1995-1996 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 2: Mechanics

1. (a) Assume that the bungee cord obeys Hooke's law,

$$\frac{F}{A} = Y\frac{\Delta l}{l}$$

$$F = (\frac{YA}{l})\Delta l = k\Delta l,$$

where A is the cross section of the elastic cord, Y is Young's Modulus, l is the length of the cord, Δl is the change in the length of the cord, and k is the elastic constant of the cord. Since Y and A are constants, kl is constant, *i.e.* $kl = k'l'$.

At $h = 0$ the velocity of the jumper has to be zero, i.e. $v = 0$, where h is the vertical distance of the jumper from the ground. Thus, using energy conservation

$$\begin{aligned} mgh_\circ &= \frac{1}{2}k'(h_\circ - l')^2 \\ &= \frac{1}{2}k(l/l')(h_\circ - l')^2 \end{aligned}$$

where $h_\circ$ is the initial vertical distance. This can be rearranged as a quadratic equation in l'.

$$l'^2 - 2h_\circ\left(1 + \frac{mg}{kl}\right)l' + h_\circ^2 = 0.$$

Solving for l' then,

$$l' = h_\circ\left(\left(1 + \frac{mg}{kl}\right) \pm \sqrt{\left(1 + \frac{mg}{kl}\right)^2 - 1}\right)$$

Using $g = 9.8$ m/s^2, $m = 55$ kg, $k = 77$ kg/s^2, $h_\circ = 60$ m, and $l = 30$ m, one gets the only physical solution, $l' = 21.4$ m. (Note: In order for l' to be physical, it must be $0 < l' \leq 30$ m.)

(b) The force exerted on the flagpole at $h = 0$ is:

$$\begin{aligned} F &= mg + k'(h_\circ - l') \\ &= mg + k(l/l')(h_\circ - l'). \end{aligned}$$

The torque exerted on the flagpole around the pivot point P is $\tau = FL'\cos\phi$. But from Figure 1, it is obvious that $L'\cos\phi = L\cos\theta$. Thus

$$\tau = FL\cos\theta.$$

The maximum torque the bolt has to withstand is

$$\tau = bT,$$

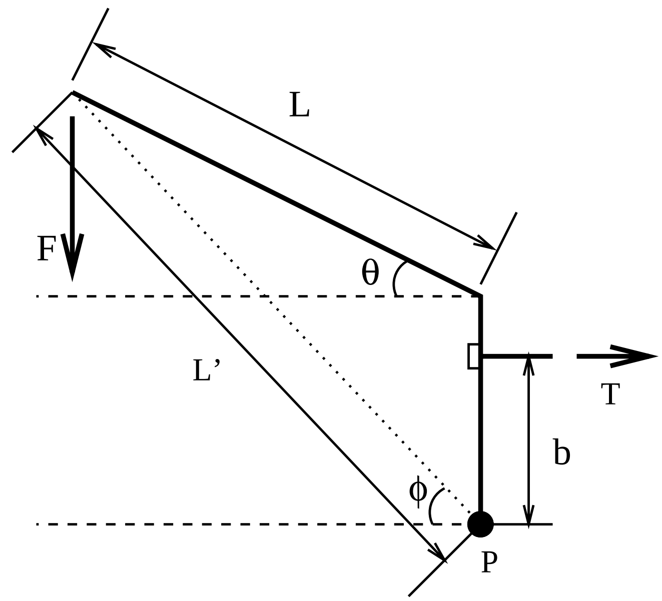

Figure 1: Figure for problem 1

where b is the distance between P and the bolt and T is the tensile strength times the cross section of the bolt.

Thus equating the above two equations,

$$\begin{aligned} T &= FL\cos\theta/b \\ &= (55\times 9.8 + 77\times(30/21.4)\times(60-21.4))\times L/0.5\ \mathrm{m\cdot kg/s^2} \\ &= 1.16\times 10^4\times L\ \mathrm{m\cdot kg/s}. \end{aligned}$$

2. (a) When rolling without slipping, the bottom point P of the yoyo has zero instantaneous velocity. Rotation around a fixed axis passing through$P$ is determined by the torque about that axis, but only F has non vanishing torque. Thus the rotation is clockwise (Figure 2(a)) and the centre of mass moves to the direction of the pull.

   (b) In order for the yoyo to stand still, the net torque of the system should be zero about the contact point P (Figure 2 (b)). Thus $mgb\sin\theta = F(b-a)$. Then

$$F = \frac{b}{(b-a)} mg\sin\theta.$$

When the yoyo is about to slip, the sum of the torques around the pivot point P', τ_{tot} is

$$\begin{aligned} \tau_{tot} &= 0 \\ &= \tau_1 + \tau_2 \\ &= mga\sin\theta - \mu mg(b-a)\cos\theta, \end{aligned}$$

where τ_1 is the torque due to the centre of mass of the yoyo being pulled down by the gravitational force and τ_2 is the torque due to the friction between the yoyo and the inclined surface.

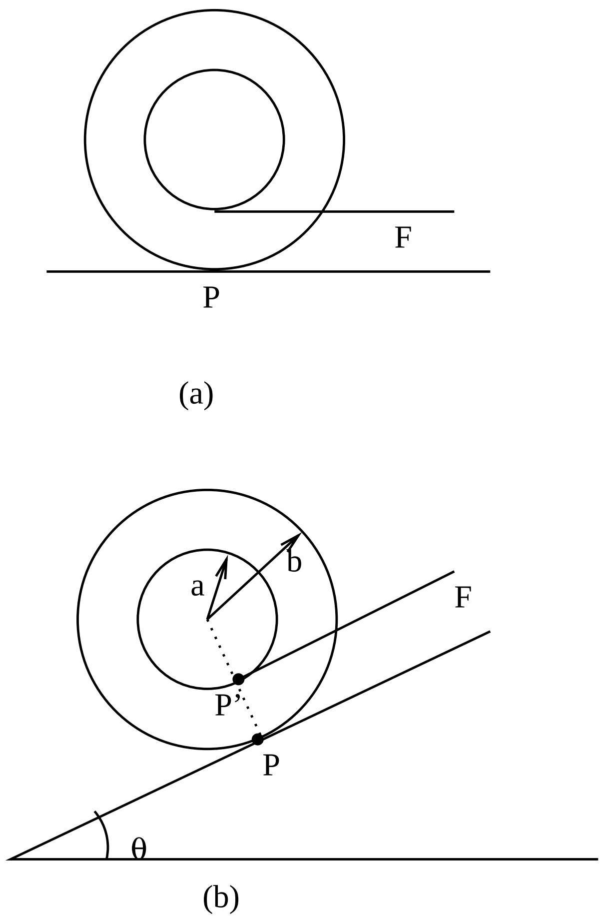

Figure 2: Figure for problem 2

Thus

$$\mu = \frac{a}{(b-a)}\frac{\sin\theta}{\cos\theta} = \frac{a}{(b-a)}\tan\theta.$$

(Note: The moment of inertia of a cylinder whose rotational axis is perpendicular to the flat surface of the cylinder and lies a distance a away from the centre is $I_R = I_M + Ma^2$. There was a typo in the problem set. Not that it would have made any difference in solving this problem...)

3. (a) From energy conservation the energy at $h = (R+a)\cos\theta$ is

$$\begin{aligned} mg(R+a)(1-\cos\theta) &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega_{sph}^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{5}mv^2 \\ &= \frac{7}{10}mv^2 \end{aligned}$$

where we used $I_{sph} = (2/5)ma^2$ and $v = a\omega_{sph}$.

(Note: The trajectory of the centre of mass of the marble rolled to an angle θ from the top of the bowl is $(a+R)\theta$ (Figure 3 (a)). And $R\theta = a\phi$. Combining these equations, one gets

$$\begin{aligned} (a+R)\theta &= a\theta + R(a/R)\phi \\ &= a(\theta+\phi) \\ &= a\theta_{sph}, \end{aligned}$$

which means $v = (a+R)\omega_\theta = a(\omega_\theta + \omega_\phi) = a\omega_{sph}$.)

For a marble about to take off from the surface of the bowl the force perpendicular to the surface of the bowl has the same magnitude as the centripetal force on the marble. Thus,

$$mg\cos\theta = mv^2/(a+R).$$

From the above two equations

$$(1-\cos\theta) = (7/10)\cos\theta$$

Solving for $\cos\theta$ gives $\cos\theta = 10/17$. Thus the marble leaves the surface of the bowl at the height of $h_c = R\cos\theta = (10/17)R$ (or $h_c = (10/17)(R+a)$, depending on what your definition of h_c is.)

(b) For a "rolling pendulum" (due to the lack of a better name) the potential energy has to be distributed between the rotational motion of the marble and the translational motion, whereas the "simple pendulum" potential energy is converted only to the translational motion. Therefore, the simple pendulum has a higher translational speed than the rolling pendulum at the same angle, *i.e.* it takes more time for the rolling pendulum to complete an oscillation than the the simple pendulum.

Brownie points:

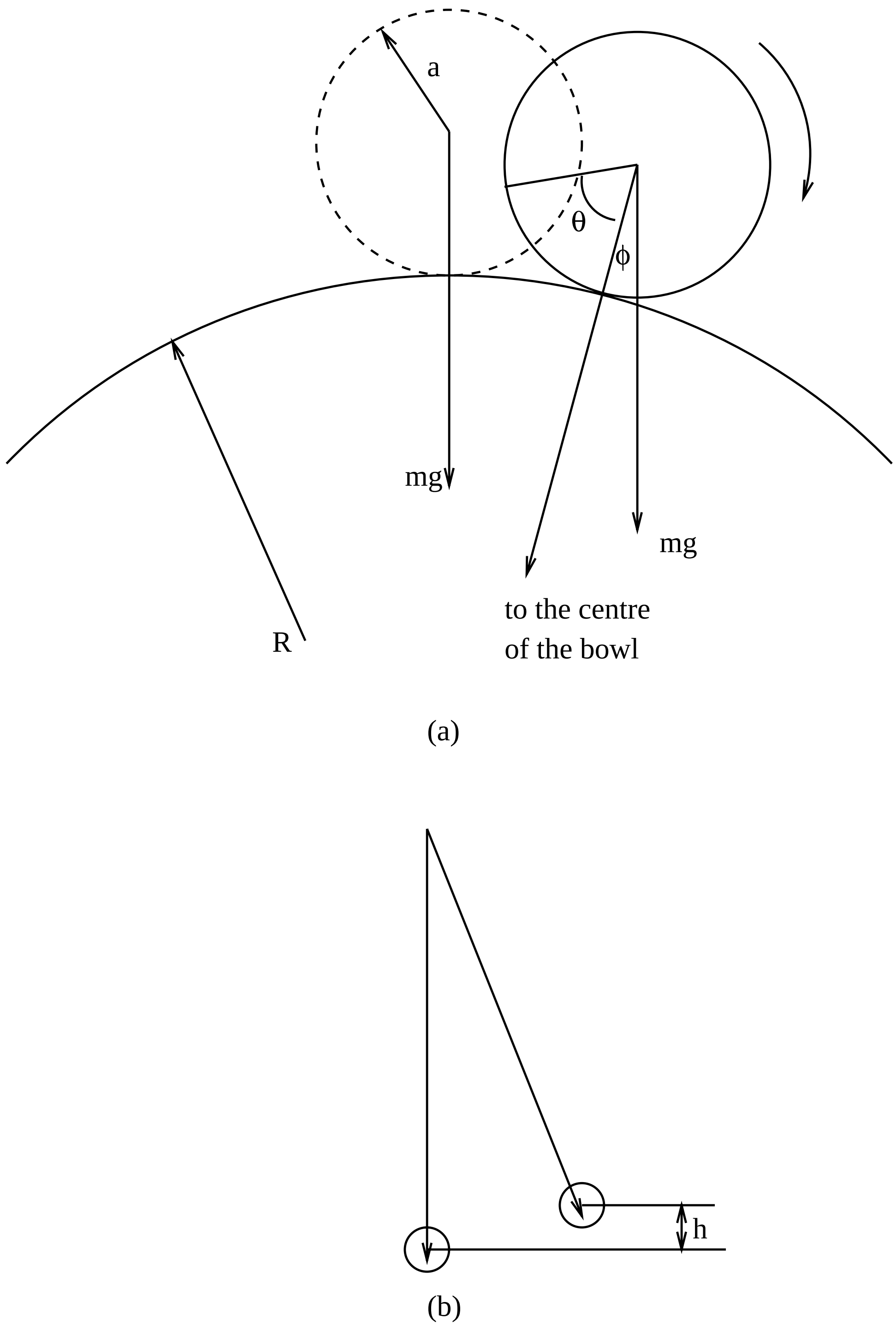

Figure 3: Figure for problem 3

For a simple pendulum, the energy conservation equation at h looks like

$$mg(h_\circ - h) = \frac{1}{2}m{v_s}^2$$

. Then

$$v_s = \sqrt{2g(h_\circ - h)}$$

, where $h_\circ$ is the release height of the pendulum.
For a rolling pendulum,

$$\begin{aligned} mg(h_\circ - h) &= \frac{1}{2}mv_r^2 + \frac{1}{2}I\omega_r^2 \\ &= \frac{7}{10}mv_r^2 \\ v_r &= \sqrt{(10/7)g(h_\circ - h)} \end{aligned}$$

.
Thus $v_r/v_s = \sqrt{5/7}$ for any given height. Which means $T_s/T_r = v_r/v_s = \sqrt{5/7}$

4. (a) The surface of the liquid is perpendicular to the direction of the sum of the forces exerted, *i.e.* centripetal force and gravitational force (Figure 4). Thus

$$\tan\theta = \frac{v^2/r}{g}$$

.
The liquid touches the rim of the the cup when $\tan\theta$ satifies the following equation:

$$\tan\theta = \frac{2}{d}(H - h_\circ)$$

.
Using the above two equations, one can obtain

$$\begin{aligned} v &= \sqrt{\frac{2gr}{d}(H - h_\circ)} \\ &= \sqrt{\frac{2 \times 9.8 \times 30}{0.08}(0.15 - 0.12)} \\ &= 14.8 \text{ m/s}. \end{aligned}$$

(b) When the inside wheels of the car are about to leave the ground (provided that the car does not slip):

$$\frac{W}{2}F_d = hF_l$$

where

W = the track width of the car (or the width of the car for our purpose),
F_d = downward force,
h = the height of the centre of mass of the car,
and
F_l = lateral force.

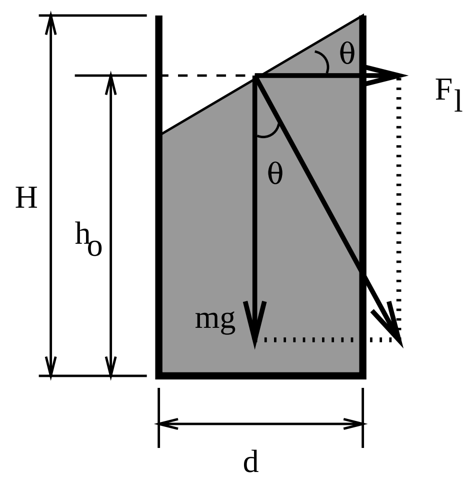

Figure 4: Figure for problem 4

The downward force is the sum of the gravitational force and the force generated by the aerodynamic device, *i.e.* $F_d = mg + cv$. Here c has units of N/(km/hr). Converting it into MKS units, it becomes N/(km/hr) = N/(1000 m/3600 s) = 3.6 N/(m/s). The lateral force is centripetal force, *i.e.* $F_l = mv^2/r$.

Rearrange it as a quadratic equation in v:

$$v^2 - \frac{rW}{2h}\frac{c}{m}v - \frac{rW}{2h}g = 0.$$

Solving for v

$$v = \frac{ac/m + \sqrt{(ac/m)^2 + 4ag}}{2}$$

,

where $a = rW/2h$.

Using $r = 30$ m, $W = 2$ m, $h = 0.4$ m, $m = 1500$ kg, and $c = 150 \times 3.6$ N/(m/s) = 540 N/(m/s), one gets

$$v_r = 43.8 \text{ m/s}.$$

However, if $\mu(mg + cv) = mv^2/r$, the car is about to slide. Rearrange it as a quadratic equation in v:

$$v^2 - \frac{\mu rc}{m}v - \mu rg = 0.$$

Solving for v

$$v = \frac{bc/m + \sqrt{(bc/m)^2 + 4bg}}{2}.$$

Using $\mu = 0.23$ one gets

$$v_s = 9.6 \text{ m/s}.$$

Since $v_r > v_s$, the car will slide before it lifts its inside wheels. Thus the maximum speed you can go through the turn is at 9.6 m/s.

5. (a) For a satellite on a circular orbit of radius r,

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

and

$$v = \sqrt{GM/r}.$$

The velocity of the satellite at the parking orbit (h = 200 km) is then $v_p = 7.772 \times 10^3$ m/s, where $G = 6.673 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$ and $M = 5.974 \times 10^{24}$kg are used.
The angular velocity of the satellite on the geosynchronous orbit is $\omega = 2\pi/(24 \times 3600 \text{ s}) = 7.272 \times 10^{-5} \text{ s}^{-1}$. The radius of the geosynchronous orbit, r_g, can be obtained from the first equation using $v = r\omega$,

$$\begin{aligned} r_g &= \left(\frac{GM}{\omega^2}\right)^{1/3} \\ &= 4.224 \times 10^7 \text{ m}. \end{aligned}$$

Thus the velocity of the satellite on the the geosynchronous orbit is $v_g = r_g\omega = 3.072 \times 10^3$ m/s.
From energy conservation

$$\frac{1}{2}mv_{pt} - \frac{GMm}{r_p} = \frac{1}{2}mv_{gt} - \frac{GMm}{r_g},$$

where

$$\begin{aligned} v_{pt} &= \text{The velocity at the perigee of the transfer orbit,} \\ v_{gt} &= \text{The velocity at the apogee of the transfer orbit,} \\ r_p &= \text{The distance of the perigee from the centre of the earth} \\ &= \text{The radius of the parking orbit,} \\ \text{and} & \\ r_a &= \text{The distance of the apogee from the centre of the earth} \\ &= r_g. \end{aligned}$$

From angular momentum conservation

$$\vec{r}_p \times \vec{v}_{pt} = \vec{r}_a \times \vec{v}_{at}$$

,

and at the perigee and the apogee, $\vec{r}$ and $\vec{v}$ are perpendicular to each other, so

$$r_p v_{pt} = r_a v_{at}$$

or

$$v_{at} = (r_p/r_a)v_{pt}.$$

Thus the energy conservation equation now looks like

$$\frac{1}{2}mv_{pt}^2\left(1-\left(\frac{r_p}{r_a}\right)^2\right) = \frac{GMm}{r_a}\left(1-\left(\frac{r_p}{r_a}\right)\right)$$

giving

$$\begin{aligned} v_{pt} &= \left(\frac{2GM}{r_a(1+r_p/r_a)}\right)^{1/2} \\ &= 1.022\times10^5 \text{ m/s}. \end{aligned}$$

So $v_{pt} - v_p = 2.248\times10^3$ m/s.
The velocity at the apogee of the transfer orbit is then

$$\begin{aligned} v_{at} &= \left(\frac{r_p}{r_a}\right)v_{pt} \\ &= 1.597\times10^3 \text{ m/s}. \end{aligned}$$

Thus the velocity change the satellite has to achieve in order to get into the geosynchronous orbit from the transfer orbit at the perigee of the transfer orbit is $v_g - v_{at} = 1.475\times10^3$ m/s.

(b) The transition takes half the time for the satellite to complete the elliptical orbit. So using the Kepler's law

$$\begin{aligned} T_{1/2} &= \frac{T}{2} \\ &= \frac{1}{2}\left(\frac{4\pi^2((r_p+r_a)/2)^3}{GM}\right)^{1/2} \\ &= 1.900\times10^4 \text{ sec} \\ &\simeq 5 \text{ hrs } 17 \text{ min} \end{aligned}$$

It takes 5 hours and 17 minutes to complete the transition.

6. (a) Of course not, silly. The possible physical effects are
   i. Coriolis force. $2m\vec{\omega}\times\vec{v}$
   ii. Centripetal force. mv^2/r.
   iii. Decrease of gravitational force due to the increase in r.
   iv. Relativistic effect (time dilation). $t = \tau/\sqrt{1-v^2/c^2}$, etc.

   International flight normally flies at an altitude of about 10 km. Thus the change in the graviational acceleration at an altitude h is $\Delta g = g(1-(R/(R+h))^2) = 3.1\times10^{-3}\times g$. The change in T is $\Delta T \simeq \sqrt{l/(g-\Delta g)} - \sqrt{l/g} \simeq 1.6\times10^{-4}\sqrt{l/g}$, where l is the length of the pendulum. But it affects the same way regardless of the direction of the flight.

   The speed of the east bound plane that Chris is on from the fixed reference frame is $v = \pi/12\times3600\times6.4\times10^6 \simeq 460$ m/s. Relativistic effects are then on the order of $\Delta T \sim T(1/\sqrt{1-(v/c)^2}-1) \sim 10^{-11}T \sim 10^{-7}$ sec.

(b) If we use the inertial system with the origin of the coordinates sitting at the centre of the earth as the reference system, the Coriolis force vanishes in this system. The effective gravitational acceleration at the earth's surface for an object with angular velocity $\vec{\omega}$ against an inertial system is

$$\vec{g}_{eff} = -\frac{GM}{R^2}\hat{R} - \vec{\omega} \times (\vec{\omega} \times \vec{R}).$$

Since Pat and Chris are moving along the equator, their angular momentum vectors $\vec{\omega}$ are perpendicular to $\vec{R}$. Thus $|\vec{\omega} \times (\vec{\omega} \times \vec{R})| = |\omega^2 R|$. Then the effective gravitational acceleration looks like

$$g_{eff} = \left(\frac{GM}{R^2} - \omega^2 R\right)$$

,

and if we replace $\frac{GM}{R^2}$ with $g_\circ$

$$g_{eff} = g_\circ - \omega^2 R.$$

The period of a simple pendulum with the length of the pendulum L is

$$T_\circ = 2\pi\sqrt{\frac{L}{g}}.$$

Since Pat is moving westward, the angular velocity is zero. Thus

$$\begin{aligned} T_{Pat} &= 2\pi\sqrt{\frac{L}{g_\circ}} \\ &= 2\pi\sqrt{\frac{L}{g + \omega^2 R}} \\ &= 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{1}{1 + \omega^2 R/g}} \\ &= T_\circ\sqrt{\frac{1}{1 + \omega^2 R/g}}. \end{aligned}$$

Whereas, Chris is moving eastward, so the angular velocity of Chris is 2ω. Thus

$$\begin{aligned} T_{Chris} &= 2\pi\sqrt{\frac{L}{g_\circ - 4\omega^2 R}} \\ &= 2\pi\sqrt{\frac{L}{g - 3\omega^2 R}} \\ &= 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{1}{1 - 3\omega^2 R/g}} \\ &= T_\circ\sqrt{\frac{1}{1 - 3\omega^2 R/g}}. \end{aligned}$$

$$T_{Chris} - T_{Pat} = T_\circ \left(\sqrt{\frac{1}{1 - 3\omega^2 R/g}} - \sqrt{\frac{1}{1 + \omega^2 R/g}} \right) = T_\circ \left(\frac{2\omega^2 R}{g} \right)$$

Using $g = g_\circ - \omega^2 = 9.8\ \mathrm{m/s^2}$, $R = 6.4 \times 10^3$ km, $T_\circ = 12$ hrs, and $\omega = 2\pi/T_\circ$ one gets $T_{Chris} - T_{Pat} \simeq 298$ seconds. Since $T_{Chris} - T_{Pat} > 0$, Pat's clock goes faster than Chris' clock. (Pat's clock would be 75 seconds faster and Chris' clock would be 224 seconds slower than the local clock at Y.)