1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 2: Mechanics

1. **Pinball pastimes**

We can consider the ball as a point concentrated in its centre of mass. This assumption is valid since the ball is not rolling and the contact between the ball and the table is frictionless.

a) When the ball reaches the top of the semi–circle, it already has speed v; thus, it also experiences a centripetal force:

$$mv^2/(0.285\text{ m}) = mg\sin 15° \qquad [1.1]$$

or

$$v = \sqrt{(0.285)(9.79)\sin 15°} = 0.850\text{ ms}^{-1} \qquad [1.2]$$

Let x be the distance between the ball with the equilibrium point of the spring. Using the conservation of energy (with the equilibrium of the spring as the reference point, x = 0):

$$\frac{1}{2}kx^2 - mgx\sin 15° = \frac{1}{2}mv^2 + mg(1.170)\sin 15° \qquad [1.3]$$

or

$$100x^2 - 0.253x - 0.333 = 0 \qquad [1.4]$$

Solving this equation, one obtains x = 0.0590 m or 5.90 cm. (The other root is not used for obvious reason).

b) After it reaches the top, the ball continues to pass A since it has momentum. Due to the inertia, the ball tends to go straight but the curve bends the path. One can also get the same answer using the symmetry argument.

c) Since the ball barely touches F, its speed v = 0 and its position from the equilibrium is s = 0.870 m.

$$\frac{1}{2}kx^2 - mgx\sin 15° = mg(0.870)\sin 15° \qquad [1.5]$$

or

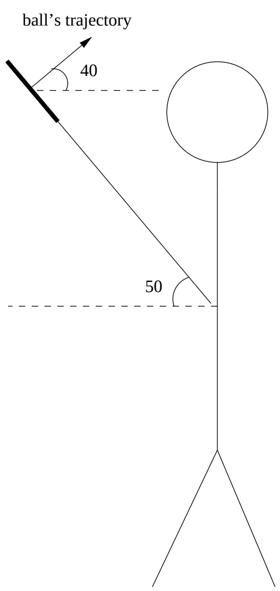
$$100x^2 - 0.253x - 0.220 = 0 \qquad [1.6]$$

Solving this equation, one obtains x = 0.0482 m or 4.82 cm.

2. **Net Result**

To maximize v_p, want to maximize the velocity component of the ball parallel to the ground. Therefore, one should hit the ball at the top of the stroke, i.e., when the racquet head is perpendicular to the ground.

a) The "power serve" hitting the centre service line with the maximum v_p (and still being a legal serve) would land at the back of the service area.



Horizontal distance travelled by ball:

$$x^2 = a^2 + b^2 = (4.12)^2 + (18.25)^2 \quad \Rightarrow x = 18.7 \text{ m}$$

vertical distance travelled by ball:

Martina: 1.57 m + 0.71 m + 0.46 m = 2.74 m
Amanda: 1.35 m + 0.56 m + 0.46 m = 2.37 m

time ball is in the air: $d = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2d}{g}}$

Martina: $t_M = 0.75$ s
Amanda: $t_A = 0.70$ s

$\therefore$ maximum v_p for:

Martina: $v = x/t_M$ = 18.7m / 0.75s = 25m/s or 90 Km/h
Amanda: $v = x/t_A$ = 18.7m / 0.70s = 26.9m/s or 97 Km/h

We, of course, have to make sure that balls with these velocities will clear the net! (see part b)

b) (Note that a more precise solution to this problem can be obtained by the methods of calculus. We sketch here an approximate solution.)

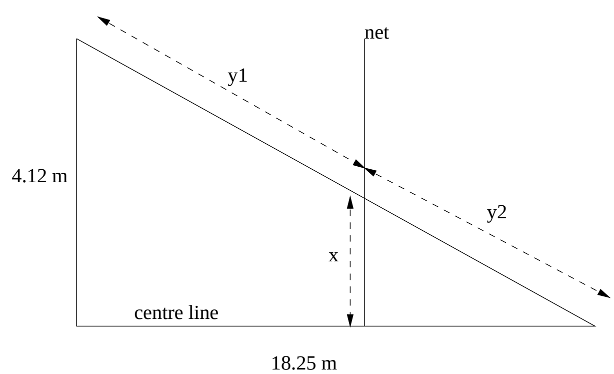
Using the same trajectory for the ball as in part (a), we need to find the distance from the server to the net.

By similar triangles:

$$\frac{4.12}{18.25} = \frac{x}{6.40} \quad \Rightarrow x = 1.44\text{m}$$

$$\therefore\ y_2^2 = (6.40)^2 + (1.44)^2 \quad \Rightarrow y_2 = 6.56\text{m}$$

$$\therefore\ y_1 = 18.7 \text{ m} - 6.56 \text{ m} = 12.14 \text{ m}$$



To find the minimum v_p for each player, time for ball to fall from point of stroke to height of the net:

$$t_{m'} = \sqrt{\frac{2(d_m - h_{net})}{g}} = \sqrt{\frac{2\ (274 - 0.91)}{g}} = 0.61 \text{ s}$$

$$t_{a'} = \sqrt{\frac{2(d_a - h_{net})}{g}} = \sqrt{\frac{2 \bullet (2.73 - 0.91)}{g}} = 0.55 \text{ s}$$

v_{min}:

Martina: 12.14m / 0.61 s = 19.9 m/s or 71.5 km/h
Amanda: 12.14m / 0.55 s = 22.1 m/s or 79.5 km/h

So, the serves in part (A) are fast enough to clear the net.

c) maximum speed is for longest-possible horizontal trajectory, as

$$v_p = \frac{\text{horizontal distance}}{\text{time for ball to fall to ground}}$$

so we want the ball to land in for rear corner of service area:

Horizontal distance travelled by ball = $((8.23)^2 + (18.25)^2)^{1/2}$ = 20.0m

Maximum speed:

Martina: 20.0m / 0.75s = 26.7 m/s, or 96.0 km/h
Amanda: 20.0m / 0.70s = 28.6 m/s, or 103 km/h

d) We need to know:

1) how high above the ground the ball is when Amanda hits it
2) how much vertical distance between the release point of the ball and the top of the net — gives time available to clear net
3) the horizontal velocity component of the ball needed to reach the net at the closest distance to the point of service such that the ball will land in the service area. (See part b)

1) ball height at release point: 1.35 + (0.56 + 0.46) sin 50° = 2.13 m
2) d = (2.13 m – h_{net}) = 1.22 m
∴ Time available to clear net = $(2d/g)^{1/2}$ = 0.50 s

3) From part (b): distance = 12.14 m
$v_{horizontal}$ = 12.14m / 0.50s = 24.3 m/s
v_{total} (tangential to the arc of the racquet): $v_{horizontal} = v_{total} \cos 40°$
v_{total} = 31.7 m

Racquet head speed × 1.8 = ball speed
$v_{racquet}$ = 17.6 m/s

angular velocity of arm = ω = v / r = 17.6 / (0.56+0.46) = 17.3 rad/s

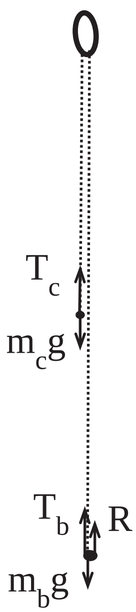
vertical velocity of ball: $v_y = v_{total} \sin 40^\circ = 20.37$ m/s
So, the maximum height above the ball's release point is

$$2gy = v_y^2 \quad \Rightarrow y = 21.1 \text{ m}$$

Max. height of ball = 1.22 m + 21.1 m = 22.4 m

3. **Mind for rocks, or rocks for brains?**

The system has been simplified since ropes don't stretch and so no slack is allowed on the line. This of course is not physically reasonable (climber would almost always be supported by belayer and so could never actually 'climb'!) but is a good first approximation.



a) Let 'c' indicate climber
Let 'b' indicate belayer
Let R be force by floor on belayer

Since belayer does not want to accelerate off the ground:

$$\Sigma F_b = 0$$

Since there is no friction $T_c = T_b$

$$\therefore \; T_b + R = m_b g$$
$$T_c + R = m_b g$$

Climber is also not accelerating.

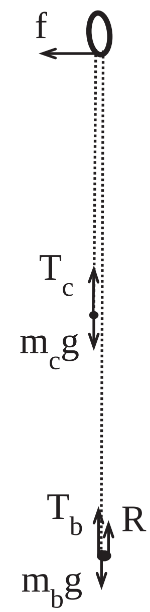
$$\therefore \; T_c = m_b g$$

Maximum m_c for a given m_b occurs for R = 0, i.e., belayer barely touching floor.
$\therefore$ as expected, max. mass of climber is your mass, as belayer
(Of course in real life, you would not want to be right on the edge of stability, but within the assumptions given the correct answer is 'maximum mass = your mass', not '< your mass' which some people stated. This is a minor point.)

b) Same pix as before except for friction on metal loop. Friction force opposes motion.
Again we are trying to find a max. m_c for a given m_b. Start by setting R = 0.

Note that forces on rope must balance
$\therefore \; m_c g = f + m_b g$
(direction of f choosen since we are opposing motion of climber dropping, not belayer dropping).

Frictional force is standard static friction, i.e.



$$f \le f_{max}$$
$$\le \mu_s (m_cg + m_bg)$$

For maximum m_c, set $f = \mu_s (m_cg + m_bg)$

$$\therefore\ m_cg = \mu_s (m_cg + m_bg) + m_bg$$
$$m_cg (1 - \mu_s) = m_bg (1 + \mu_s)$$
$$\therefore\ m_c = m_b (1+ \mu_s) / (1- \mu_s)$$

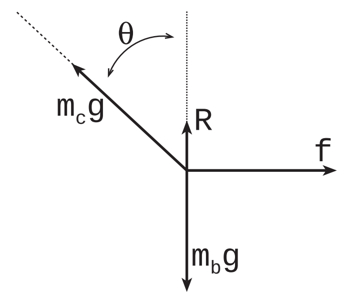
but question states that $m_c = 2m_b$ for maximum m_c

$$\therefore\ 2 = (1+ \mu_s) / (1- \mu_s)$$

$\Rightarrow$ $\mu_s = 1/3$ <— cofficient of friction between loop and rope.

c) Since there is no friction in the rope, $T_c = T_b$. Since climber does not fall $T_c = m_cg$, $\therefore$ the forces on the belayer look like:

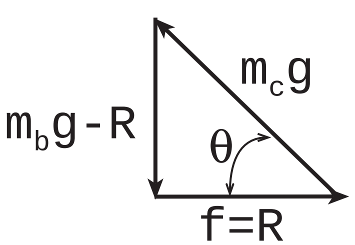
$$f \le \mu_s R$$
$$\le 1.0\ R$$



For maximum m_c, f = 1.0 R

Since there is no acceleration, Σ F = 0. Standard way would be to break this up into horizontal and vertical forces but more elegant way is to consider vector sum of forces: (since Σ F = 0, can draw triangle)

And write: $(m_bg - R)^2 + (1.0\ R)^2 = (m_cg)^2$



but we know: $(a + b)^2 \ge a^2 + b^2$

$$\therefore\ (m_bg - R)^2 + R^2 \le (m_bg - R + R)^2 \le (m_bg)^2$$
$$\therefore\ (m_bg)^2 \ge (m_cg)^2$$

$\therefore$ max. m_c is m_b

$\therefore$ friction of floor does not help!

4. **Dense and denser**

a) Effective buoyancy force = g ($m_{displaced\ air}$ + $m_{displaced\ fluid}$), but the masss of the displaced air is very small, so can be ignored.

Net force = 0, $\therefore$ $g\ V_c /2\rho_{fluid} = g\ V_c\ \rho_c$

$$\therefore\ \rho_{fluid} = 2\ \rho_c$$
$$= 1.26\ g/cm^3$$

b) This problem explores what happens when you take a model and go to a silly extreme. You are required to make one simplification. The question states only the pressure for one point in the chamber. Of course, as the gas becomes more dense the pressure increase due to gravitational forces (ρgh) becomes more important. It is this very pressure gradient that causes the cylinder to rise. For this part of the question, since the pressure is increased only a small amount one can assume that the gas it displaces would have a minor pressure gradient across the length of the cylinder. Thus one can assume a constant density and that Diane's measurement was taken at a height similar to the height of the cylinder.

Let x be the distance that the cylinder is submerged.

Net force = 0

$$\therefore\ g\, V_c\, \rho_c = g\,(x\, V_c\, \rho_{fluid} + (1 - x) V_c\, \rho_{air})$$

You cannot ignore the displaced air.

$$\rho_c = (l - x)\, \rho_{air} + x\, \rho_{fluid}$$

ρ_{air} is calculated from PV=nRT .

$$\therefore\ x = 4.4\ \text{cm}$$

Cylinder moves up 6 mm.

Note: cylinder floats **higher** in fluid. Diane should have **reduced** air pressure, not increased it.

c) Cylinder + fluid 'float' to top of chamber.

5. 'I crush your moon – crush crush!'

a) Knowing the period of the moon's orbit, we can figure out its distance from the earth, because the period depends on the orbit radius *r* and orbital speed v:

$$T_{orbit} = \frac{distance}{speed} = \frac{2\pi r_{M,m}}{\mathrm{v}} \qquad r_{M,m} = \text{earth-moon separation, v = speed of moon} \qquad [5.1]$$

The orbital speed and radius are also related by the centripetal-force formula:

$$F_{Mm} = \frac{m\mathrm{v}^2}{r_{M,m}} \qquad m = \text{mass of moon} \qquad [5.2]$$

Where the centripetal force is gravitational attraction:

$$F_{Mm} = \frac{GMm}{r_{M,m}^2} \qquad M = \text{mass of earth},\ m = \text{mass of moon},\ G = \text{gravitational const.} \qquad [5.3]$$

Equations [5.2] and [5.3] both give the force, so with right-hand-sides equal, we can eliminate the moon mass *m*, to get:

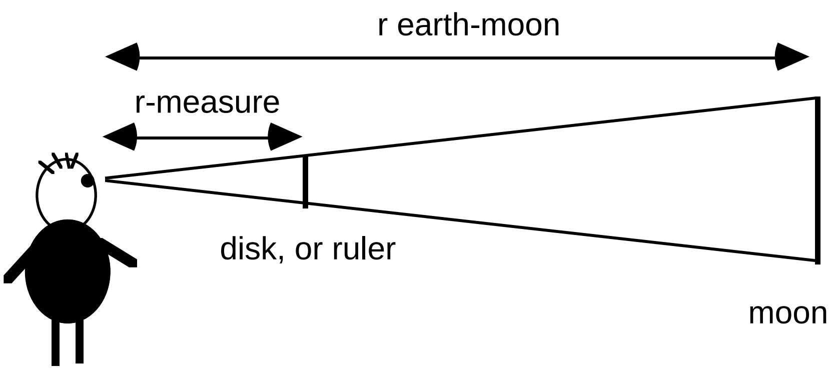
$$\frac{GM}{r_{M,m}} = v^2 = \left(\frac{2\pi r_{M,m}}{T_{orbit}}\right)^2,$$ where the last equality uses [5.1].

Then we have

$$r_{M,m} = \left(\frac{GM\,T_{orbit}^2}{4\pi^2}\right)^{1/3}$$

For $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M = 5.98 \times 10^{24}$ kg, $T_{orbital}$ = 27.333 days * (8.64×10^4 seconds/day) = 2.36×10^6 s, this gives $r_{M,m} = 3.83 \times 10^8$ m (*cf.* 3.844×10^8 m).

b) Ah, now that we have the distance $r_{M,m}$ we can figure out the diameter of the moon by constructing similar triangles:



By holding a small paper disk, and moving it toward and away from the eye until it exactly covers the moon, then recording both the disk size d_{disk} and the distance held away from the eye $r_{measure}$, the earth-moon distance from (a) can be used to figure out the moon's size:

$$\frac{d_{moon}}{r_{M,m}} = \frac{d_{disk}}{r_{measure}}$$ or finally

$$\boxed{d_{moon} = d_{disk}\frac{r_{M,m}}{r_{measure}}}$$

When I did this, I found roughly d_{disk} = 5 mm at $r_{measure}$ = 50 cm, and so $d_{moon} = (0.5/50)3.83 \times 10^8$ m $= 3.83 \times 10^6$ m. Compare this with a reference-table value of 3.476×10^6 m.

<u>Small Shortcut:</u> Instead of using the values of G and of M, you might note that at the earth's surface ($r_{earth,m} = r_{earth}$) we must have $F_{Mm} = m\,g$, where g = 9.8 kg m^{-1} s^{-2}, so:

$$F_{Mm} = \frac{GMm}{r_{M,m}^2} = mg\frac{r_{earth}^2}{r_{M,m}^2},\ i.e.,\ GM = gr_{earth}^2, \text{ constants easier to remember} \qquad [5.4]$$

<u>Synodic vs. sideral:</u> How *should* you measure a complete rotation of something in space? Suppose the earth did not rotate at all. If it made one orbit of the sun, every spot on earth would have daytime and night-time, both, so the earth would see one revolution or one day in terms of the sun, though in terms of the 'fixed' stars it had not rotated (no sidereal days at all). So the *synodic* and *sidereal* periods for the earth are not the same. The same holds for the *moon* and its orbit of the earth.

As the moon orbits the earth, the earth orbits the sun. If the moon 'start-finish line' for an orbit is an imaginary line between the earth and some very distant star, then it will orbit and come back to this line again in a *sideral period*. If the start-finish line is a line between the earth and the sun, then while the moon is orbiting, this line will change along with the *earth's* orbiting, so the finishing-line also will move through some angle in space. With a retreating finish-line, a complete revolution in these terms will take longer.

(Of course, during the time it takes to catch up, the line will have moved a little bit again, making a bit more to catch up, and so on, and so on. The problem ends up like figuring out from 8:00 pm exactly when the minute hand ends up opposite the hour hand: in 10 minutes it is opposite where it *used* to be, but the hour hand has moved an angle 1/12th as much as the minute hand just moved. So the minute hand reaches the *corrected* place 1/12 of 10 minutes later, but then the hour hand in that time moved 1/12th of 1/12th of 10 minutes... The answer can be found using the sum of the series $(1/12)^n$, which has a well-known formula.)

6. What's the buzz?

a) Need some approximation

$$\text{angle } \theta\text{: } \frac{0.7\text{cm}}{1.0\text{cm}} = \sin\theta = 0.7$$

$$\Rightarrow \theta = 0.775 \text{ radians} = 44.4^\circ$$

Momentum: 0.3 g = 3×10^{-4} kg; mv = 2×10^{-4} kg · 1 m s^{-1} = 3×10^{-4} kg · m s^{-1}. The momentum resolves *along* the rigid rod and along the ⊥ to the rigid rod.

$$\frac{p_\perp}{p} = \frac{0.7\text{cm}}{1\text{cm}} = p_\perp = 0.7\ p$$

Angular momentum is conserved; with the free-pivot action, it sees no torque and so is conserved **without** contribution from the ceiling, earth, etc..

$$L = P_\perp \bullet r = 0.7\ p \bullet 10^{-2}\ \text{m},$$

$$= 2.1 \times 10^{-6}\ \text{kg m}^2\ \text{s}^{-1}$$

It's undefined what the final angle is when he reaches the other side — he could swing around until he is parallel to the ceiling, or maybe until his 1 cm-long legs, halfway down his body, hit the roof. Either is good for the sort of rough estimates we are making on behalf of flies! You figure.

Say, take 1st case: the final angle is π radians. Then he swings through $(\pi - 0.775)$ radians = 2.37 radians or 135.6°

Arc length: $\Delta\theta \cdot r = 2.37 \text{ rads} \cdot 1 \text{ cm} = 2.37 \text{ cm} = 2.37 \times 10^{-2}$ m

Speed: $v_\perp = 0.7 \cdot v = 0.7 \times 1 \text{ m s}^{-1} = 0.7 \text{ m s}^{-1}$

$$t = \frac{d}{v} = \frac{2.37 \times 10^{-2}\text{m}}{0.7 \text{ ms}^{-1}} = 3.39 \times 10^{-2} \text{ s} = 33.9 \text{ ms}$$

we'll accept 15 – 35 ms, about one frame of a video camera.

b) Again depends on final angle, leg length, etc., but approx $v_\perp = 0.7 \text{ m s}^{-1}$; we'll accept answers 0.35 – 0.7 depending on final angle

$$v = 0 \Rightarrow -\frac{u^2}{2d} = a$$

c) $v^2 = u^2 + 2ad$

$$a = -\frac{(0.7)^2}{21.5 \times 10^{-3}} = \frac{0.49}{3 \times 10^{-3}} = 0.16 \times 10^3 = 1.63 \times 10^2 = 163 \text{ m s}^{-2}$$

d) $F = M_{fly} \bullet a$
$= 0.3 \times 10^{-3} \text{ kg } (-163 \text{ m s}^{-2})$
$= -49 \times 10^{-3} \text{ kg m s}^{-2}$
$= -4.9 \times 10^{-2} \text{ kg m s}^{-2}$
$= -4.9 \times 10^{-2}$ N

$F = M_{human} \bullet a$
$= 70 \text{ kg } (-163 \text{ m s}^{-2})$
$= -1.14 \times 10^4 \text{ kg m s}^{-2}$
$= -1.14 \times 10^4$ N

<u>PART II</u>

30 km/hr = 30×10^3 m / 3600 s = 8.3 ms^{-1}

e) $v^2 = u^2 + 2ad$

$v = 0 \Rightarrow$

$$a = -\frac{(8.3\text{ms}^{-1})^2}{21.5\text{m}} = -23 \text{ m s}^{-2}$$, much less than for the fly

$F = m \bullet a = (1{,}000 \text{ kg}) (-23 \text{ m s}^{-2})$
$= -2.3 \times 10^4 \text{ kg m s}^{-2}$

f) not very significant that belts stretch a little, if assume constant acceleration force (unless you let the passenger bounce) total distance is 1.75 m

$\Delta x = 1.75$ m

50 km/h = 50×10^3 m / 3600 s = 13.9 m s^{-1}

$$a = -\frac{u^2}{2d} = -\frac{(13.9\ \mathrm{m\ s^{-1}})^2}{2(1.75\ \mathrm{m})} = -55.2\ \mathrm{m\ s^{-2}}$$

$$F = m \bullet a = 80\ \mathrm{kg} \bullet (-55.2\ \mathrm{m\ s^{-2}})$$
$$= -4.416\ \mathrm{kN}$$

Then 0.35 m • 0.05 m = 0.0175 m^2 area for belt. Pressure is

(4.416 kN / 0.0175 m^2) = 2.52×10^5 Pa

[1 ATM = 1.01325×10^5 Pa]

– about 2.5× atmospheric pressure; can bruise heavily where the belt cuts in

– or, say my fist is 7 cm × 9 cm = 63 cm^2 = 0.0063 m^2, so to produce the same pressure I would need a force of about 1.6 kN, the weight of about 162 kg — much more than I weigh, and probably a really decent heavyweight prizefighter punch if the followthrough is strong.

Or, to compare another way, the whole force on the person is F = 4.416 kN corresponds to the weight of ~450 kg – almost half the weight of a small car, so this is a serious force. How long does it last?

$$t = -\left(\frac{13.9\ \mathrm{m\ s^{-1}}}{2(-55.2)\ \mathrm{m\ s^{-2}}}\right) = 0.126\ \mathrm{s} = 126\ \mathrm{ms}$$

Perhaps this is something like being directly hit by a motorcycle? You can hit the belt, or you can hit the windshield, but much better the belt! Buckle up — it saves lives.