1994-1995 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 2: Mechanics

1. (a) If the real depth is s and the velocity of sound is v_s, then the measured time would be $t_m = (2s/g)^{1/2} + s/v_s$ and the measured depth would be $s_m = gt_m^2/2$. Neglecting squares of small terms, we obtain

$$\frac{s_m - s}{s} \simeq \sqrt{\frac{2gs}{v_s^2}} = \sqrt{\frac{2(9.8)(30)}{340^2}} \doteq 7.2\%.$$

That is, the depth calculated without taking into account the finite speed of sound would be about 7 percent too large.

(b) An approximate value for the final velocity is[1] $(2gs)^{1/2} \simeq 24\,\text{m/s}$, showing that the Reynolds number is of the order of 10^4 most of the time; the drag force is therefore $C_D\pi\rho r^2 v^2$, with $C_D \simeq 0.4$. The equation of motion is thus $ma = mg - \beta v^2$, with $\beta = C_D\pi\rho r^2$. For the zeroth approximation, we neglect the last term so that $a = g$ and thus $v = gt$. For the first order approximation, we insert this value of v into the (small) last term to give $ma = mg - \beta g^2 t^2$, or $a = g - \beta g^2 t^2/m$. Now, if you knew calculus, you would immediately integrate this to obtain $s = gt^2/2 - \beta g^2 t^4/12m$. You might also just know that whenever you have a term At^n in the acceleration, it gives you an $At^{(n+1)}/(n+1)$ term in the velocity and an $At^{(n+2)}/(n+1)(n+2)$ term in the displacement. You should really learn calculus.

Anyway, neglecting the air drag thus introduces an error equal to the last term, $\beta g^2 t^4/12m$. The relative error is thus $\beta g t^2/6m \simeq \beta s/3m$. From the data given, we find that this is about 6%, thus the air drag and the finite speed of sound give approximately equal errors. Both errors make the calculated depth too large.

(c) For the table tennis ball, the drag force is much more important relative to the weight. The velocities are smaller, but still large enough to make $R > 10^3$, i.e. C_D is still about 0.4. In this case, the terminal velocity is given approximately by[2] $(mg/\beta)^{1/2} \doteq 5.3\,\text{m/s}$. The ball will reach this speed after a few meters, so that the zeroth approximation, $s_m = gt^2/2$, is a gross overestimate of the depth—by a factor of about 2.6 for $s = 30\,\text{m}$. A better first approximation is to assume that it travels at its terminal velocity for the whole distance. In this approximation, the error due to the finite speed of sound is clearly given by the ratio of the terminal velocity to the speed of sound. The error due to the initial period of acceleration cannot be evaluated without using the exact solution of the equation of motion. This is too complicated to be evaluated in the present context.

2. (a) The dominant consideration is the conversion of the kinetic energy of the running man to gravitational potential energy with something less than 100% efficiency. If, in his approach, he attains a speed of 10 m/s, the corresponding rise is 5 m. Smaller terms arise from

   i. the fact that his centre of mass is already 1 m above the ground when he starts,
   ii. the work done by his legs on take-off and by his arms in climbing up the pole (? = 0.5 m; consider the "vertical" of a standing person),
   iii. the fact that his centre of mass actually passes *below* the bar (? = 10 cm).

Adding these terms together gives approximately 6.6 m. The difference between this and the observed 5.5 m is due to an efficiency of less than 100%—or to errors in the estimated quantities. In any case, there is clearly no hope of Mr. Fibreglass making good on his boast of 9 m since 10 m/s is an excellent sprint speed and the other terms can not be improved by much.

[1] Equate initial potential energy with final kinetic energy and solve for speed.

[2] Write force equation for terminal body, including gravity and drag force but neglecting the small buoyant force, set acceleration to zero, then solve for speed.

(b) Work done is the change in potential energy of my body, $W = mgh$. Uncertainty is

$$\Delta W/W = \sqrt{(\Delta m/m)^2 + (\Delta g/g)^2 + (\Delta h/h)^2}.$$

The $(\Delta g/g)^2$ is negligible. The result is $34.85 \pm 0.4\,\text{kJ}$. Note that the uncertainty in the stair height dominates. A measurement of the actual distance would improve things immensely.

The average power output is $P = W/t$, while the uncertainty is

$$\begin{aligned}\Delta P/P &= \sqrt{(\Delta W/W)^2 + (\Delta t/t)^2} \\ &= \sqrt{(\Delta m/m)^2 + (\Delta g/g)^2 + (\Delta h/h)^2 + (\Delta t/t)^2}.\end{aligned}$$

The result is $348.5 \pm 8\,\text{W}$. This is nearly half a horsepower. Don't be impressed. I only did this for a couple of minutes. A horsepower is the average power a horse can put out over a work day. I've been told that for longer times (15 minutes to 2 or 3 hours) for people of average mass (65 to 75 kg), useful power output is roughly 275-300 W. By that criteria, given how winded I was, I am out of shape!

(c) If the rotor, of radius R, gives a downward velocity v to a column of air initially at rest, the momentum transferred per second will be μv where μ, the mass of the air accelerated per unit time, is $\pi R^2 v \rho$. If the helicopter is to be airborne, this must equal Mg, where M is the mass of the helicopter. Inserting the values given, we obtain $v \doteq 4.5\,\text{m/s}$. Neglecting energy losses, the energy transferred per second to the bulk motion of the air is $\mu v^2/2$ which equals 4.5 kW in this case. For the system to be practical, the pilot must thus generate about 5 kW continuously. From the answer to the previous question, it appears that the maximum rate of energy production by a human is of the order of 1/3 kW. Thus we are a far cry from being able to run a helicopter by human power.

3. (a) For the moon in an orbit of radius r, we have[3] $GMm/r^2 = mr\omega^2$, assuming $M \gg m$, where G is the universal gravitational constant and $\omega = 2\pi/T$ is the angular frequency with T being the period. For the rock of mass μ, in orbit of radius $r - a$, Newton's Law gives us

$$\frac{GM\mu}{(r-a)^2} - \frac{Gm\mu}{a^2} + F = \mu(r-a)\omega^2$$

where F is the force of contact between the rock and the moon. If the rock is to be lifted off, $F = 0$. Eliminating ω between the two equations, we obtain

$$\frac{M}{m} = \frac{r^3}{a^3}\frac{(r-a)^2}{3r^2 - 3ra + a^2} \simeq \frac{1}{3}\left(\frac{r}{a}\right)^3$$

as required.

(b) For any spacecraft in a circular orbit, $GMm/r^2 = mr\omega^2$. For a body on the surface of the Earth, $GM_E/r_E^2 = g \doteq 10\,\text{m/s}^2$. With the numbers given, we find $\omega = 10.4\,\text{rad/s}$, i.e. the period is approximately $\pi/5$ seconds—somewhat uncomfortable for the intrepid astronaut! We can also deduce that the linear velocity in orbit is approximately $10^7\,\text{m/s} \simeq 0.03c$—high, but not relativistic.

i. For an astronaut of mass m_a at the mass centre of the craft, $GMm_a/r^2 = m_a r\omega^2$ is also true, i.e. she will, indeed, float.

ii. If r changes to $r - x$, the gravitational force becomes $GMm_a/(r-x)^2$. If sideways motion across the vehicle is prevented, and if the vehicle is large compared with the mass of the person, then ω will remain constant, that is the ship will force the astronaut to orbit with

[3] The gravitational force provides the centripetal force which is set equal to the mass times centripetal acceleration.

the same period. The force needed to maintain the astronaut in the new, smaller orbit will be $m_a(r-x)\omega^2$. This is no longer equal to the gravitational force. The difference, approximately $3m_a\omega^2x$, will accelerate the person towards the centre at a rate $3\omega^2x$. This will give rise to a velocity[4] $\sqrt{3}\omega x$. Starting from the end of the tunnel ($x = 50$ m), this gives rise to a maximum speed of 900 m/s (3240 km/h in more homely units). On the way to this unfortunate conclusion there will be a Coriolis acceleration sideways across the tunnel equal to $2\sqrt{3}\omega^2x$. This will reach the value g before the victim has moved more than a few centimetres.

iii. The orbit as described is indeed stable. The differential radial force calculated above—the so-called tide-raising force—will provide a restoring couple if the attitude is disturbed, as can be seen by drawing a diagram. Also, one can appeal to tides themselves; they are at a maximum when the graviational source is overhead.

(c) Initially Newton's Law gives

$$mv^2/r = Gm^2/4r^2. \qquad (1)$$

The final system is most likely to break up if the explosion increases the velocity of one of the one star in the direction in which it is moving at the time of the explosion (consider the diagram where before and after denote the situation an instant before and after the explosion). Call the increased velocity v_f. The centre of mass now has a velocity $(v - v_f)/2$ and the velocity of either star with respect to the centre of mass is, say,

$$(v + v_f)/2 = v'. \qquad (2)$$

The system will break up if the new kinetic energy with respect to the centre of mass coordinates is greater than the work needed to separate the components to infinity, i.e. if $mv'^2 > GM^2/2r$. Using relations (1) and (2), the required condition becomes

$$v_f = (2\sqrt{2} - 1)v = (2\sqrt{2} - 1)(Gm/4r)^{1/2}.$$

The above argument assumes that the mass lost as a result of the explosion is so small that it has no effect on the result. If this quantity is denoted by Δm, then an analysis along exactly the same lines gives the condition $v_f/v = 2(2 - \Delta m/m)^{1/2} - 1$, thus showing that the assumption was valid.

4. We choose the positive sense of ω so that the velocity of the point of contact of the ball with the table, v_c, is given by $u + a\omega$. The frictional force $F = \mu mg$ is assumed constant. The motion of the centre is thus given by $u = u_0 - \mu gt$ and the rotation about the centre by $\omega = \omega_0 - a\mu mgt/I$. Since $I = 2ma^2/5$ for a sphere, the latter expression becomes $\omega = \omega_0 - 5\mu gt/2a$. These results combine to give $v_c = u_0 + a\omega_0 - 7\mu gt/2$. This is equal to zero (i.e. slipping stops and rolling begins) when $t = 2(u_0 + a\omega_0)/7\mu g = \tau$, say. The value of u at $t = \tau$ is $(5u_0 - 2a\omega_0)/7$; this gives the speed of rolling after slipping has stopped. If $u_0/a\omega_0 < 2/5$, the value is negative and the ball will roll backwards towards the cue. This corresponds to the imposition of "backspin" as specified. If $u_0/a\omega_0 > 2/5$, then the ball will roll forwards.

To enquire whether both motions are possible, we consider an impulse Q given to the ball at a point $a - x$ above the table in the direction making an angle θ with the horizontal (see the diagram). Then $mu_0 = Q\cos\theta$ and $I\omega_0 = Qp$, with $p = a\sin(\theta + \phi)$ and $\sin\phi = x/a$. If we use the relation $5u_0 = 2a\omega_0$ to determine the boundary condition between the two modes of behaviour, then, after a little algebra, we get $\cos\theta = \sin(\theta + \phi)$, i.e.

$$\tan\theta = \left(\frac{1 - x/a}{1 + x/a}\right)^{1/2}.$$

Real values of θ can be found for any x/a—except that $x \to a$ would be a little difficult, in practice!

[4] Think of simple harmonic motion with a spring constant over mass $\omega_0^2 = k/m_a = 3\omega^2$; the maximum speed, obtained while passing through the centre, will be $x\omega_0$.

5. (a) The annihilation energy of a particle of rest mass m_1 will make available an energy $\Delta E = m_1 c^2$. If this energy is given to another particle of rest mass m_2, which thereby acquires momentum p, we have the relation[5]

$$(\Delta E + m_2 c^2)^2 = p^2 c^2 + (m_2 c^2)^2.$$

In the particular case envisaged in the problem, $m_1 = m_2 = m$ is the mass of a proton or antiproton. This leads to the result $p = \sqrt{3}mc$. If there are initially N protons and N antiprotons, then this process can be repeated N times[6]. If all N ejected particles can be persuaded to leave the spaceship in the same direction, the effect will be to give a total momentum $\sqrt{3}Nmc$ to the ship.

(b) If we consider the mutual annihilation of two particles, each of rest mass m, a similar argument shows that the (scalar) total of the momentum thereby produced is $2mc$. There will, in fact, be two photons proceeding in opposite directions (since the vector total must be zero). With the aid of a deep paraboloidal mirror made of a material that can reflect high-energy gamma rays (!) all such photons can be collimated into a beam carrying a momentum $2Nmc$. The spaceship will have an equal momentum in the opposite direction. This is larger than the other result.

The above arguments are strictly valid only if the increment in velocity of the spaceship is much less than c. If this is not the case, the total number N can be conceptually subdivided into small groups for which it is true, and the total effect of the momentum increments between successive inertial frames can then be calculated as an integral (sum). The advantage of (5b) over (5a) applies at every stage and therefore also overall.

6. (a) As a first guess, one thing we can do is to get a maximum on the period. We don't feel the effect of the centripetal acceleration of our orbiting the sun very easily on Earth, so it must be less than, say, a percent of g. Since the acceleration goes as the inverse square of the period, the period must be less than 10 percent the Earth's year, that is less than a month. It is difficult to say anything more since we don't have an indication of how restrictive our maximum is.

More quantitatively, the centripetal acceleration of a particle moving in a circle of radius r at a constant speed v is

$$a_c = \frac{v^2}{r} = \left(\frac{2\pi r}{T}\right)^2 \frac{1}{r}$$

where T is the period. Thus $T = 2\pi\sqrt{r/a_c}$. Using $a_c = g$ we get $T \doteq 7.77 \times 10^5\,\mathrm{s} \doteq 9.0$ days. That seems like an awfully short time to go around the sun! Note that we've ignored the gravitation attraction of the star, which is small compared to g.

(b) Consider a narrow section of the ring as shown in the diagram. The net force on the section is $2F_T \sin(\Delta\theta)$ towards the centre. This is the centripetal force and must equal $ma_c = mg$ where m is the mass of the section. This is equal to $2r\Delta\theta M/2\pi r$ where r is the radius of the ring and M is the total mass. The total mass is the volume times the density, $M = 2\pi r A\rho$, where A is the cross-sectional area. Letting $\sin(\Delta\theta) \to \Delta\theta$ for small angles, we find that the tension is given by $F_T = rA\rho g$. This is just the "weight" of the ringworld on Earth (if we could weigh it) divided by 2π! To get the tensile strength we just divide by the cross-sectional area; $S = r\rho g$. Notice how we don't need to know A. If we take the density of aluminum as a possible density for the material, then the required tensile strength is $4 \times 10^{15}\,\mathrm{N/m^2}$. This is seven orders of magnitude greater than most metals.

[5] The left-hand side is the square of the total energy before, while the right-hand side is the square of the total energy after, using the relationship between energy and relativistic momentum.

[6] Actually, the more complicated process of annihilating a proton-antiproton pair giving the energy to another proton and antiproton pair which doesn't annihilate but rather is ejected in the same direction is performed $N/2$ times, but the effect is the same.

(c) The rotation of the ring will give it orientational stability, but not positional. The positional stability is depends only on the net gravitational attraction of the star.

If the ring is pushed so that it moves perpendicularly to its plane, then one can see a net gravitational force attracting the ring back to equilibrium. If the ring is pushed in its plane, the situation is not as easy to see. Consider the force due to four pieces as shown in the diagram. The size of the force goes as $1/r^2$ which depends on the position of the piece on the circle. The "vertical" component (see figure) also depends on the position but in a opposing way. The result is that one can't find out whether there is a net gravitational force in the plane by simple arguments. One must use math and it turns out there is a net restoring force.

Remarkably enough, there are cases where one can have an instability. A Dyson Sphere (a spherical shell of similar dimensions surrounding a star) would have no net force on it due to a star inside, thus it would be unstable. The argument for this is mathematical as well.

I got the idea for the ringworld question from a book by Larry Niven called, curiously enough, *Ringworld*. It is followed by *Ringworld Engineers*, and may be found in the science fiction section of libraries and bookstores. The book describes the material as "Very dense, with a tensile strength on the order of the force that holds nuclei together." Anyone care to check on whether that would be enough?