

Solutions to problem set 5

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1. The problem may be solved graphically. We first notice that the circuit can be viewed as a coupling of a linear block (battery and resistor R , to the left of the AB line), and a non-linear one (D_1, D_2 & r , to the right of AB). This linear/non-linear break-up is easier to analyze from a graphic point of view. For both blocks we can construct the (U, I) characteristics, and expect to

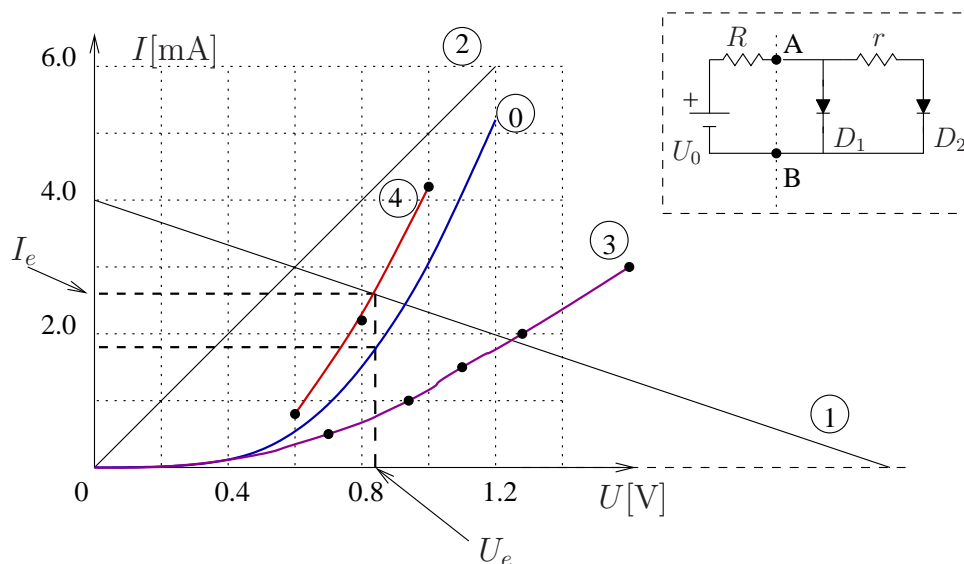


Figure 1: Illustration to problem #1.

find a (perhaps unique) point (U_e, I_e) as their intersection. For the linear block the relationship $I(U)$ is simple, and reads

$$I = \frac{1}{R}(U_0 - U), \quad U \leq U_0. \quad (1)$$

Indeed, this is just the Kirchhoff's rule for the loop, where U is the voltage drop across the diode D_1 , and also the common "boundary" voltage with the non-linear block (i.e. U_{AB}). The corresponding characteristic is denoted (1) in Figure 1.

We determine next the $I(U)$ characteristic for the non-linear block, which consists of two branches: D_1 and D_2 & r . For D_1 we already have the characteristic (curve (0)). In the D_2 & r branch, the resistor has its own characteristic (line (2)), given by

$$I = \frac{1}{r}U. \quad (2)$$

Since r and D_2 are connected in series, the total voltage drop on this branch can be determined by taking the sum of voltage drops on r, D_2 , for several values of the common I . The resulting

characteristic is denoted (3). The total current from the battery (I_{total}) into the non-linear block can be determined by adding, at some given values of U , the currents as they are read from the (0) and (3) curves. Note that it is sufficient to construct the characteristics just in the region where the intersection(s) occur. From the graph we find $U_e \approx 0.82 \text{ V}$ and $I_e \approx 2.6 \text{ mA}$. The current through D_1 at this U_e is known (from curve (0)), namely $I_1 \approx 1.8 \text{ mA}$. Hence, the current through D_2 is $I_2 = 0.8 \text{ mA}$.

2. The central idea is to recognize that the distribution of charge on a given sphere depends on the electric field created at the location of interest by all other charges in the system.

After the first step the potential of spheres 1 and 2 is the same. We have

$$\frac{Q_1^{(1)}}{4\pi\epsilon r} + \frac{Q_2^{(1)}}{4\pi\epsilon a} + \frac{q}{4\pi\epsilon a\sqrt{2}} = \frac{Q_1^{(1)}}{4\pi\epsilon a} + \frac{Q_2^{(1)}}{4\pi\epsilon r} + \frac{q}{4\pi\epsilon a}, \quad (3)$$

and the conservation of charge

$$Q_1^{(1)} + Q_2^{(1)} = 2q. \quad (4)$$

Above, $\epsilon = \epsilon_0\epsilon_r$ is the permittivity of the environment in which the spheres are located (it simplifies away, anyway) and the superscript (1) denotes the step. Since $r \ll a$ we can (i) neglect the details of the distribution of charge on the spheres (hence we could assume spherical symmetry when applying Gauss's theorem to determine the potential created by the charge on its own sphere) and (ii) neglect r/a at powers higher than 1 (see below).

After the second step (shorting of 2 and 3) we have in a similar manner (omit the $1/4\pi\epsilon$ factor and denote $\alpha = r/a$)

$$Q_1^{(1)}\alpha + Q_2^{(2)} + Q_3^{(2)}\alpha = Q_1^{(1)}\alpha/\sqrt{2} + Q_2^{(2)}\alpha + Q_3^{(2)}, \quad (5)$$

and the conservation of charge

$$Q_2^{(2)} + Q_3^{(2)} = q + Q_2^{(1)}. \quad (6)$$

Finally, after shorting 3 and 1 we obtain

$$Q_1^{(3)}\alpha/\sqrt{2} + Q_2^{(2)}\alpha + Q_3^{(3)} = Q_1^{(3)} + Q_2^{(2)}\alpha + Q_3^{(3)}\alpha/\sqrt{2}, \quad (7)$$

and the conservation of charge

$$Q_1^{(3)} + Q_3^{(3)} = Q_3^{(2)} + Q_1^{(1)}. \quad (8)$$

The system of 6 equations looks a bit scary, but after some straightforward algebra we find

$$Q_1^{(1)} = q(1 + \gamma), \quad (9)$$

$$Q_2^{(1)} = q(1 - \gamma), \quad (10)$$

$$Q_2^{(2)} = q \left(1 - \frac{3}{2}\gamma - \gamma^2 \right), \quad (11)$$

$$Q_1^{(3)} = Q_3^{(3)} = q \left(1 + \frac{3}{4}\gamma + \frac{\gamma^2}{2} \right), \quad (12)$$

where

$$\gamma = \frac{\alpha}{2 - 2\alpha} \left(1 - \frac{1}{\sqrt{2}} \right) \approx \alpha \frac{2 - \sqrt{2}}{4}, \quad (13)$$

and the approximation followed from the assumption $\alpha \ll 1$.

The final answer is then

$$Q_1 \approx q \left[1 - \frac{3r}{2a} \frac{2 - \sqrt{2}}{4} \right], \quad Q_2 = Q_3 \approx q \left[1 + \frac{3r}{4a} \frac{2 - \sqrt{2}}{4} \right]. \quad (14)$$

Note that the order in which we connect the spheres does matter! For example, we get a different answer if we start the redistribution of charges by short-circuiting 2-3, then 3-1 and last 1-2.

3. The behaviour of the bar upon the quick discharge of the capacitor resembles the behaviour of the ballistic galvanometer. For the ballistic galvanometer due to the (relatively) large momentum of inertia of the coil the (maximum) displacement of the galvanometer's needle is proportional to the total charge flowing through the circuit, rather than being proportional to the current. The same here: the capacitor quick discharge is effectively a kick, which the bar integrates, and displays as mechanical oscillation. The “integrates” comes from the recognition that $I = \Delta q / \Delta t$, which upon integration in time provides us with the total charge q .

The discharge of the capacitor is associated with a current I flowing through the circuit. The instantaneous force acting on the bar is $F = IB\ell$ and the direction of the force is either straight up or straight down, depending on the capacitor polarity. Over a small time interval Δt , during which we may assume the current to be constant, the impulse of the force is

$$F\Delta t = IB\ell \Delta t = B\ell \Delta q, \quad (15)$$

where Δq is the charge flowing through the circuit. The change in the (linear) momentum of the bar is then

$$\Delta p = m\Delta v = F\Delta t = B\ell \Delta q. \quad (16)$$

Basically, this is the same action mechanism as with the ballistic galvanometer (in this case force impulse instead of torque impulse). Over the time of discharge, the total change in the moment of the bar is

$$p = mv = \Sigma m\Delta v = \Sigma B\ell \Delta q = B\ell q, \quad (17)$$

where $q = CU$ is the total charge that passed through the circuit (which equals the initial charge on the capacitor).

According to the statement of the problem, the discharge of the capacitor and hence the impulse of the force F occur so fast that the mechanical oscillations do not have time to wind up; F is then the only force exerting a sizeable mechanical work. This behaviour justifies the name “ballistic”; that is the impulse-like behaviour of the motion (a kick at the beginning determines the evolution of the system – see projectile motion setups).

The bar on the springs is a pendulum, where the equivalent stiffness of the two springs (parallel “connection”) is $2k$. Let us take the initial position as being the equilibrium position, with zero

potential energy. The initial kinetic energy associated with the impulse $F\Delta t$ equals the potential energy at the extreme displacement A during the mechanical oscillations:

$$\frac{1}{2}mv^2 = \frac{1}{2}2kA^2. \quad (18)$$

Replacing $mv = B\ell CU$, we obtain the amplitude of the mechanical oscillation:

$$A = \frac{B\ell CU}{\sqrt{2km}}. \quad (19)$$

Numerical application: with $B = 1 \text{ T}$, $k = 10 \text{ N/m}$, $\ell = 0.1 \text{ m}$, $m = 0.1 \text{ kg}$, $C = 10^{-3} \text{ F}$, and $U = 100 \text{ V}$, one obtains $A \approx 7 \text{ mm}$.

4. The change in magnetic field \vec{B} induces an electrical field \vec{E} . By the geometry of the problem, the field lines of the induced \vec{E} are tangent to the ring. The absolute value of the electric field on the contour of radius R can be easily found as

$$E = \frac{|\mathfrak{E}_{induced}|}{2\pi R} = \frac{1}{2\pi R} \left| \frac{\Delta\Phi}{\Delta t} \right|, \quad (20)$$

where $\Delta\Phi$ is the change in the magnetic flux.

For a small piece of ring, of length $\delta\ell$, which bears the charge

$$\delta Q = \frac{Q}{2\pi R}\delta\ell, \quad (21)$$

and has the mass

$$\delta m = \frac{m}{2\pi R}\delta\ell, \quad (22)$$

one can compute the electric force F and apply Newton's second law. Indeed, we have

$$F_{\delta\ell} = \delta QE = \frac{Q\delta\ell}{2\pi R} \frac{1}{2\pi R} \left| \frac{\Delta\Phi}{\Delta t} \right|, \quad (23)$$

and therefore

$$a = \frac{F_{\delta\ell}}{\delta m} = \frac{Q}{2\pi Rm} \left| \frac{\Delta\Phi}{\Delta t} \right|. \quad (24)$$

From (24) it is clear that the acceleration does not depend on the location of the elementary piece $\delta\ell$. Hence, over time interval Δt the speed of all ring parts increases by $\Delta v = a\Delta t$, which after a bit of algebra can be written as

$$\Delta v = \frac{QR}{2m} |\Delta B|. \quad (25)$$

The total change in velocity is just the sum of Δv 's, namely

$$v = \Sigma\Delta v = \frac{QRB_0}{2m}. \quad (26)$$

Therefore, the angular velocity the ring gains after the change in the magnetic field is

$$\omega = \frac{v}{R} = \frac{QB_0}{2m}. \quad (27)$$

Regarding the direction of rotation: it depends on the sign of the charge Q , which was not given in the problem. Let us assume that Q is positive. The rotation of the charged ring should produce a magnetic flux that opposes the tendency of the external flux associated with B_0 (otherwise we would end up with a perpetuum mobile; Lenz's rule makes sure this doesn't happen). Since B_0 is increasing as shown in the figure, the ring's B_r should be increasing but in the opposite direction, which leads us to conclude that the (positively charged) ring rotates clockwise (view from above).

Numerical application: for a reasonably sized macroscopic ring and charge one may take for example $Q = 0.2 \times 10^{-6} \text{ C}$, $m = 0.1 \text{ kg}$, $B_0 = 1 \text{ T}$, which gives $\omega = 10^{-6} \text{ rad/s} \approx 5 \text{ rotations/year}$. The effect is theoretically pretty, but hardly observable. However, one may think of the ring as being microscopic (electron moving on the orbit), in which case $Q = |e|$, $m = m_e$, and $B_0 = 1 \text{ T}$ some external macroscopic field created in the laboratory. Now the effect might be quite detectable, as a small change in the frequency of the electromagnetic radiation emitted/absorbed by the atom!

5. Let us consider for the beginning the simpler case of a plate falling with constant velocity v . Due to the motion of the plate in the magnetic field the electrons are subject to Lorentz's force $F_L = evB$ (where the e is the magnitude of the charge of the electron). As a consequence the electrons will move in the plate until the electric force $F_E = eE$ (where E is due to the redistribution of charge in the plate) balances Lorentz's force. Therefore $E = vB$.

The thickness of the plate is much smaller than its radius, therefore the electric field inside the plate may be considered uniform (neglect the edge effects). The plate behaves as a plane capacitor, with the potential difference $U = Ed = Bvd$, and with the charge

$$Q = CU = \frac{\varepsilon_0 S}{d} Bvd = \varepsilon_0 S B v. \quad (28)$$

Here $S = \pi R^2$ is the area of one of the plate's faces. We notice that the charge depends on the velocity v . If the velocity changes so does the charge, that is a current will flow. We have

$$I = \frac{\Delta Q}{\Delta t} = \varepsilon_0 S B \frac{\Delta v}{\Delta t} = \varepsilon_0 S B a, \quad (29)$$

where a is the acceleration of the plate. The force on the plate due to the interaction of the current I with the magnetic field B is $F = BId$ and is oriented upward (opposite direction to the direction of motion). From Newton's second law we now have

$$ma = mg - Bld \Rightarrow ma = mg - \varepsilon_0 B^2 a \pi R^2 d. \quad (30)$$

The acceleration of the plate is therefore

$$a = g \frac{1}{1 + \frac{B^2 d \varepsilon_0 \pi R^2}{m}}. \quad (31)$$

For the numerical estimation we compute the ratio

$$\delta = \frac{|a - g|}{g} \approx \frac{\varepsilon_0 B^2 \pi R^2 d}{m}, \quad (32)$$

where it is assumed the r.h.s. ratio is small (which it is, for realistic parameters; if $x \ll 1$ then $1/(1+x) \approx 1-x$; this one follows from the identity $(1-x)(1+x+x^2+x^3+\dots) = 1$). Note that $d\pi R^2/m = 1/\rho$, where ρ is the density of the plate. For a plate of aluminum ($\rho_{\text{Al}} = 2.7 \text{ g/cm}^3$), with $B = 1 \text{ T}$ (a pretty strong electromagnet!), and vacuum permittivity $\varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$, one obtains $\delta \approx 3 \times 10^{-12} = 3 \times 10^{-10}\%$. This effect is virtually undetectable. The effect is even more undetectable if we use $B = 0.5 \times 10^{-4} \text{ T} = 0.5 \text{ Gauss}$ (the intensity of Earth's magnetic field).