2007-2008 Physics Olympiad Preparation Program
University of Toronto

Problem set 6: AC Circuits and general
Due: March 31, 2008

The Canadian Association of Physicists (CAP) High School Exam takes place on
Friday, April 11$^{\text{th}}$.
To be selected for the National Finals, the successful participation in the CAP Exam is essential.
Please visit http://www.cap.ca/edu/HSPrize/2008HighSchool.html for details

The 2008 POPTOR Weekend takes place during the weekend of May 16 – 18, 2008

1. Let us denote by $x(U)$ the change in the coordinate of the movable plate relative to its initial position ($x_0 = 0$ for $\Delta V \equiv U = 0$). We find the force (F) of attraction between the capacitor's plates, charged to the potential difference U. Note that F does not depend on the capacitor being attached or not to the voltage source, but only on the value of U.

If the capacitor is charged to voltage U and disconnected from the source, the magnitude of the charge on each plate is $Q = UC$, and the energy "stored" in the field is $W = Q^2/2C$. By changing the distance between the plates by Δx the energy changes by

$$\Delta W = \frac{1}{2}Q^2\left(\frac{1}{C'} - \frac{1}{C}\right) = \frac{1}{2}Q^2\left(\frac{d-\Delta x}{\varepsilon_0 S} - \frac{d}{\varepsilon_0 S}\right) = -\frac{Q^2}{2\varepsilon_0 S}\Delta x, \tag{1}$$

where C' is the capacitance of the capacitor when the distance between the plates is $d - \Delta x$. In absolute value this energy change is equal to the work of force F over the distance Δx (assume $\Delta x/d \ll 1$, so that the change in F can be neglected – as being a second order effect). Hence

$$F\Delta x = \frac{Q^2}{2\varepsilon_0 S}\Delta x \;\rightarrow\; F = \frac{Q^2}{2\varepsilon_0 S} = \frac{U^2C^2}{2\varepsilon_0 S} = \frac{U^2\varepsilon_0 S}{2d^2}. \tag{2}$$

We see that the force between the plates is inversely proportional to the square of the distance between the plates. Let us now assume the capacitor-voltmeter is connected to a voltage source U_x, in which case the coordinate of the moving plate is x (measured from $x_0 = 0$). The force of attraction between the plates is

$$F_x = \frac{U_x^2\varepsilon_0 S}{2(d-x)^2}, \tag{3}$$

and it is balanced by the elastic force of the spring. We have

$$kx = \frac{U_x^2\varepsilon_0 S}{2(d-x)^2}, \tag{4}$$

therefore

$$U_x^2 = \frac{2k}{\varepsilon_0 S}x(d-x)^2. \tag{5}$$

Clearly, the maximum measured voltage is reached when $f(x) = x(d-x)^2$ is maximized. For $x \in (0, d)$ the maximum of interest is reached for $x_M = d/3$ (set the first derivative of U_x^2 with respect to x to 0, then check that the second derivative evaluated at x_M is negative). Therefore

$$(\Delta V)^2_{max} = \frac{8kd^3}{27\varepsilon_0 S}, \tag{6}$$

and the maximum potential difference this capacitor-voltmeter can measure is

$$|\Delta V_{max}| = \frac{2d}{3}\sqrt{\frac{2kd}{3\varepsilon_0 S}}. \tag{7}$$

Numerical application: with $d = 1\,\mathrm{cm}, S = 10^{-2}\,\mathrm{m}^2$, $\varepsilon_0 = 8.8 \times 10^{-12}\,\mathrm{F/m}$ and $k = 1000\,\mathrm{N/m}$ we obtain $\Delta V_{max} = 0.6 \cdot 10^5\,\mathrm{V}$! In practice, it could be troublesome to measure such a high potential difference (because of electric breakdown of the air, a.s.o.).

2. With the values of C and R as given, the time constant of the RC circuit is $\tau = 1\,\mathrm{s}$, which is much larger than the period of the applied rectified DC voltage. In practice, this means we may neglect the change in the discharge current over one period of the applied voltage (see Figure 1). For the change in the voltage drop on the resistor we have

$$\Delta U = \frac{\Delta q}{C} = \frac{I\Delta t}{C} = \frac{U\Delta t}{RC}. \tag{8}$$

The quality of the filtration of the alternating component is characterized by the voltage pulsation coefficient, and is given by

$$k = \frac{\Delta U}{U_0} = \frac{\Delta t}{RC}. \tag{9}$$

Since the capacitor discharges only a little over Δt, we may consider the charging to occur at the moment when the voltage applied to it is maximal, therefore $\Delta t = T = 1/f$, where f is the frequency of the alternating voltage (and current) rectified by the diode. Numerically,

$$k = \frac{1}{fRC} \approx 0.017. \tag{10}$$

Notes:

1) Since $RC = \tau \gg T \sim \Delta t$, the intuitively clear assumption of neglecting the capacitor discharge over T can be formalized by using the approximation $e^{-\Delta t/\tau} \approx 1 - \Delta t/\tau$. Then the law of discharge of the capacitor $U = U_0 e^{-\Delta t/\tau}$ gives $|\Delta U| = U_0 \Delta t/\tau$.

2) In the RC circuit presented above the capacitive reactance of the capacitor is

$$X_C \equiv \frac{1}{\omega C} \ll R, \tag{11}$$

where $\omega = 2\pi f$. In AC-circuits jargon this RC circuit consists of a resistor R short-circuited by the reactive resistance X_C connected in parallel. The pulsating component of the current prefers “to route” through the capacitor rather than the resistor.

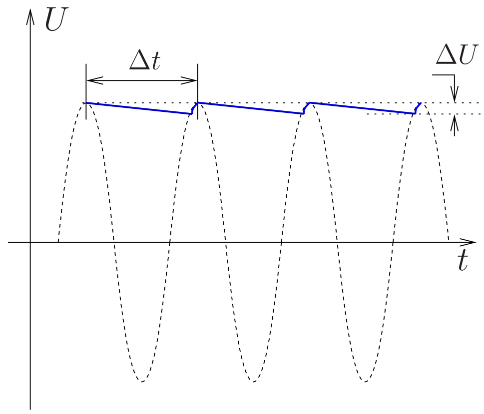

Figure 1:

3) The same philosophy should apply for the connection of the inductance L in *series* with the resistor, in which case the filter is effective if the inductive reactance X_L obeys

$$X_L \equiv \omega L \gg R. \tag{12}$$

In this case most of the pulsating component of the voltage will fall over the inductance rather than the resistor, effectively lowering the "alternating" component on R.

(ii) By adding the coil the amplitude of the voltage pulsation on the resistor should become even lower. The inductance of the coil should be relatively large[1] $R \ll 2\pi f L$ (this statement is equivalent to saying that time constant of the RL circuit is much larger than the period of the pulsating current). We shall again rightfully assume the current through R to be constant. The voltage on the capacitor will follow the same pattern as in part (i), however the voltage on the resistor will experience much smaller fluctuations, since the voltage on the coil will "cancel" most the fluctuations of the voltage (ΔU) on the capacitor. With these assumptions, at some time $t \in [0, T]$ between two voltage maxima (in the rectified current) the EMF on the coil changes linearly, namely

$$\mathcal{E} = \frac{\Delta U}{T} t, \tag{13}$$

while for the current through the coil we obtain (by integrating $\mathcal{E} = -L\Delta I/\Delta t$)

$$I = I_0 - \frac{\Delta U}{LT} \frac{1}{2} t^2. \tag{14}$$

The pulsation in the voltage on R is in this case determined by the the pulsation in the current, namely $\Delta U' = R\Delta I$. We have

$$\Delta I = I_{max} - I_{min} = \frac{1}{2} \frac{\Delta U}{LT} T^2 = \frac{1}{2} \frac{UT^2}{LRC}, \tag{15}$$

[1] There was a typo in this problem: the inductance was quoted as $L = 0.1$ H instead of the correct value, $L = 100$ H. There will be no penalty for using the wrong L, although the corresponding result (with $L = 0.1$ H) makes little sense.

where we accounted for $\Delta U = UT/RC$. Finally, with $L = 100\,\mathrm{H}$ one obtains

$$k' = \frac{\Delta U'}{U_0} = \frac{1}{2}\frac{T^2}{LC} = 1.4 \cdot 10^{-3}. \tag{16}$$

We see that the coefficient of pulsation with the LC filter decreased by a factor of $k/k' = 12$ in comparison to the capacitor filter only.

Note that the attenuation of the alternating component by the LC filter can be thought of as a cascade connection of two filters. We have $k_C = f_C/f = 1/fRC$, $k_L = f_L/f = R/fL$ and therefore $k_{total} = k_C k_L = 1/f^2 LC$, which coincides with (16) save for the SOFT effect (Systematically Omission of Factor Two). The effectiveness of the filter increases as f increases, and does not depend on the load (provided we have $X_C \ll R \ll X_L$).

3. Let us assume that a resistive external circuit is connected to the battery, and let U denote the voltage (potential difference) between the battery's terminals. We shall find the voltage-current characteristic of the battery. Clearly, only the electrons with (kinetic) energy $W \geq eU$ do reach the inner wall of the sphere. In the stationary regime U is constant, and the charge of the sphere is constant too. Therefore, the number of electrons arriving the sphere's interior wall is the same with the number of electrons passing through the external circuit, and the same number of electrons gets back to the radioactive sample in the center of the sphere.

Note that all the electrons make it to the wall of the sphere if $eU \leq W_{min}$, therefore the amount of charge the electrons carry in one second is (numerically) equal to the maximum current the battery provides, that is

$$I_{max} = e\nu, \tag{17}$$

where ν is the number of electrons hitting the sphere in one second (= the number of radioactive decays).

For higher values of the voltage U, such that $W_{min} < eU < W_{max}$, only the electrons with $W > eU$ make it to the sphere – the rest of them just fall back onto the radioactive sample in the center. The current through the external circuit is then determined by the fraction η of the electrons with $W > eU$, which is given by

$$\eta = \frac{W_{max} - eU}{W_{max} - W_{min}} < 1. \tag{18}$$

Above, we accounted for the uniform energy distribution of the electrons. The current flowing through the external load is then $I = e\nu\eta$. The maximum voltage difference $EMF = U_{max}$ at the battery's terminals is obviously obtained when the current goes to 0 (that is when the external circuit has infinite resistance, i.e. an open circuit). We have $eU_{max} = W_{max}$, therefore $U_{max} = W_{max}/e$. As soon as U_{max} is achieved, the charge of the sphere (and of the radioactive sample) remains constant.

The voltage-current characteristic of the battery is shown in Figure 2 (solid thick line). Also shown are the voltage-current characteristics of the external circuit for several values of the resistance (thin straight lines). Note that for $R \leq R_0 = W_{min}/e^2\nu$ the current does not change its value,

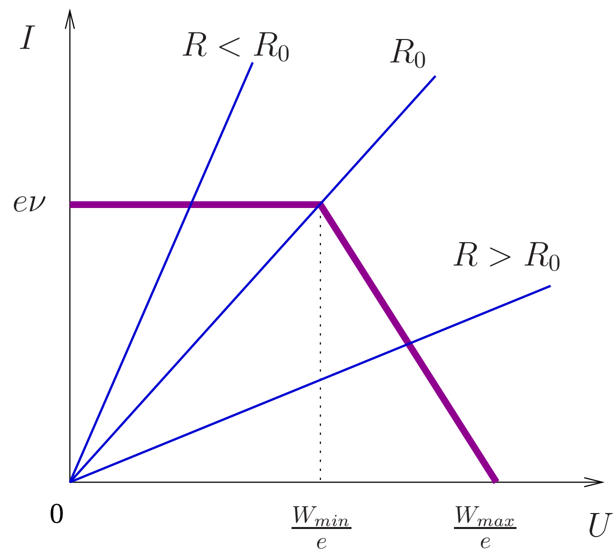

Figure 2:

regardless of the resistance of the external circuit! In such a case the battery can be considered to be a "current source" (instead of voltage source, as a battery is usually thought of[2])

Numerical application: with tritium as the radioactive material, $W_{max} = 18.6\,\text{keV} = 18.6 \cdot 10^3 \cdot 1.6 \cdot 10^{-19}\,\text{J}$, $|e| = 1.6 \cdot 10^{-19}\,\text{C}$, therefore $EMF = 18.6\,\text{kV}$!

4. The vector diagram of the potential differences across the elements of the circuit shown in Figure 3 is presented in Figure 4. Note that since the same current flows through R and C, the vectors corresponding to the voltages on R and on C are perpendicular, $\vec{U}_R \perp \vec{U}_C$. The two resistors r are equal, and the current through them is the same, hence $|\vec{U}_r| = \frac{1}{2}|\vec{U}|$. The potential difference between the points A and B is the magnitude of the vector $\vec{U}_{AB}$. Since the angle at vertex B is 90°, a well known geometrical property tells us that $|\vec{U}_{AB}| = |\vec{U}_r| = |\vec{U}/2|$, and A is the center of the circle circumscribing the triangle of vectors.

Hence the magnitude of $\vec{U}_{AB}$ does not depend on the value of R, since it always equals the radius of the circumscribed circle, which is half of $|\vec{U}|$.

As R changes the vector $\vec{U}_{AB}$ will rotate around A, although it's magnitude will not change. Therefore, this circuit is suitable for changing the phase between the input voltage $\vec{U}$ and the one picked up in between A and B.

5. Because of the motion of the metal ball in the uniform magnetic field **B** the electrons are subject to the action of the magnetic force. As a result the free electrons spread on the exterior of the ball, in such a way that the electric field created by the charge (re)distribution inside the ball is uniform and compensates the effect of the magnetic force. Once the equilibration is achieved,

[2] A current source can be constructed using a voltage source U with an internal resistance r much larger than the resistance of the external circuit R, in which case the current is approximately U/r and does not depend on R.

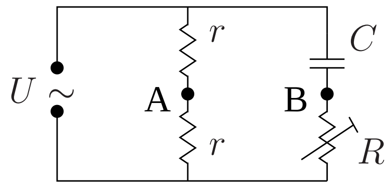

Figure 3:

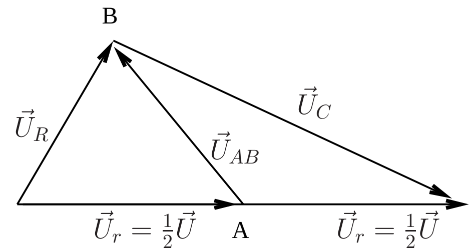

Figure 4:

the redistribution of the electrons stops. The intensity of the electric field $\mathbf{E}$ in such circumstances obeys

$$q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = 0, \tag{19}$$

where q is the charge of the electron. From (19), which states that the Lorentz's force is 0, we obtain

$$\mathbf{E} = \mathbf{B} \times \mathbf{v}. \tag{20}$$

Therefore we conclude that inside the ball the electric field is uniform and has the magnitude

$$|\mathbf{E}| = |\mathbf{B}||\mathbf{v}| \sin\alpha. \tag{21}$$

The maximum value of the potential difference $\Delta\varphi_{max}$ can be found by taking the longest path inside the ball, that is the diameter parallel to the vector $\mathbf{E}$. Finally, we get

$$\Delta\varphi_{max} = |\mathbf{E}|2r = 2r|\mathbf{B}||\mathbf{v}| \sin\alpha \tag{22}$$

With $\alpha = \pi/4, B = 1\,\mathrm{T}, v = 1\,\mathrm{m/s}$ and $r = 1\,\mathrm{cm}$, the maximum potential difference is $\Delta\varphi_{max} \approx 14\,\mathrm{mV}$, which is easily detectable (provided you are able to keep the test leads on the ball, and don't induce any spurious signal in the test wires).

Last revised: April 3, 2008