PROBLEM SET #1 – GENERAL

1. **Laser–Tag**

This is a standard refraction-type question. Since Daniel is in a medium with a different (i.e. higher) refractive index than air, the laser beam will 'bend' around the corner.

Let x be the minimum distance Daniel must move to target Yiorgos. The only other variable to worry about is at what angle Daniel will aim (i.e. where will the beam hit the window). For minimum x, Daniel will aim to hit the very edge of the window.

From Snell's law

$$n_i \sin\theta_i = n_r \sin\theta_r$$

We know that after the Gas-Bomb is used, $n_i = 2n_r$

$$\sin\theta_r = \frac{4}{\sqrt{4^2 + 1^2}} = \frac{4}{\sqrt{17}}$$

Also

$$\sin\theta_i = \frac{x}{\sqrt{x^2 + 2^2}} = \frac{x}{\sqrt{x^2 + 4}}$$

$$\therefore \quad 2n_r \frac{x}{\sqrt{x^2 + 4}} = n_r \frac{4}{\sqrt{17}}$$

$$17x^2 = 4\,(x^2 + 4)$$

$$13x^2 = 16$$

$$x = 1.1 \text{ m} \qquad \text{(Negative root extraneous).}$$

Daniel must move at least 1.1 m to the right to target Yiorgos.

(Common error: In Snell's law, the incident and refracted angle are measured between the ray and the *normal* to the interface, not the ray and the interface.)

b) Nothing went wrong, if Daniel was looking along the beam with the laser-gun held high to aim. Daniel sees his laser beam due to light scattering back to him from dust or other particles in the air. This scattered light, along nearly the same path, but backwards, will also obey Snell's law, and as such the light that reaches his eye will follow a similar path as the laser beam. Therefore, to Daniel it looks straight.

(Many of you came up with ingenious ways that would result in the laser beam not bending, and, if it was physically possible, got part-marks for your efforts. As you can see from the solution, none of these were necessary.)

2. Apple shoot

The total momentum of the system remains constant in a collision. Thus,

$$mv_0 = mv + MV \tag{2.1}$$

Here, v is the velocity of the bullet and V that of the apple after the collison. Both objects fall a distance h to the ground; their time of flight is $t = \sqrt{2h/g} = 0.78$ s. During time t, the apple covers a horizontal distance of s = 10 m, so its horizontal velocity is V = s/t = 12.8 m/s. Thus, the conservation of momentum equation yields v = 104 m/s for the horizontal velocity of the bullet immediately after the collision. Since the bullet's time of flight is also t, it travels a distance vt = 105 m horizontally from the column.

Initially, the kinetic energy of the system is $E_i = mv_0^2/2 = 1250$ J. After the collision, the kinetic energy is $E_f = mv^2/2 + MV^2/2 = 93.2$ J. Thus,

$$1250 - 93.2 = 1156.8 \text{ J} \tag{2.2}$$

is lost as heat (92.5% of the original energy). Note that the collision is *not* completely inelastic. In a completely elastic collision, the kinetic energy is conserved. In a completely inelastic collision, the bullet would remain inside the ball.

3. Stealth

In the world of science-fiction, a 'cloaked' ship has to be perfectly transparent; or, next-best, absorbing all radiation that strikes it but emitting only the same radiation as the molecules it displaced would have. Our current knowledge of physics doesn't allow this, in steady state, but let's assume the second case.

a) We can assume that the sun radiates uniformly over a sphere.

What is the energy the Stealth absorbs per second?

The sun radiates $P = 3.8 \times 10^{26}$ W over it's entire sphere. At $r = 1.5 \times 10^8$ km, this corresponds to a flux of $P/(4\pi r^2) = 1.3 \times 10^3$ W m^{-2}.

∴ Stealth absorbs a total power of 1.3×10^3 W m^{-2} • 20 000 m^2 = 2.6×10^7 W.

Over 10 minutes, Stealth absorbs 2.6×10^7 J s^{-1} • 10 min • 60 s min^{-1} = 1.6×10^{10} J.

b) With no radiation losses: $\Delta t = 1.6 \times 10^{10}$ J / { (2.5×10^8 g)(2 J g^{-1} °C^{-1})} = 32° C,
∴ t_{final} = (18 + 32) = 50°C)

c) Unfortunately, the question does not give enough information to really answer this problem. We need the *total* surface area A_{surf}, not just the cross-sectional area. We can make some guesses at this, though. Since the cross-sectional area = 20 000 m^2, the surface area must be at least 40 000 m^2. There is no upper limit to this number.

(Common error: many people just plugged in the only area value they were given, i.e. 20 000. A ship that has a cross-sectional area of 20 000 cannot have a surface area of 20 000, this answer is impossible.)

At steady-state, *power absorbed = power radiated*. Using Stefan's law: $H = e\sigma A_{surf} T^4$ (T in Kelvin), and since ship is perfectly absorbing, emissivity $e = 1$, ship absorbs 1.3×10^3 W/m^2, $\therefore$ at steady-state $1.3 \times 10^3 \text{ W m}^{-2} \bullet 20\ 000 \text{ m}^2 = \sigma A_{surf} T^4$,

$$T = \left(\frac{2.6\times10^7}{\sigma A_{surf}}\right)^{1/4}$$

Since we do not know A_{surf}, we can determine only a rough range of T.

Maximum for A_{surf} = 40 000 m^2 (the minimum):

$$T = \left(\frac{2.6\times10^7}{\sigma A_{surf}}\right)^{1/4} = 389 \text{ K} = 116° \text{ C}$$

Since A_{surf} can be any larger surface area, T_{min} can approach zero K.

A possible ship shape would be a sphere. The radius of such a sphere is

$\pi r^2 = 20\ 000 \text{ m}^2$
$\Rightarrow r = 80 \text{ m}$
$A_{surf} = 4\pi r^2 = 80\ 000 \text{ m}^2$
giving $T = 275 \text{ K} \rightarrow 2°\text{C}$

No matter what your answer for T, you can determine the corresponding peak wavelength of the radiation distribution. Wien's Displacement law is the quickest way to the answer. It states:

$$\lambda_{max} T = 0.002898 \text{ m}\bullet\text{K}$$

Substitute for T (in Kelvin) and solve for wavelength in meters. 275K gives λ_{max} = 20.5 μm.

4. Wesley's test

A typical first guess at the contents of the black box is a triangular configuration of resistors of resistances R_{ab}, R_{bc} and R_{ca} shown in Fig. 4.1(a). Since Wesley is such a stereotypical thinker, he immediately grasps at this solution. When the resistance across terminals *ab* is measured, the resistance of the parallel circuit formed by resistors R_{ab} and $R_{bc} + R_{ca}$ is

$$R_{ab\text{-}total} = \left(\frac{1}{R_{ab}} + \frac{1}{R_{ca} + R_{bc}}\right)^{-1} \quad [4.1]$$

Similarly, the resistances measured across terminals bc and ca are

$$R_{bc\text{-}total} = \left(\frac{1}{R_{bc}} + \frac{1}{R_{ca} + R_{ab}}\right)^{-1} \quad [4.2]$$

and

$$R_{ca\text{-}total} = \left(\frac{1}{R_{ca}} + \frac{1}{R_{ab} + R_{bc}}\right)^{-1} \quad [4.3]$$

respectively.

Since all three measurements give the same result, $R_{ab\text{-}total} = R_{bc\text{-}total} = R_{ca\text{-}total}$. From the similar equations (4.1), (4.2) and (4.3) it can easily be shown that

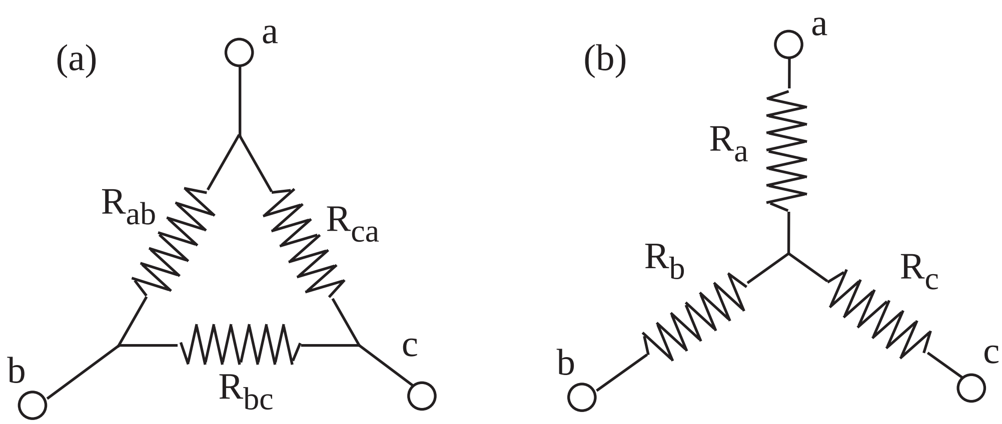
$$R_{ab} = R_{bc} = R_{ca} \equiv R_p \quad [4.4]$$

The total resistance for each pair of terminals, measured in terms of R_p, is

$$R^{total} = \left(\frac{1}{R_p} + \frac{1}{2R_p}\right)^{-1} = \frac{2}{3} R_p \quad [4.5]$$

and so, if R^{total} is to be 8Ω, then $R_p = 12\ \Omega$.

Figure 4.1: Two black box circuits.



Is the answer unique? This is the tough part of the problem (assuming you got past the 'guessing' part). Well, if the configuration above were the only solution the answer would certainly be yes. However, there is another possibility, which is shown in Fig. 4.1(b). The resistors are connected together and to each terminal is associated a single resistor named R_a, R_b, and R_c. In this circuit, a measurement across terminals ab will give the resistance of the series circuit formed by R_a and R_b. As above, it can be shown that equal measurements across ab, bc, and ca must require $R_a = R_b = R_c = R_s$ and for $R^{total} = 8\ \Omega$, $R_s = 4\ \Omega$.

Wesley, that impetuous fellow, immediately thought that the answer was unique. Too bad. Shows that one should put one's mind into gear before thinking. Would you have been able to confidently answer this question?

5. Nonlinear electronics

a) For a linear device, R = V / I. But this is not a linear device, so this relationship is not useful. A more general definition is $R_{effective} = dV/dI$, the local rate of change of voltage with current — the slope of the tangent. When the device *is* linear, V vs. I is a straight line through the origin, so this definition comes back to R=V/I then.

Even if you have no calculus experience, you can find an intuitive solution for V = 10 V and V = 4 V. At both these points, an increase or decrease in voltage corresponds to *no* change in current, *i.e.*, R_{eff} is huge, approaches ∞.

b) We do not know the voltage drop across either component or the current. Start by listing the unknowns.

V_R : Voltage accross resistor
$V_?$: Voltage across unknown device
i_R: Current through resistor
$i_?$: Current through unknown device

Right away, we know $i_R = I_?$ *the current flows through one then the other*
$V_R + V_? = 10$ Volts
$V_R = i_R \bullet R$ *the resistor is a linear device*
$I_? = F(V_?)$ where F() is the function shown in fig. 2, PS#1.

So we have 4 equations with 4 unknowns. Thus we can find a solution(s).

Start with $V_R = i_R\, R$

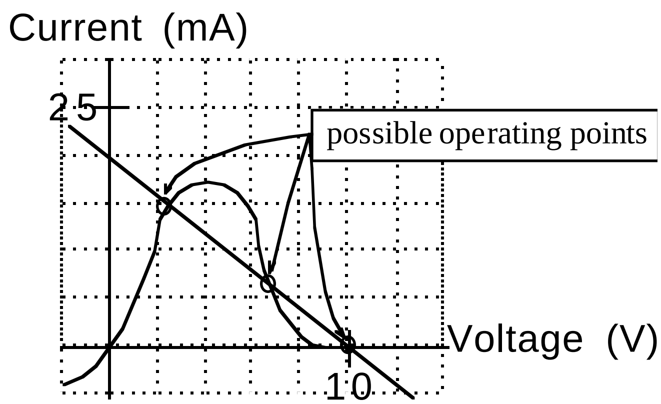
but $V_R = 10 - V_?$

$\therefore\ 10 - V_? = i_R\, R$

but $i_R = I_?$

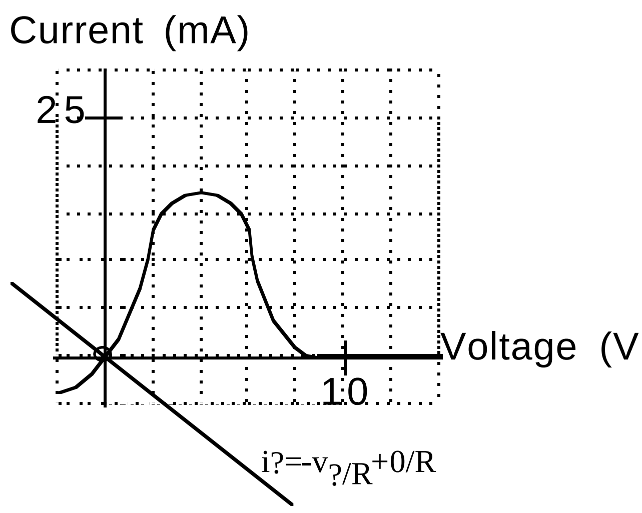
$\therefore\ 10 - V_? = I_?\, R \quad \Rightarrow \quad i_? = -V_?/R + 10/R$

Now we have 2 equations with 2 unknowns, but we must solve it graphically. The intersections between the two lines are possible operating points.

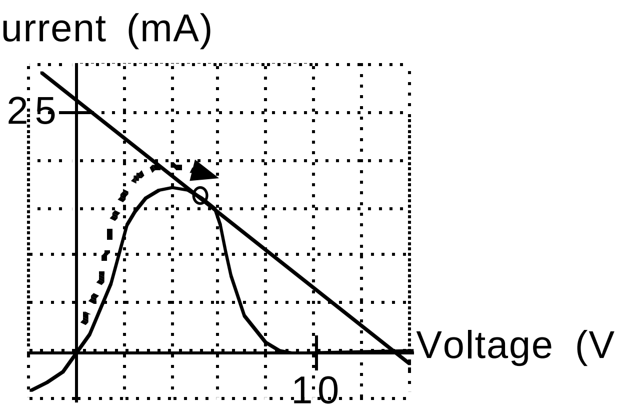


The actual voltage/current across/through the unknown device depends on how Marie-Eve started the circuit (as shall be discussed in the section below).

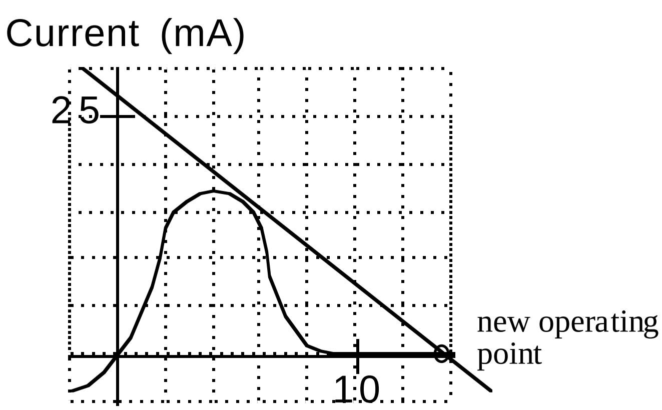
c) At V = 0 V



This line moves as Marie-Eve increases the voltage. Since we assume there is no noise in the system, the operating point moves up the curve:

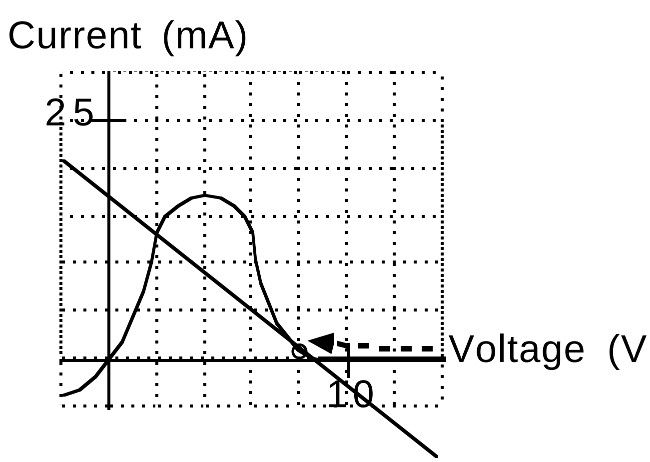


At this point ($V_{var.\ P.s.}$ = 13 V), the operating point jumps to

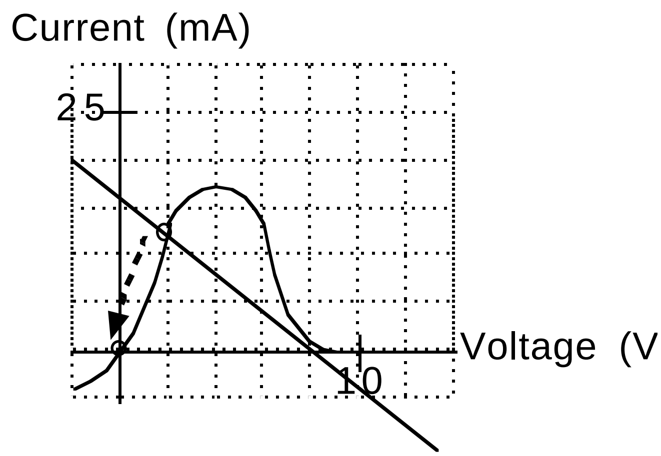


$V_{var\ p.s.}$ continues up to 15 V with $I_?$ = 0A

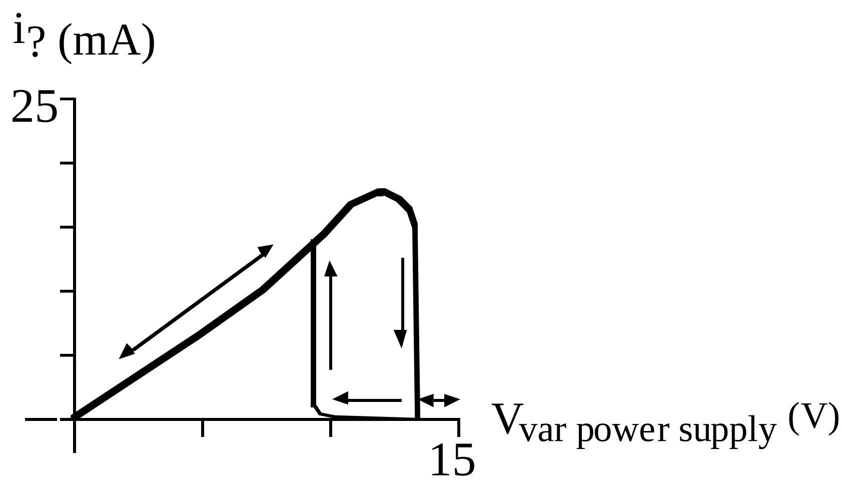
As Marie-Eve decreases from 15 V:



At this point ($V_{var\ p.s.} = 9V$), the operating point jumps to:



Let's put this all together.



This is an excellent case of hysteresis!

6. TP Pull

Comment: see the POPTOR web pages, under Problem Set Extras for a photograph of such a roll-dispenser. Also a friction-lock, using stacks of many plates to increase the drag torque, in the quick-release adjustments of an adjustable office-chair.

a) Knowing the normal force on each of two faces of the roll, we can find the total frictional (drag) force:

$$\left.\begin{array}{l} F_n = 13N \\ \mu_s = 0.27 \end{array}\right\} \Rightarrow F_D = \mu_s F_N = 0.27 \times 13N$$

$$= 3.51N \text{ @ each end}$$

The question is properly a matter of **torques**, since it is *rotational* forces

Torque from drag: $T_D = 2 \cdot F_D \cdot r_o = 2.351$ N $\cdot$ 2 cm
$= 14.0$ N·cm (0.14 N·m)

Torque from pulling: $T_{pull} = F_{pull} \cdot b$ where $F_{pull} \leq 3.5$ N. $T_{pull} = T_{drag}$ while pulling, and T_{drag} is constant in the static case (i.e. testing from rest at different 'b')

So, 14.0 N·cm = $F_{pull} \cdot b$ unless it breaks b; b is minimized where F_{pull} is *maximized;* the last b for not breaking is

$$\boxed{b = \frac{14.0\text{N}\cdot\text{cm}}{3.5\text{N}} = 4 \text{ cm}}$$

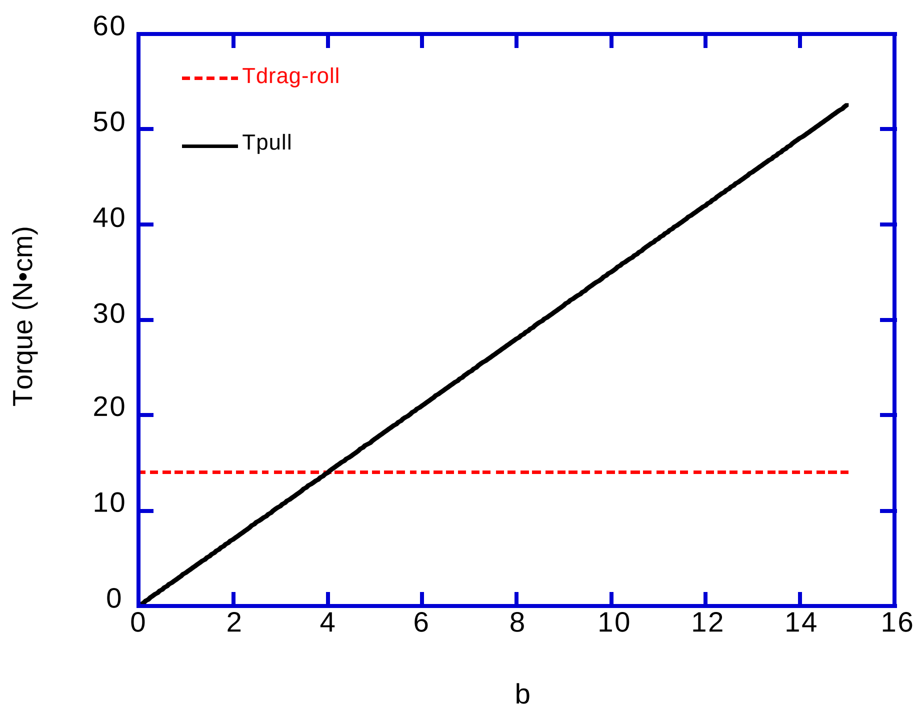


Figure 6.1: Comparing the constant drag-torque from the friction plates on the cardboard roll (red dashed line) to the maximum-deliverable pulling-torque, as the roll size increases (black solid line). The paper is pulled only with as much force as is necessary to overcome the drag-torque, but as the roll becomes smaller, even the maximum tension on the paper (burst-strength) is not enough, and the paper breaks at b = 4 cm, as the lines cross.

b) $T_D = 2\, \mu_S F_N \cdot r_o = 2\,(0.18)\,13\,(2) = 9.4$ N $\cdot$ cm (0.094 N $\cdot$ m)

$$T_{pull} = F_{pull} \cdot b$$

$$b = \frac{9.4 N \bullet cm}{3.5 N} = 2.7 \text{ cm}$$

$T_D = T_{pull} \Rightarrow$ 2.7 cm, i.e. 0.7 cm left on roll.

c) Approximate, rather than finding the spiral length from an integral: consider separate layers, starting from the core (not a perfect spiral anyway ...)

length layer = $2\pi r$ 2 cm $\leq r \leq$ 5.75 cm (d/2)
increment in r is Δr = 1mm/(10 layers) = 0.1 mm layer^{-1}
= 0.01 layer^{-1}
Thus we have (4 cm – 2 cm) * 100 layers cm^{-1} = 200 layers

$$\sum_{n=0}^{200} 2\pi(r_o + n\,\Delta r)$$

where Δr = 0.01 cm. Using $\sum_{n=1}^{j} n = \frac{1}{2} j(j+1)$ to simplify, we get

$$= \sum_{n=0}^{200} 2\pi_o + 2\pi\,\Delta r \sum_{n=0}^{200} n$$

$$= 200(2\pi r_o) + 2\pi\,\Delta r\,(200(201)/2)$$
$$= 200\ 2\pi\ 2 \text{ cm} + 2\pi\ 0.01\ (100)\ (201) \text{ cm} = 3776 \text{ cm}$$
$$= 37.76 \text{ m}$$

Likewise for b = 2.7 cm (2.7 - 2) * 100 = 70 layers

$$\text{length} = 70\ (2\pi r_o) + 2\pi\,\Delta r\ (70(71)/2)$$
$$= 70\ (2\pi)\ 2 \text{ cm} + 2\pi\ 0.01 \bullet 35 \bullet 71 \text{ cm} = 1036 \text{ cm}$$
$$= 10.36 \text{ m}$$

Compare this with the answer you can get by dividing the area of the side of the roll by the thickness of a single layer — giving a length. A = ($\pi b^2 - \pi r_o^2$), Δr = 0.01 cm; for b = 4 cm, A = 37.70 cm^2, so A/Δr = 37.70 m; for b = 2.7 cm, A = 10.34 cm^2, so A/Δr = 10.34 m. This is probably just as good as the approximation above (thanks, Gordon Cook!)

d) In this case the frictional force is the same (area increases but normal **pressure** decreases) BUT the torque changes. The torque changes because the frictional force is applied at a changing **effective** radius.

To best do the problem requires adding up all the torques contributed by all the **similar** regions (r = const).

<u>Plan 'A'</u>: If you can **integrate**, it is not hard:

Consider a roll of diameter b, and core size r_o. With a normal force F_N evenly distributed over this area, the force per unit area is $P_N = F_N/(\pi b^2 - \pi r_o^2)$. Then for an annulus (ring) very thin of width dr set at radius r, the force on the annulus is

$$dF_{drag} = \mu_s P_N dA = \mu_s P_N 2\pi r dr$$

then the torque-contribution due to this annulus is

$$dT_{drag} = r \cdot dF_{drag} = \mu_s P_N 2\pi r^2 dr$$

The whole drag-torque on each side is then the sum of torques over all annuli

$$\frac{1}{2} T_{drag} = \int_{T(r_o)}^{T(b)} d\,T_{drag} = \int_{r_o}^{b} \mu_s P_N$$

$$2\pi^2 \; dr$$

$$= \mu_S P_N 2\pi \int_{r_o}^{b} r^2 \, dr = \mu_S P_N$$

$$2\pi \left[\frac{1}{3} r^3\right]_{r_o}^{b}$$

$$= \mu_s P_N \frac{2\pi}{3} (b^3 - r_o^3)$$

So, since $F_n = P_n \pi(b^2 - r_o^2) = P_n \pi(b - r_o)(b + r_o)$, then we can write $\pi P_n(b - r_o) = \frac{F_n}{(b + r_o)}$ and substitute for P_N

$$T_{drag} = \frac{4}{3} \mu_S F_N \frac{(b^3 - r_o^3)}{b^2 - r_o^3}$$
$$= \frac{4}{3} \mu_S F_N \frac{(b^2 + br_o + r_o^2)}{(b + r_o)}$$

now as you can see, the drag torque is not constant but scales with the outer diameter of the roll, b. To find out if the paper breaks — ever, never, or always — you have to compare this drag torque to the pulling torque you can produce from the maximum (breaking) tension of the paper applied as a force at the radius b of the roll.

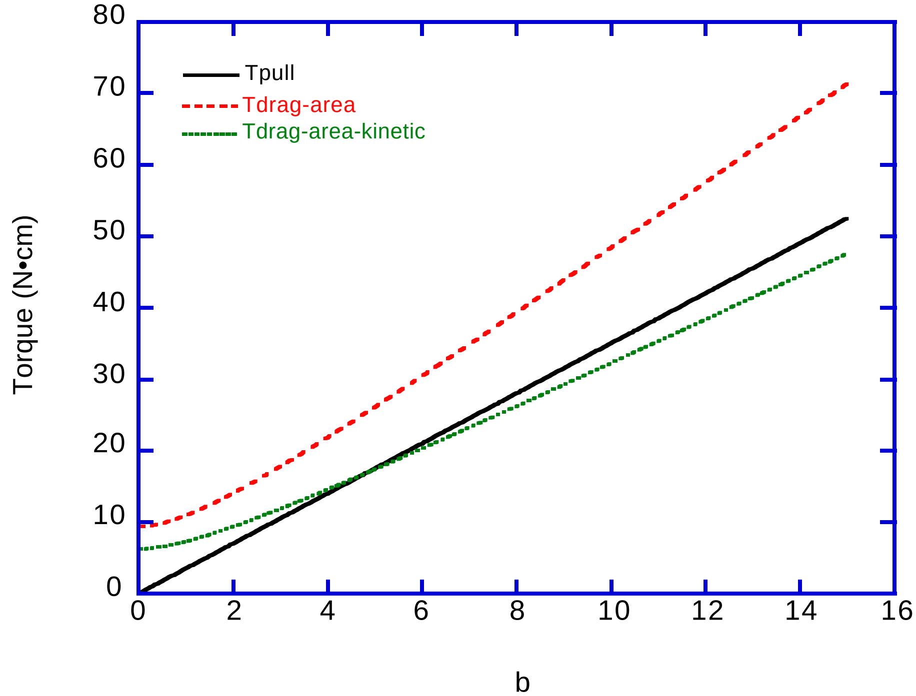


Figure 6.2: Comparing the *changing* drag-torque from the friction plates on the decreasing roll (dashed lines: red–μ_{static}, green–$\mu_{kinetic}$) to the maximum-deliverable pulling-torque (solid black line), as the roll size increases. In the static case, the maximum pulling torque is never enough — the paper always breaks; with a rolling start (the kinetic case), there can be barely a solution for a little while until the paper roll becomes too small.

Plan 'B': If you do not yet integrate, you can also do it by **series summation**:

$$\Delta F_{drag}(r) = \mu_s \ P_N \ \Delta(\text{Area}) = \mu_s P_N 2\pi r \Delta r$$

the torque about the cylinder axis from just this ring is then

$$\Delta T_{drag}(r) = r \cdot \Delta F_{drag} = \mu_s P_N \ 2\pi r^2 \Delta r$$

we can take each ring of width Δr to be an additional layer, then $\Delta r = 0.01$ cm as in part (c) above. Then for a diameter b there are $(b - r_o)/\Delta r$ layers, each contributing a torque from the drag at its particular radius; adding all the torques up on **one** side gives us **half** the **total** drag torque, so

$$\frac{1}{2} T_{drag} = \Delta T_{drag}(r_o) + \Delta T_{drag}(r_o + \Delta r) + \Delta T_{drag}(r_o + 2\Delta r) + \ldots$$

$$= \sum_{n=1}^{(b-r_o)/\Delta r} \Delta T_{drag}(r_o + n\Delta r) ,$$

$$= \sum_{n=1}^{(b-r_o)/\Delta r} \mu_s \; P_N \; 2\pi(r_o + n\Delta r)^2 \; \Delta r,$$

$$= \mu_s \; P_N \; 2\pi\Delta r \sum_{n=1}^{(b-r_o)/\Delta r} (r_o + n\Delta r)^2 ,$$

$$= \mu_s \; P_N \; 2\pi\Delta r \sum_{n=1}^{(b-r_o)/\Delta r} (r_o^2 + 2r_o \Delta r \bullet n + (\Delta r)^2 n^2) ,$$

$$= \mu_S \; P_N \; 2\pi\Delta r \sum_{n=1}^{(b-r_o)/\Delta r} r_o^2 + \sum 2r_o \Delta r \bullet n + \sum(\Delta r)^2 \; n^2$$

$$= \mu_S \; P_N \; 2\pi\Delta r \{ r_o^2 \sum 1 + 2r_o \Delta r \sum n + (\Delta r)^2 \sum n^2 \}$$

Now

$$\sum_{n=1}^{N} 1 = N$$

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

where N is number of layers = 100 (b - r_o), n is layer index-number. So

$$T_{drag} = 4\pi\mu_s P_N\Delta r \{r_o^2 N + 2 r_o \Delta r \frac{N(N+1)}{2} + (\Delta r)^2 \frac{N(N+1)(2N+1)}{6} \}$$

where $N = (b - r_o)/\Delta r$

$\Delta r = 0.01$ cm

b = diameter of roll (cm)

r_o diameter of core (cm)

If now you let the layers become very thin, so that $\Delta r \to 0$ and $N \to \infty$, this becomes

$$T_{drag} = 4\pi\mu_S \; P_N \; r_o^2 (N\Delta r) + r_o(N\Delta r)^2 + \frac{1}{6} \bullet 2(N\Delta r)^3$$

with $N\Delta r = (b - r_o)$,

$$= 4\pi\mu_S \quad P \quad n\, r_o^2 (b - r_o) \quad + r_o(b - r_o)^2 \quad + \frac{1}{3}(b - r_o)^3$$

$$= \frac{4}{3}\ \pi\mu_s \quad P_N(b - r_o)^3 \quad + 3r_o(b - r_o)\ \ 2 + 3r_o^2(b - r_o)$$

$$= \frac{4}{3}\ \pi\mu_s \quad P_n\ (b - r_o)\ (b - r_o)^2 \quad + 3r_o(b - r_o) \quad + 3r_o^2$$

Then with $\pi P_n(b - r_o) \quad = \dfrac{F_n}{(b + r_o)}$ as above,

$$T_{drag} = \frac{4}{3}\mu_s\, F_n \frac{(b - r_o)^2 + 3r_o(b - r_o) + 3r_o^2}{(b + r_o)}$$

$$= \frac{4}{3}\mu_s\, F_n \frac{(b^2 + br_o + r_o^2)}{(b + r_o)}$$

The **same answer** as the integral, because this **is** in fact doing the integral from first principles.