1994-1995 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 1: General Physics

1. Dimensional analysis can be used to check to see if an expression is dimensionally correct or to get the form of an expression if we don't know it. We are going to do the latter.

   (a) Hmm... What could the period of a simple pendulum possibly depend on? There's the mass of the (ideal point) bob (or is that Bob, founder of the Church of the Sub-Genius?), m, the length of the (essentially massless ideal) string, l, the angle of the swing, θ, and the acceleration of gravity, g. We might think to include things like frictional forces, but they're small compared to gravity and besides, that would be too complicated. Let's assume the period T is a function of these four variables, each raised to some power:

   $$T = Cm^w l^x \theta^y g^z.$$

   C is a dimensionless constant, and w, x, y, and z are exponents for which we wish to solve (no dangling prepositions here). The dimensional equation for this relationship is

   $$[T] = [M]^w [L]^x [L/T^2]^z;$$

   the angle θ (measured in radians) has no dimensions. Simplifying,

   $$[T] = [M]^w [L]^{x+z} [T]^{-2z}.$$

   To have dimensional consistency, we must have

   $$\begin{aligned} 0 &= w \\ 0 &= x + z \\ 1 &= -2z \end{aligned}$$

   which gives us $w = 0$, $z = -1/2$, and $x = 1/2$. Thus the desired equation is

   $$T = C\sqrt{l/g}f(\theta)$$

   It turns out that $C = 2\pi$ and $f(\theta) \simeq 1$ for small angles, but we can't get that from this analysis.

   (b) The general equation for this problem is $a_c = Cm^x r^y v^z$, which gives us the dimensional equation $[L/T^2] = [M]^x [L]^y [L/T]^z$. Simplifying, $[L][T]^{-2} = [M]^x [L]^{y+z} [T]^{-z}$, which gives us the three equations $0 = x$, $1 = y + z$, and $-2 = -z$, which are solved by $x = 0$, $z = 2$, and $y = -1$. Thus $a_c = Cv^2/r$. It turns out that $C = 1$.

   (c) The general equation for this problem is $T = Cp^x \rho^y e^z$, which gives us the dimensional equation $[T] = [M/LT^2]^x [M/L^3]^y [ML^2/T^2]^z$. Simplifying, $[T] = [M]^{x+y+z} [L]^{-x-3y+2z} [T]^{-2x-2z}$, which gives us the three equations $0 = x + y + z$, $0 = -x - 3y + 2z$, and $1 = -2x - 2z$, which are solved by $x = -5/6$, $y = 1/2$, and $z = 1/3$. Thus $T = C\left(\frac{\rho^3 e^2}{p^5}\right)^{1/6}$.

2. For these ones, we quote numerical values of things from memory or make good guesses, multiply numbers in our head, rounding them off, and in general not worry about being too precise. It may also help to do the problem algebraically first, then put in the numbers, since this will allow us to plug in more precise numbers if we wish.

(a) In the spirit of rapid estimation, we will give the result as a series of approximate calculations. This way we only have to remember a few numbers at a time.

Presumably the depth of liquid, d, will not be large compared to the radius of the Earth, R_E, so we can use the formula for the volume of a thin shell[1]; $V = 4\pi R_E^2 d$. Thus $d = V/4\pi R_E^2$. What is V? Well, there are nearly 6 billion people on the planet. Their average mass is, say, about 50 kilograms thus the total mass of people is about 300 billion kilograms. The density of a human body is close to that of water[2], about a tonne per cubic metre, so the total volume is about 300 million cubic metres.

The radius of the Earth is about 4 thousand miles (a good number to remember) or over six thousand kilometres. Squared, we have about 40 trillion square metres. Multiplying by $4\pi \simeq 12.5$ we have the surface of the Earth being about 500 trillion square metres. Dividing the volume by this we get a depth of about a half a micron (μm). I don't even know if this would get your socks dirty, but you could easily get by without the rubber boots and just wearing shoes.

The politically-correct restatement of the problem is: if every person on the planet decided to go skinny-dipping in the ocean, then by how much would the ocean level rise? The answer is about 4/3 the answer to the non-politically correct question since the oceans cover about 3/4 of the planet.

What about the number of Skydome^{TM}'s? We just divide the volume found above by the volume of a building. The rough area of the football field is 60 m by 130 m or about 8 thousand square metres. The total area is maybe four times that, and the total height is, perhaps, 30 m. Thus the total volume is on the order of one million cubic metres. Dividing the volume of liquid by this we get something on the order of several hundred Skydome^{TM}'s.

(b) You may have heard that the federal debt per capita is about \$20,000. If you hadn't heard that, then you should read more newspapers! With 27 million people in Canada, this works out to about \$550 billion. Wow! A loonie has a mass of about 10 grams, I believe. On Earth, 454 grams weighs a pound, so lets say that 50 loonies weigh one pound. Thus the federal debt weighs about 11 billion pounds in loonies.

Stacking them, they are about 2 mm think so that makes about 1.1 trillion metres or a million kilometres. That's to the moon and back easily!

Converting them to a pound of flesh, we have that a pound is about half a kilogram and the density is a tonne per cubic metre. This works out to 270 billion kilograms or 270 million cubic metres. This nearly doubles the volume of liquid for the previous problem, thus it nearly doubles the depth.

3. (a) What is wrong with the phrase "the car was traveling at a high rate of speed"? Well, the word rate usually refers to a change in a quantity measured with respect to an interval of time. The definition of speed is the distance traveled per unit of time, i.e. the rate of change of position with respect to time. Thus the word rate in the above phrase is superfluous. One could say "at a high speed" or one could say "at a high rate" where it is implicit that the rate is position with respect to time, i.e. speed. Note also that we can't really interpret "at a high rate of speed" as acceleration since in that case it would be "at a high rate of change of speed".

(b) As in the previous question, the fast and slow refer to rates as well. What most people mean is "five minutes ahead" or "two minutes behind". One can use rate-implying words when talking about clocks, but in the following context. Suppose we set a watch to coincide with an atomic

[1] You can derive this in the following way. Let r be the radius of a sphere, and $r + \Delta r$ be the radius of a slightly larger sphere. The volume of the shell between the two spheres is the difference in volumes which is $4\pi((r+\Delta r)^3 - r^3)/3 = 4\pi(r^3 + 3r^2\Delta r + 3r\Delta r^2 + \Delta r^3 - r^3)/3$. This is then equal to $4\pi(3r^2\Delta r + 3r\Delta r^2 + \Delta r^3)/3 = 4\pi r^2 \Delta r(1 + \Delta r/r + \Delta r^2/3r^2)$. We then use the fact that $\Delta r/r << 1$ to get the volume $V \simeq 4\pi r^2 \Delta r$.

[2] People float with the aid of air in their lungs; without that they would sink. Thus people are denser than water, but not by much.

clock. Thirty days later we check and the watch is ahead one minute. We have gained 1 minute over a period of 30 days. Thus the watch is about one minute per month *fast*. Strictly speaking the units can be canceled to get a unitless number, but it is more meaningful to people to use the above form. Also note that I have seen people with degrees in physics make the mistake of saying fast when they mean ahead!

(c) The normal high and normal low are, in more precise language, the average high and low. That is, they are the averages of the daily highs and lows for a particular day of the year, averaged over the past 100 years or so for which there are recorded temperatures. What is left out is the variance of the highs and lows or rather the distribution of the temperatures.

It is possible that the average temperature on August 28 is 25 ℃ but there may be just as many August 28's which were 23, 24, 26, and 27. Does it then make sense to use the word normal to mean average? In such a case is 25 normal? Probably not.

It is really a case of experts not wanting to use precise language because it may end up confusing people, or so they think.

4. We know that $N/M = N_A/M_A$ where N is the number of molecules in the sample and M is the mass of the sample and the subscript refers to a mole of the substance. Thus Avogadro's Number is $N_A = NM_A/M$. To get M_A, we take the formula and the molecular masses and find the total; $(38 \times 1 + 19 \times 12 + 2 \times 16) = 298\,\mathrm{g}$. To get N, we take the volume of the sample, V, and divide it by the volume of a single molecule, v. Getting v is the tricky part.

The $COOH$ end of the molecule is hydrophilic while the rest of the molecule is hydrophobic. Thus the molecules form a layer standing side-by-side vertically on the surface of the water. The diagram shows approximately what the molecule looks like if we assume that the carbons are at the centres of tetrahedrons with the neighbouring atoms on the corners. There is an end-on view and a side view. The distance between two atoms joined to a common carbon is x—this is the length of an edge of the tetrahedron. It turns out that the end-on area is nearly square—the zig-zag in the carbon chain compensates for the difference between the edge length in one direction and the distance from edge-centre to opposite edge-centre in the other direction. In all three directions a distance of x is added to account for the fact that neighbouring molecules will not be butted up against one-another. The x puts the hydrogen atoms on one molecule a distance x from those on the neighbouring molecules. Thus the volume of each molecule is about $v \simeq 4t^3/121$ where t is the layer thickness. There is a measure of uncertainty in this last equation since we don't know how much space there is between molecules.

The total volume is $V = At$, thus $1/v \simeq 121A^3/4V^3$ and $N \simeq 121A^3/4V^2$. Now $A = \pi(d/2)^2$ and $V = M/\rho$; with a little algebra we get

$$N_A \simeq \frac{121\pi^3\rho^2 d^6 M_A}{256M^3}.$$

Putting in the numbers,

$$\begin{aligned} N_A &\simeq \frac{121\pi^3(0.8\,\mathrm{g/cm^3})^2(84\,\mathrm{cm})^6(298\,\mathrm{g})}{256(0.81\times10^{-3}\,\mathrm{g})^3} \\ &\doteq 2\times10^{24}. \end{aligned}$$

So Avogadro's number is about 10^{24} molecules per mole. We are off by a factor of 3 from the accepted value, which is ok considering how many uncertain numbers there were.

5. A telephoto lens, like a telescope, makes far objects appear near by giving rise to an angular magnification. Let us approximate a horse (call him Boxy) by a rectangular body with a vertical leg at each corner. Suppose the separation of the forelegs and of the back legs is 0.5 m, and the distance between the front and back legs is 2.0 m. If we observe the horse head-on when it is 10 m away, the gap between

its forelegs will subtend an angle of 1/20 rad at the eye, while the gap between the hind legs subtends 1/24 rad. This difference gives most of the impression of the length of the horse.

If the same horse is 100 m away the same two angles become 1/200 rad and 1/204 rad, which are not very different. However, if we use a telescope with an angular magnification of $10\times$, the angles become 1/20 rad and 1/20.4 rad. The apparent size of the front is the same as that of a horse standing 10 m away, but now its hind legs appear to be only 0.2 m further back. A *very* short horse!

6. (a) Break the problem into two parts. Consider the mesh with 1 A going in at X and 1 A coming out at infinity. By symmetry, the wire XY must carry 1/4 A of current. Alternately, consider the mesh with 1 A going in at infinity and 1 A coming out at Y. Again, by symmetry, 1/4 A runs through the wire XY. Add the two cases together and you get the stated problem, with the solution that 1/2 A flows along wire XY.

   (b) Similarly, the two cases each have 1/6 A flowing along XY, giving a solution of 1/3 A.

   (c) Referring to the diagram, you can see that because the circuit is infinite, it can be written as a circuit with itself being one of the components. If R' is the resistance of the entire circuit, then, using the rules for combining resistors in series and parallel, we have

$$\frac{1}{R'} = \frac{1}{R} + \frac{1}{2R + R'}.$$

   Multiplying by all three denominators and rearranging we end up with a quadratic equation for R', ${R'}^2 + 2RR' - 2R^2 = 0$. We use the quadratic formula to solve, with the negative root thrown out. The result is $R' = (\sqrt{3} - 1)R$.