2008-2009 Physics Olympiad Preparation Program

University of Toronto

Problem set 2: Mechanics

Solutions

1. An 80 kg cyclist riding a 16 kg bicycle is travelling along a city street at 18 km / hour. A taxi pulls in front of him and stops suddenly to pick up a passenger. By the time the cyclist assesses the situation and begins to apply his brakes, he is 5 m away from the taxi. The cyclist knows from experience that if he jams on the brakes and his wheels skid, he will have less braking force, so he avoids this. The brake pads have a surface area of 8 mm by 70 mm and a coefficient of friction of 0.5, the bicycle has 700 mm diameter wheels and the diameter of the middle of the rim (where the brake pads are applied) is 650 mm.

   a) If the braking force is distributed equally between the front and back wheels, calculate the force that must be applied to each pad in order to stop in time.

   b) If the brake lever (on the handlebar) moves 2 cm for each brake pad to move 1 mm, calculate the force which must be applied to the brake lever.

a) Some deceleration required:

$$v^2 - v_0^2 = 2\,a\,s$$

$$v_0 = 18 \text{ km / hour} = 5 \text{ m / s}$$

$$a = \frac{v^2 - v_0^2}{2s} = \frac{0 - 25}{2 \bullet 5} = -2.5 \text{ m / s}$$

$$F = m\,a = -96 \bullet 2.5 = -240 \text{ N}$$

i.e., 120 N per wheel opposite to the direction of rotation

The brake pads apply a torque to the rim, but at a shorter distance from the axle than the outside diameter of the wheel (contact point with the road).

$$\tau_{OD} = \tau_{rim}$$

$$F_{p\text{-}r} = \frac{F_{OD} \bullet R_{OD}}{R_{rim}} = \frac{120 \bullet 0.350}{0.325} = 129 \text{ N}$$

$$F_{p\text{-}r} = \mu \bullet F_{perpendicular}$$

$$F_{perpendicular} = F_{p\text{-}r} / \mu = 129 / 0.5 = 258 \text{ N}$$

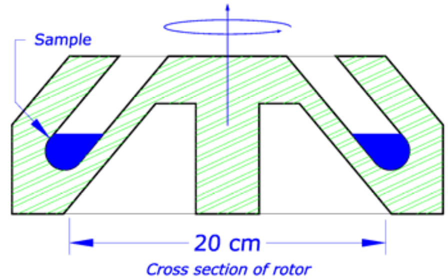
b) The combination of handlebar lever and caliper assembly has a mechanical advantage of 20. (Think of a lever with 20 units on one side of the fulcrum and 1 unit on the other.)

$$F_{lever} = F_{perpendicular} / 20 = 12.9 \text{ N}$$

2. Four samples of a colloidal aqueous mixture each weighing 12.0 g are placed in the rotor of a high speed centrifuge, equally spaced around the circumference of the rotor. The samples are located at 10 cm from the axis of rotation of the rotor.

   a) If the centrifuge motor delivers a constant torque of 0.25 Nm and the empty rotor has a moment of inertia of 0.06 kg m^2, how long does it take for the rotor to accelerate to its operating state of 18,000 rpm (rotations per minute). (For this calculation, neglect any change in position of the sample during acceleration.)

   b) When the centrifuge is up to speed, calculate the force exerted on the sample by the rotor.



a) From $\omega - \omega_0 = \alpha t$ and $\tau = I\alpha$

$$t = (\omega - \omega_0) \bullet I_{total} / \tau$$

where $I_{total} = I_{rotor} + I_{samples} = I_{rotor} + 4m_{sample}r^2 = 0.06 + 4 \bullet 0.012 \bullet (0.1)2 = 0.0605 \text{ kg m}^2$

and $\omega = 18{,}000 \text{ (rpm)} / 60 \text{ (seconds/minute)} \bullet 2\pi \text{ (radians/revolution)} = 1885 \text{ s}^{-1}$

$$t = 1885 \bullet 0.0605 / 0.25 = 456.2 \text{ s}$$

b) Force on the sample:

$$F_{central} = m\,\omega^2 r = 0.012 \bullet 1885^2 \bullet 0.1 = 4{,}264 \text{ N}$$

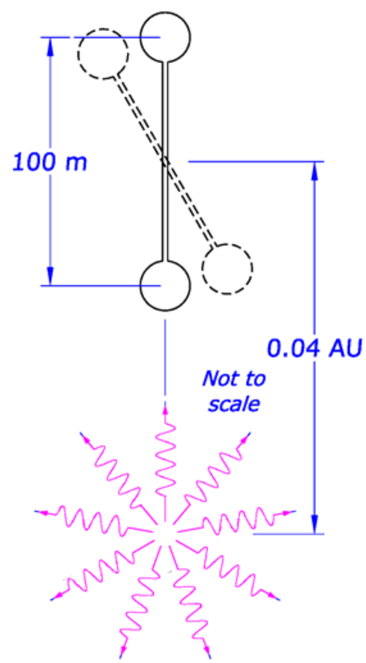
3. An intergalactic observation station is in a circular orbit of radius 0.04 AU around a black hole of mass 10 solar masses. The observation station is constructed in the shape of a dumbbell with two spheres 20 m in diameter, connected by a hollow cylindrical beam of small cross section and 80 m long..

   a) Calculate the orbital period and the tangential velocity of the station.

   b) The station is aligned in orbit so that the long axis of the dumbbell is aligned radially with respect to the centre of the black hole. If each of the spheres weighs 100 tonnes, calculate the tension on the connecting beam.

   c) If the alignment of the station noted in (b) is to be a stable configuration, any change in that alignment would have to result in forces which cause the station to return to its stable alignment. Calculate the magnitude and direction of the gravitational forces which would occur if the dumbbell axis were rotated 30° out of position. Comment on the stability of this configuration.

   d) The cylindrical beam is made of a titanium alloy which has a yield strength of 830 MPa (mega Pascals). If the beam has a diameter of 2.00 m and a wall thickness of 5 cm, calculate how close the station can come to the black hole and not be torn apart by the differential gravitational forces.



a) For a circular orbit, with the central mass much greater than the satellite mass:

$$F_{central} = F_{gravitational}$$

$$m\,\omega^2\,r = G\,M\,m / r^2$$

$$\omega = [G\,M / r^3]^{1/2} = [6.67 \times 10^{-11} \bullet 20 \times 10^{30} / (0.04 \bullet 1.5 \times 10^{11})^3]^{1/2}$$

$$\omega = 7.86 \times 10^{-5}$$

$$\text{Period} = 1 / f = 1 / 2\,\pi\,\omega = 2{,}025 \text{ s}$$

$$\text{Tangential Velocity} = \omega\,r = 4.72 \times 10^5 \text{ m / s}$$

b) The tension in the beam is due to the difference in gravitational force between the inner and outer sphere.

$$T = \frac{GMm_1}{(R-\Delta R)^2} - \frac{GMm_2}{(R+\Delta R)^2}$$

$$= \frac{GMm}{R^2}\left[\frac{1}{(1-\Delta R/R)^2} - \frac{1}{(1-\Delta R/R)^2}\right]$$

as $\Delta R / R << 1$, we can use the binomial expansion on both fractions

$$= \frac{GMm}{R^2}\left[1 - 2(\Delta R/R) + 3(\Delta R/R)^2 \ldots - 1 - 2(\Delta R/R) - 3(\Delta R/R)^2 \ldots.\right]$$

$$= -\frac{4GMm\,\Delta R}{R^3}$$

$$= -\frac{4 \bullet 6.67 \times 10^{-11} \bullet 20 \times 10^{30} \bullet 100 \times 10^3 \bullet 50}{(0.04 \bullet 1.5 \times 10^{11})^3} = -0.124 \text{ N}$$

c) To calculate the displacement angle α between one of the spheres, the black hole and the radial axis, the distance from the sphere to the radial axis $D = \Delta R \sin(30°) = 25$ m. Then

$$\sin(\alpha) = \frac{D}{R - \Delta R\cos(30°)} \approx \frac{D}{R}$$

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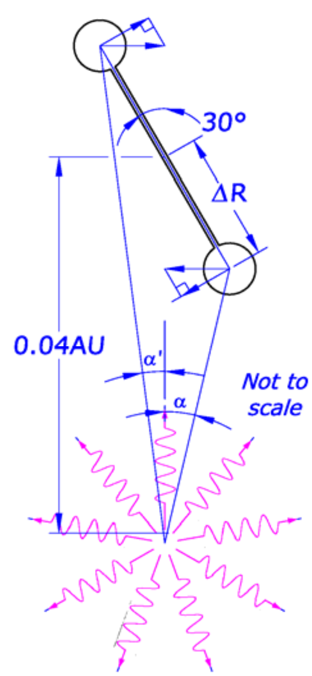


The restoring force perpendicular to the radial axis, $F_{\perp RA}$ (a component of the force of the black hole on one sphere) is

$$F_{\perp RA} = \frac{GMm}{R^2}\sin(\alpha) \approx \boxed{\frac{GMmD}{R^3}} = 1.54 \times 10^{-2} \text{ N}$$

The restoring force F_R is a component of $F_{\perp RA}$, perpendicular to the axis of the observation station is given by

$$F_{jR} = F_{\perp RA}\cos(30°) = 1.33 \times 10^{-2} \text{ N}$$

A similar restoring force acts on the other sphere. As $\alpha' \approx \alpha$, this force is equal to the force on the first sphere, so the total restoring force is 2.66×10^{-2} N

Clearly, this total restoring force would be insufficient to deal with the non-radial impacts or varying pressures on the station from gas particles and larger debris being accelerated into the black hole. Some form of active attitude control would be needed.

d) The maximum tension force F_{max} that can be applied to the beam is given by

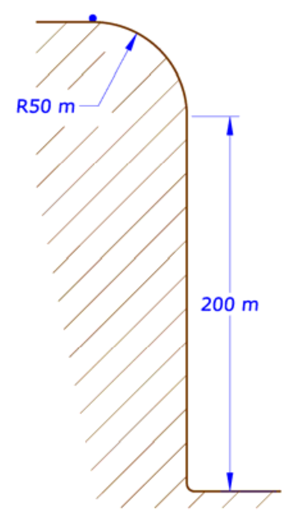
$$F_{max} = \text{Cross sectional area of beam x yield strength}$$
$$= \pi (1 - 0.95^2) \bullet 830 \times 10^6$$
$$= 2.54 \times 108 \text{ N}$$

The equation for differential force as a function of distance R was calculated in (b). The distance of closest approach is given by:

$$F_{max} = \frac{4GMm\,\Delta R}{R^3} \quad \text{or} \quad R^3 = \frac{4GMm\,\Delta R}{F_{max}}$$

$$R_{min} = 4.72 \times 10^6 \text{ m}$$

4. You are hiking in the coastal range, and after about an hour along an uphill, heavily wooded trail, you come out onto a smooth expanse of rock which after a few metres begins to slope downward and gradually blends into what appears to be a sheer vertical drop. Indeed, the top of this rock formation could be approximated by a ¼ cylinder (with axis horizontal). While you are enjoying the view and getting your water bottle out of your backpack, a small, hard rubber ball (diameter 60 mm) escapes from your pack and starts rolling towards the cliff. By the time that you notice it, it is too late to do anything but watch it go.



a) Assuming that the approach to the cliff really is cylindrical in cross section, with radius 50 m and that the ball starts from the top of the cylindrical surface, calculate how far away the ball is when it becomes airborne.

b) If the ball weighs 150 g and the vertical face of the cliff is 200 m high, does the ball reach terminal velocity on the way down (assume that the drag coefficient for a sphere is 0.47). How long does it take from when it leaves the surface to reach the base of the cliff?

a) To solve this, consider the potential and kinetic energies. Let θ be the angle between the vertical and the position vector of the ball (from the centre of the cylinder to the centre of the ball).

$$PE = KE$$

$$m g h = \tfrac{1}{2} m v^2 + \tfrac{1}{2} I \omega^2$$

where h is the vertical distance the ball falls. If the radius of the cliff edge is R and the radius of the ball is r,

$$h = (R + r)(1 - \cos\theta)$$

using $v = r\omega$ and $I = (2/5)\, m r^2$ for a sphere,

$$m\,g\,(R + r)\,(1 - \cos\theta) = \tfrac{1}{2}\,m\,v^2 + (1/2)\,(2/5)\,m\,r^2\,\omega^2$$
$$= \tfrac{1}{2}\,m\,v^2 + (1/5)\,m\,v^2 = 0.7\,m\,v^2 \qquad (1)$$

The component of the gravitational force on the ball pointing to the centre of the cliff edge keeps the ball touching the surface. When the force required to keep the ball moving in a circular path equals this component of the gravitational force, the ball leaves the surface.

$$F_{Rgrav} = F_{central}$$
$$m\,g\cos\theta = m\,v^2 / (R + r) \qquad (2)$$

substituting from eqn (1)

$$m\,g\cos\theta = \frac{m\,g\,(R + r)\,(1 - \cos\theta)}{0.7\,(R + r)}$$

$$\cos\theta = 1 / 1.7\,; \quad \theta = 53.97° \text{ or } 0.942 \text{ radians}$$

distance along cliff edge surface $= R\,\theta = 50 \bullet 0.942 = 47.1$ m

b) $$v_t = [2mg/\rho A C_d]^{1/2}$$

$$v_t = \sqrt{\frac{2 \bullet 0.15 \bullet 9.8}{1.293 \bullet \pi \bullet 0.03^2 \bullet 0.47}} = 41.36 \text{ m / s}$$

From eqn (2), we can obtain a value for v when the ball leaves the surface:

$$v^2 = g\cos\theta\,(R + r)$$
$$v = 16.98 \text{ m / s}$$
$$v_h = v\cos\theta = 9.99 \text{ m / s}$$
$$v_v = v\sin\theta = 13.73 \text{ m / s}$$

Vertical distance to fall s includes the part of the round edge after the ball leaves the surface and the height of the vertical drop.

$$s = (R + r)\cos\theta + 200 = 229.4 \text{ m}$$

In the vertical direction, if we momentarily disregard the air resistance which is the cause of the terminal velocity,

$$v_{fv}^2 = v_{iv}^2 + 2\,g\,s = 13.73^2 + 2 \bullet 9.8 \bullet 229.4$$
$$v_{fv} = 68.45 \text{ m / s}$$

So it appears that the ball might reach terminal velocity.

Calculating the time to reach the ground takes a bit of work. Once the ball is free of the surface, it will accelerate downward due to the force of gravity. However, if we are thinking in terms of terminal velocity, the drag caused by the air will exert a force F_d on the ball in the direction opposite to gravity.

$$F_d = \tfrac{1}{2}\,\rho A C_d\,v^2$$

where the parameters are the same as in the formula for the terminal velocity.

$$F_{total} = ma = F_g - F_d$$

$$a = g - \rho A C_d v^2 / 2m$$

$$\frac{dv}{dt} = g - \frac{\rho A C_d}{2m} v^2$$

To simplify the algebra, let

$$b = -\frac{\rho A C_d}{2m}$$

$$dt = \frac{dv}{g + bv^2}$$

$$t = \int_{v_{vi}}^{v_{vf}} \frac{dv}{g + bv^2} = \frac{1}{\sqrt{-gb}} \tanh^{-1} \left[\frac{bv}{\sqrt{-gb}} \right]_{v_{vi}}^{v_{vf}}$$

$$t = 4.221 \tanh^{-1} (0.02418\, v_{vf}) - 1.456$$

$$v_{vf} = 41.36 \tanh \left[\frac{t + 1.456}{4.221} \right] = \frac{ds}{dt}$$

However, we only have information on the vertical distance that the ball falls in free flight, s = 229.4 m, so we will have to integrate this equation again.

$$s = 41.36 \int_{t=0}^{t} \tanh \left[\frac{t + 1.456}{4.221} \right] dt$$

$$\text{as} \quad \int F(u)\, du = \int \frac{F(u)}{u'}\, dt, \quad \text{where } u' = \frac{du}{dt}$$

$$\text{let} \quad u = \frac{t + 1.456}{4.221}\,; \qquad \frac{du}{dt} = \frac{1}{4.221}$$

$$s = 41.36 \cdot 4.221 \left[\ln \{\cosh(u)\} \right]_{t=0 \;\; 1.456/4.221}^{t \;\; (t+1.456)/4.221}$$

$$229.4 = 174.6 \ln \cosh \left[\frac{t + 1.456}{4.221} \right] - 10.19$$

$$1.3723 = \ln \cosh \left[\frac{t + 1.456}{4.221} \right]$$

taking the exponential and arccosh of both sides,

$$2.0489 = \frac{t + 1.456}{4.221}$$

$$t = 7.19 \text{ s}$$

substituting this value for t back into the equation for v_{vf} above, gives a value of $v_{vf} = 40.01$ m /s, so the ball does not quite reach terminal velocity.

If you didn't want to do all that calculus, a close estimate can be obtained by numerically modeling the acceleration using an iteration of the equation $v_f^2 = v_i^2 + 2 a \Delta s$ in steps of Δs. For example, if you use Δs = 20 m, t = 7.2 s and you don't quite reach the calculated terminal velocity. This calculation can be easily done using a spreadsheet. A sample Excel file will be available on the web site.

5. One method of doping silicon in the fabrication of semiconductors is to project a beam of the required ions onto a silicon wafer in an ion implanter. A beam of ^{11}B ions (mass 11.009 AMU, charge +1) is accelerated through a voltage drop of 80 kV towards a silicon wafer where the boron ions are absorbed in the wafer. If the electrical current carried by the beam of boron ions is 10 mA, calculated the force exerted by the boron ions on the wafer. If the silicon wafer is 150 mm diameter and 0.675 mm thick, calculate the temperature rise of the wafer in 1 second.

The force exerted by the ions is equal to their change in momentum per unit time.

$$F = \Delta(m v) / \Delta t$$

$$E_{Boron\text{-}11} = \tfrac{1}{2} m v^2 = 80 \text{ keV} = 80 \times 10^3 \bullet 1.602 \times 10^{-19} \text{ (J/eV)} = 1.28 \times 10^{-14} \text{ J}$$

$$m_{Boron\text{-}11} = 11.009 \bullet 1.66 \times 10^{-27} \text{ (kg/AMU)} = 1.83 \times 10^{-26} \text{ kg}$$

$$v^2 = \frac{2E}{m} = \frac{2 \bullet 1.28 \times 10^{-14}}{1.83 \times 10^{-26}} = 1.40 \times 10^{12}$$

$$v = 1.18 \times 10^6 \text{ m / s}$$

$$\Delta(m v) \text{ per Boron-11} = 2.16 \times 10^{-20} \text{ kg m s}^{-1}$$

The number of Boron-11 atoms per second is determined from the electrical current. Charge +1 means that each boron atom carries one elemental charge.

10 mA = 0.01 Coulomb / s

1 elementary charge is 1.602×10^{-19} Coulombs, so

10 mA = 6.24×10^{16} elementary charges / s = N, the number of boron atoms / s

For Δt = 1 s, the force on the wafer is

$$N \Delta(m v) = 6.24 \times 10^{16} \bullet 2.16 \times 10^{-20} = 1.35 \times 10^{-3} \text{ N}$$

Heat transferred is $\Delta Q = m\,C\,\Delta T$, where ΔQ is the energy deposited by the boron beam and m and C are the mass and heat capacity of silicon.

$$m_{wafer} = \rho\,V = 2330 \bullet \pi \bullet 0.075^2 \bullet 0.675 \times 10^{-3} = 2.78 \times 10^{-2} \text{ kg}$$

$$\Delta T = \frac{\Delta Q}{mC} = \frac{1.28 \times 10^{-14} \bullet 6.24 \times 10^{16}}{2.78 \times 10^{-2} \bullet 705} = 40.8 \text{ C}°$$

INFOBITS:

1 AU (Astronomical Unit) is the mean distance from the earth to the sun, $\approx 1.5 \times 10^{11}$ m

1 solar mass = 2×10^{30} kg

Terminal velocity $v_t = [2mg/\rho A C_d]^{1/2}$ where
m is the mass of the object,
ρ is the density of the medium,
A is the cross sectional area of the object and
C_d is the drag coefficient for the shape of the object

1 tonne = 10^3 kg

Density of air 1.293 kg/m^3

Density of silicon 2330 kg/m^3

Specific heat of silicon 705 J kg^{-1} °K^{-1}