1994-1995 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 5: Electricity and Magnetism

1. All forces mentioned in the following are with respect to conductor B (refer to Figure 1). The magnetic force on a conductor in a uniform external magnetic field is given by

$$F_{\text{mag}} = I_B \vec{l} \times \vec{B}_A,$$

where $\vec{l}$ is a displacement vector along the conductor B, and $\vec{B}_A$ is the magnetic field from conductor A. The above expression can be simplified to

$$F_{\text{mag}} = I_B l B_A \sin\theta = I_B l B_A,$$

since $\vec{B}_A$ is perpendicular to $\vec{l}$ ($\theta = 90°$).

¿From the Biot-Savart Law, the magnetic field $\vec{B}_A$ at a distance r from a wire carrying a current I_A is given by

$$B_A = \frac{\mu_o I_A}{2\pi r},$$

where μ_o is the permeability of free space.

The magnetic force per unit length on conductor B is therefore

$$\frac{F_{\text{mag}}}{l} = \frac{\mu_o I_A I_B}{2\pi r}.$$

The gravitational force per unit length on conductor B, F_g, neglecting the gravitational fields of conductor A and the non-conducting guides, is

$$\frac{F_g}{l} = \lambda_B g,$$

where $\lambda_B \equiv$ the linear density of conductor B and g is the acceleration due to gravity.

At equilibrium, we set the forces to be equal, which gives:

$$\begin{aligned}
\frac{F_{\text{mag}}}{l} &= \frac{F_g}{l} \\
\frac{\mu_o I_A I_B}{2\pi r} &= \lambda_B g \\
\lambda_B &= \frac{\mu_o I_A I_B}{2\pi r g} \\
&= \frac{(12.566 \times 10^{-7}\ \text{N/A}^2)(100\ \text{A})(147\ \text{A})}{2\pi(0.02\ \text{m})(9.8\ \text{m/s}^2)} \\
&= 1.5 \times 10^{-2}\ \text{kg/m} = 0.15\ \text{g/cm}.
\end{aligned}$$

2. Calculate the change in *total* resistance with respect to the change in temperature, ΔR_t:

$$\begin{aligned} R(T) &= R_o(1+\alpha_t \Delta T) \\ \Longrightarrow \frac{R(T)-R_o}{\Delta T} &= R_o \alpha_t \\ \Longrightarrow \frac{\Delta R_t}{\Delta T} &= R_o \alpha_t . \end{aligned}$$

Given that $\alpha_t = 0$ at 20°C,

$$\frac{\Delta R_t}{\Delta T} = 0.$$

For each of the two individual resistance regions:

$$\frac{\Delta R_1}{\Delta T_1} = (R_o)_1 \alpha_1$$

$$(R_o)_1 = \frac{\rho_1 l_1}{A},$$

where A is the cross sectional area of the octagonal bars.
Therefore,

$$\frac{\Delta R_1}{\Delta T_1} = \frac{\rho_1 l_1 \alpha_1}{A}.$$

Similarly,

$$\frac{\Delta R_2}{\Delta T_2} = \frac{\rho_2 l_2 \alpha_2}{A}.$$

Since the two materials are in a series configuration:

$$\frac{\Delta R_t}{\Delta T} = \frac{\Delta R_1}{\Delta T_1} + \frac{\Delta R_2}{\Delta T_2} = 0.$$

Since $\Delta T_1 = \Delta T_2$, $\Delta R_1 = -\Delta R_2$ and

$$\rho_1 l_1 \alpha_1 = -\rho_2 l_2 \alpha_2$$

$$\Longrightarrow \frac{\alpha_1}{\alpha_2} = -\frac{\rho_2}{\rho_1} \cdot \frac{l_2}{l_1} = \frac{-1}{(3.2)(2.6)} = -0.12.$$

3. (a) The capacitances per unit length due to each dielectric are given by:

$$c_1 = \frac{2\pi\epsilon_1}{\ln(r_2/r_1)}$$

and

$$c_2 = \frac{2\pi\epsilon_2}{\ln(r_3/r_2)}.$$

The total capacitance per unit length is calculated by treating the above two capacitances in series. The total capacitance per unit length is therefore given by:

$$c = \frac{c_1 c_2}{c_1 + c_2},$$

$$c = \frac{2\pi\epsilon_1\epsilon_2}{\epsilon_2 \ln(r_2/r_1) + \epsilon_1 \ln(r_3/r_2)}.$$

(b)

$$c_1 = \frac{2\pi \frac{4.5}{36\pi} \times 10^{-9} \text{ F/m}}{\ln 2} \doteq 0.36 \text{ nF/m},$$

$$c_2 = \frac{2\pi \frac{3}{36\pi} \times 10^{-9} \text{ F/m}}{\ln 2} \doteq 0.24 \text{ nF/m},$$

$$\frac{V_2}{V_1} = \frac{c_1}{c_2} = \frac{0.36}{0.24} = 1.5,$$

$$V_1 + V_2 = 1200 \text{ V}.$$

Solving for V_1 gives $V_1 = 480$ V. Therefore

$$q = c_1 V_1 = 172.8 \text{ nC/m}.$$

The maximum electric field intensity occurs at the inner surface of either dielectric. Hence, at $r = r_1$:

$$E_{\max}(r_1) = \frac{172.8 \times 10^{-9} \text{ C/m}}{2\pi(0.010 \text{ m})(4.5)(10^{-9}/36\pi \text{ F/m})} \doteq 69.1 \text{ kV/m}.$$

Similarly for $r = r_2$:

$$E_{\max}(r_2) = \frac{172.8 \times 10^{-9} \text{ C/m}}{2\pi(0.020 \text{ m})(3.0)(10^{-9}/36\pi \text{ F/m})} \doteq 51.8 \text{ kV/m}.$$

4. First calculate the initial potentials on the sphere surfaces. For the smaller sphere:

$$V_1 = \frac{1}{4\pi\epsilon}\frac{q}{r} = \frac{(9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)(3 \times 10^{-8} \text{ C})}{(0.06 \text{ m})} = 4.5 \times 10^3 \text{ V}.$$

For the larger sphere:

$$V_2 = \frac{(9 \times 10^9\ \mathrm{N \cdot m^2/C^2})(3 \times 10^{-8}\ \mathrm{C})}{(0.12\ \mathrm{m})} = 2.25 \times 10^3\ \mathrm{V}.$$

Since $V_1 > V_2$, the direction of charge motion is towards the 12 cm radius ball.

When the conducting wire is connected, the spheres seek to assume equal potentials. Therefore,

$$\frac{1}{4\pi\epsilon}\frac{q_1}{r_1} = \frac{1}{4\pi\epsilon}\frac{q_2}{r_2},$$

$$\frac{q_1}{r_1} = \frac{q_2}{r_2},$$

$$q_1 = \frac{q_2 r_1}{r_2}.$$

Since $q_1 + q_2 = 6 \times 10^{-8}$ C,

$$q_2\left(\frac{r_1}{r_2} + 1\right) = 6 \times 10^{-8}\mathrm{C},$$

and the final charge on the larger sphere becomes

$$q_2 = 4 \times 10^{-8}\ \mathrm{C},$$

whereas the final charge on the smaller sphere becomes

$$q_1 = 6 \times 10^{-8}\ \mathrm{C} - 4 \times 10^{-8}\ \mathrm{C} = 2 \times 10^{-8}\ \mathrm{C}.$$

The magnitude of transferred charge is therefore 1×10^{-8} C.

The final potential of each sphere is given by:

$$V_{\mathrm{smaller}} = \frac{1}{4\pi\epsilon}\frac{q}{r} = \frac{(9 \times 10^9\ \mathrm{N \cdot m^2/C^2})(2 \times 10^{-8}\ \mathrm{C})}{(0.06\ \mathrm{m})} = 3 \times 10^3\ \mathrm{V}$$

and

$$V_{\mathrm{larger}} = \frac{(9 \times 10^9\ \mathrm{N \cdot m^2/C^2})(4 \times 10^{-8}\ \mathrm{C})}{(0.12\ \mathrm{m})} = 3 \times 10^3\ \mathrm{V}.$$

5. In the frame of the lab, the total energy is $E + mc^2$, where E is the energy of the incident proton and mc^2 is that for the proton at rest. Let the incident proton have momentum p.

   The invariant mass, M, of this lab system is then given by

$$M_{\mathrm{lab}}^2 = \frac{E_{\mathrm{total}}^2}{c^4} - \frac{p^2}{c^2}.$$

In the centre of mass (CM) frame, the 4 reaction products are at *rest* for this problem, since we are only calculating the minimum, or threshold energy, for this reaction. The total energy in this system is therefore $4mc^2$ and the invariant mass is

$$M_{\rm CM}^2 = \frac{\left(4mc^2\right)^2}{c^4}.$$

Since $M_{\rm lab}^2 = M_{\rm CM}^2$ and $E^2 = p^2c^2 + m^2c^4$, we get

$$\frac{(E+mc^2)^2}{c^2} - p^2 = \frac{(4mc^2)^2}{c^2},$$

$$(E+mc^2)^2 - E^2 + m^2c^4 = (4mc^2)^2,$$

$$E^2 + m^2c^4 + 2Emc^2 - E^2 + m^2c^4 = 16m^2c^4,$$

$$2Emc^2 = 14m^2c^4,$$

and

$$E = 7mc^2.$$

6. First calculate the number of electrons per unit volume. Take Avogadro's number to be $N = 6.02\times10^{26}$ atoms/kmol, the density of copper to be 8.96×10^3 kg/m^3, and the atomic weight of copper to be 63.54 g/mol. Assuming one conduction electron per atom, the number of electrons per unit volume is

$$N_e = \left(6.02\times10^{26}\ \frac{\text{atoms}}{\text{kmol}}\right)\cdot\left(\frac{1\ \text{kmol}}{63.54\ \text{kg}}\right)\cdot\left(8.96\times10^3\ \frac{\text{kg}}{\text{m}^3}\right)\cdot\left(1\ \frac{\text{electron}}{\text{atom}}\right)$$

$$N_e = 8.49\times10^{28}\ \text{electrons/m}^3.$$

The rhombic cross sectional area is $\frac{1}{2}(2\ \text{mm})(4\ \text{mm}) = 4\times10^{-6}\ \text{m}^2$.
The number of electrons in a 100 mm length is

$$N = (4\times10^{-6}\ \text{m}^2)(0.100\ \text{m})(8.49\times10^{28}\ \text{electrons/m}^3) = 3.40\times10^{22}\ \text{electrons}.$$

A 10 A current requires that

$$(10\ \text{C/s})(1.6\times10^{-19}\ \text{C/electron})^{-1} = 6.25\times10^{19}\ \text{electrons/s}$$

pass a fixed point.
The percentage leaving the 100 mm length per second is

$$\frac{6.25\times10^{19}}{3.40\times10^{22}}\cdot(100) = 0.184\%.$$