 3:

$$\frac{\delta\ell_1}{\delta\ell_2} = \frac{m_2 r_1}{m_1 r_2}. \quad (9)$$

Note that (9) expresses the relationship between ℓ , m , r for two cases with the same light-absorbing “power”. The same relationship holds for two, three or more (n) layers, again in the assumption

that the particles do not overshadow each other. In the first case the visibility distance is related with $\delta\ell$ as $d_1 = n\delta\ell_1$, while in the second case we have $d_2 = n\delta\ell_2$. Then

$$\frac{d_1}{d_2} = \frac{n\delta\ell_1}{n\delta\ell_2} = \frac{m_2 r_1}{m_1 r_2}. \quad (10)$$

Therefore, we obtain

$$m_2 = m_1 \frac{r_2 d_1}{r_1 d_2} = 60 \text{ g/m}^3. \quad (11)$$

The statement of the problem included the error in the input quantities, so we have to do a bit of error calculus. Since (11) contains just multiplication, the easiest way is to compute the *relative* error of the end result, using the relative errors in the “ingredients”. The factors are independent of each other, therefore the relative errors (a.k.a. “fractional errors”) add in quadrature ¹:

$$\frac{\delta m_2}{m_2} = \sqrt{\left(\frac{\delta m_1}{m_1}\right)^2 + \left(\frac{\delta d_1}{d_1}\right)^2 + \left(\frac{\delta d_2}{d_2}\right)^2 + \left(\frac{\delta r_1}{r_1}\right)^2 + \left(\frac{\delta r_2}{r_2}\right)^2}. \quad (12)$$

We obtain $\delta m_2/m_2 = 0.5$, hence $m_2 = (60 \pm 30) \text{ g/m}^3$.

4. To construct the image of the square it is convenient to use the rays passing through the foci – see Figure 4. The image of the upper side (AB) of the square should be on the ray ABGF, or on its continuation to the left of the lens. The image A' of point A can be found by constructing the ray passing through the focal points (rays AFMA' and ABGF). Note that MA' is parallel to the optical axis, and that A' is real. The image of C should be on the optical axis. Since C and A are located

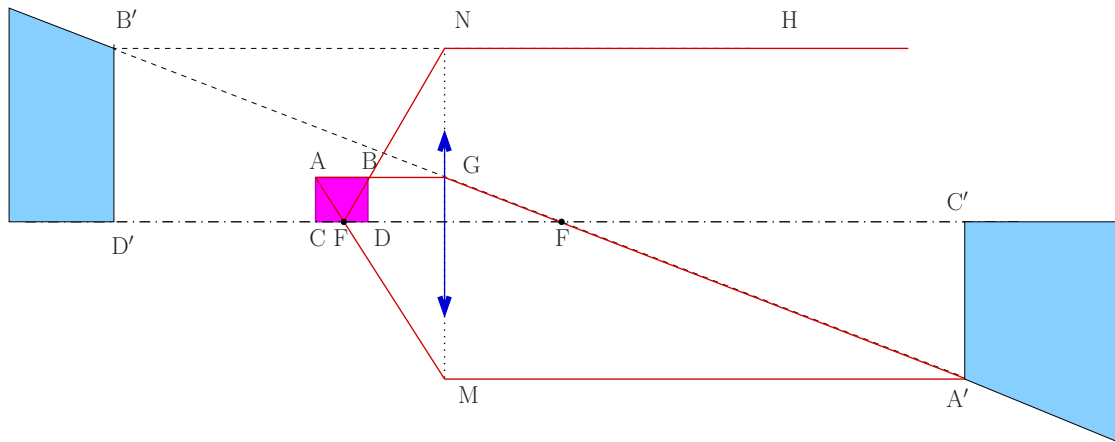


Figure 4: Construction of the image.

on the same side of F, it is easily seen that C' should be in the same region as A'. Its position can be found by drawing the perpendicular from A' to the main optical axis (objects perpendicular to the main optical axis should preserve this property after the transformation through the lens). With

¹For details see <http://www.upscale.utoronto.ca/PVB/Harrison/ErrorAnalysis/Propagation.html>

the same approach the image of B is found to be imaginary, and situated at the intersection of the continuations of the rays passing through foci, the rays FBNH and BGF. The image of D is also imaginary, and is found in the same manner as C'.

Note that to construct the rays NH and MA' it seems like we quietly assumed that the lens reaches as far as the locations of N and M. However, this is not an issue – the lens doesn't have to be that long. The difference large vs. small lens shows up only in the intensity of the image, not in its location (assuming both versions of the lens have the same focal distance).

In summary, the image of the square ABCD through the lens has two parts: the real image bounded by A'C', optical axis and the extension of the ray GF (the light blue region), and an imaginary image bounded by B'D', optical axis and the continuation of FG ray. The half of the square farther from the lens than the focal plane has a real image, the other half has an imaginary image.

5. We first show that for small α 's the laser beam exits the slanted face of the prism under a small angle γ given by

$$\gamma \approx \alpha(n - 1). \quad (13)$$

Above and herefrom the angles are expressed in radians. By applying Snell's law to the setup in Figure 5 we obtain

$$n = \frac{\sin(\alpha + \gamma)}{\sin \alpha} \approx \frac{\alpha + \gamma}{\alpha}, \quad (14)$$

from which (13) follows easily. The region where the interference occurs is shown in Figure 6.

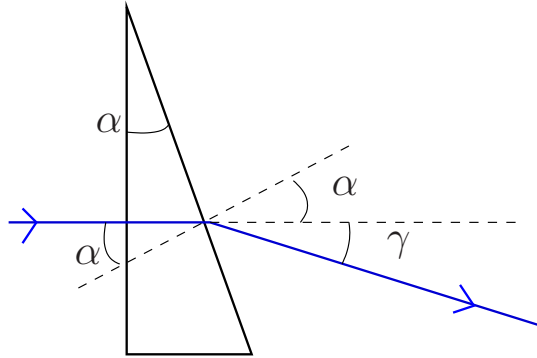


Figure 5:

The two plane-parallel beams intersect under the angle 2γ , and since the laser light is coherent we have interference. In the plane that crosses the horizontal axis at $x = 0$ we choose a point C, of coordinate $(0, y)$. We consider the superposition of the two beams at this point, assuming a constant amplitude E_0 for each of the beams. The phase of the upper beam is

$$\varphi_1 = -\frac{2\pi}{\lambda}|CA| = -\frac{2\pi}{\lambda}y \sin \gamma \approx -\frac{2\pi}{\lambda}y\gamma, \quad (15)$$

where λ is the wavelength. The phase of the bottom beam is then

$$\varphi_1 = +\frac{2\pi}{\lambda}|CB| \approx \frac{2\pi}{\lambda}y\gamma, \quad (16)$$

therefore the phase difference is

$$\Delta\varphi = \varphi_2 - \varphi_1 = \frac{4\pi}{\lambda}y\gamma. \quad (17)$$

The intensity of light in the resultant wave at point C is then

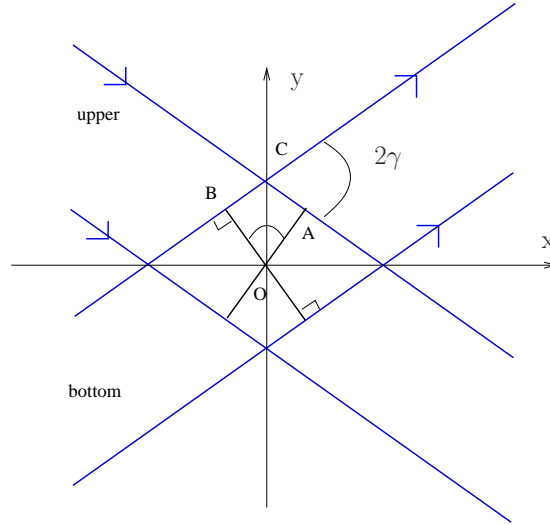


Figure 6:

$$I = E_0^2 + E_0^2 + 2E_0^2 \cos\left(\frac{4\pi}{\lambda}y\gamma\right) = 2E_0^2 \left[1 + \cos\left(\frac{4\pi}{\lambda}y\gamma\right)\right]. \quad (18)$$

Since γ is small, the distance between successive intensity maxima is related to $\lambda/2$ by $\gamma\ell = \lambda/2$, hence

$$\ell = \frac{\lambda}{2\gamma} = \frac{\lambda}{2\alpha(n-1)}. \quad (19)$$

As the particles move along y -axis with the speed v , they periodically (due to the space periodicity of the interference picture) intersect the maxima and minima of light intensity. The time period of the resulting fluctuations in the scattered light (as registered by the photodiode) equals the time it takes the particles to move across ℓ . Therefore,

$$T = \frac{\ell}{v} = \frac{\lambda}{2v\alpha(n-1)}. \quad (20)$$

The velocity of the particle is then

$$v = \frac{\lambda}{2T\alpha(n-1)} = \frac{\lambda f}{2\alpha(n-1)} = 6.3 \times 10^{-2} \text{ m/s}. \quad (21)$$