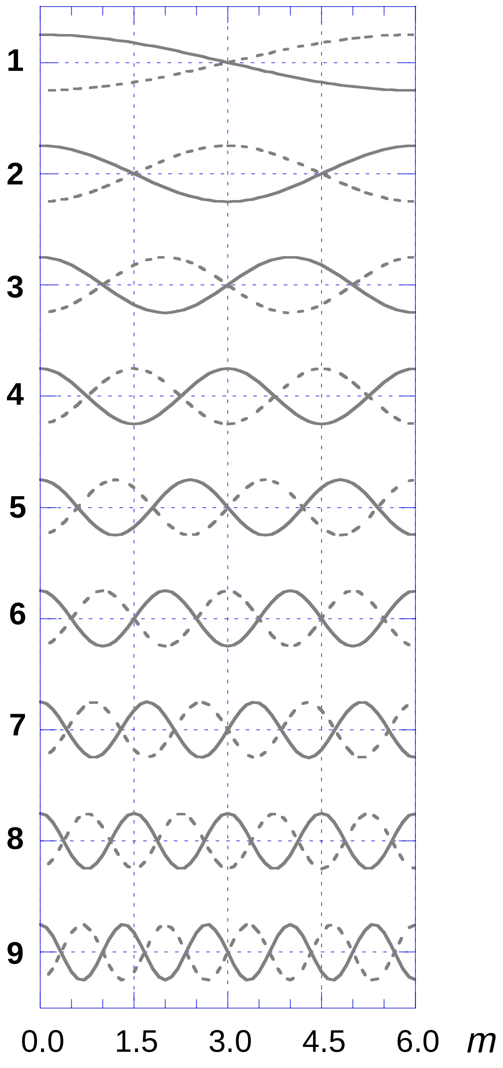
1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 4: Optics and Waves

1. Jesse's anti-music

This problem here is a simple standing-wave problem. Each speaker is at the antinode position. To have minimum sound, Jesse has to be at one of the nodes of the standing wave pattern.

Wave patterns	Mode	n	
	fundamental	1	L = 1/2 λ
		$\boxed{2}$	$\boxed{L = \lambda}$
		3	L =3/2 λ
		4	L = 2 λ
		5	L = 5/2 λ
		$\boxed{6}$	$\boxed{L = 3 \lambda}$
		7	L = 7/2 λ
		8	L = 4 λ
		9	L = 9/2 λ

where L=6.0 m is the distance between the two speakers and λ the wavelength.

a) Analytically, the appropriate wavelengths can be obtained by imposing the antinode boundary conditions:

$$\cos([2\pi/\lambda]\, L) = \pm 1, \text{ thus } \quad [2\pi/\lambda]\, L = n\,\pi \text{ for } n = 1, 2, 3, \ldots$$

At position x =1.5 m, the intensity must be a minimum, meaning

$$\cos([2\pi/\lambda]\, x) = 0 \;\Rightarrow \cos([2\pi/\lambda]\, Lx / L) = 0 \;\Rightarrow \cos([2\pi/\lambda]\, L\, [x / L]) = 0$$

or

$$0 = \cos([n\,\pi]x / L) = \cos(0.25\, n\, \pi) \quad \text{for the acceptable } n.$$

Thus $n = 2, 6, \ldots$, and

$$f = v / \lambda = v\,(2\,m - 1) / L \qquad \text{for} \qquad m = 1, 2, 3, \ldots.$$

So the frequencies would be minimum for f = 55.24 Hz, 165.73 Hz, 276.21 Hz, ….

b) No. He, in fact, would hear some frequencies louder (at maximum with intensity four times larger than its original without his own speaker).

c) If Jesse has only one speaker, he could not do any better no matter what direction his speaker faces. If he has more speakers he could arrange them as such that more frequencies can be cancelled out. Nevertheless, there is some complication due to the interference between his own speakers.

2. A new twist on light

This problem can be answered with different levels of rigour. To try and be as ‘correct’ as possible, while avoiding integration. I will define intensity as the time average of the square of the electric field:

$$I = \langle |\vec{E}(t)|^2 \rangle$$

This is necessary since for visible light fields, the electric field is oscillating very fast ($\approx 6 \times 10^{14}$ Hz), much faster than a detector can follow. This method can also be used for a slowly-varying field by taking the time average over a length of time corresponding to detector response ($\approx 10^{-12} - 10^{-6}$ s). We do not require ‘I’ as a function of time for this question.

PLEASE NOTE: Many of you may have taken a simpler approach (more likely to be found in an OAC-level course) by writing:

$$I = |\vec{E}|^2 \qquad \text{(ignore oscillating nature of } \vec{E}(t)\text{)}$$

This is o.k., but not as rigourous as my solution.

a) Consider a polarizer with a set axis and a general (instantaneous) electric field vector. where $|\vec{E}_{||}| = |\vec{E}_o|$. Transmitted electric field is $\vec{E}_{||}$. Thus transmitted intensity is:

$$I_T = \langle |\vec{E}_{||}|^2 \rangle$$

$$= < \left|\vec{E}_o\right|^2 \cos^2\theta >$$

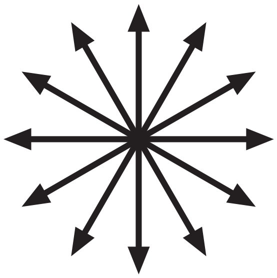
$$= < \left|\vec{E}_o\right|^2 > \cos^2\theta$$

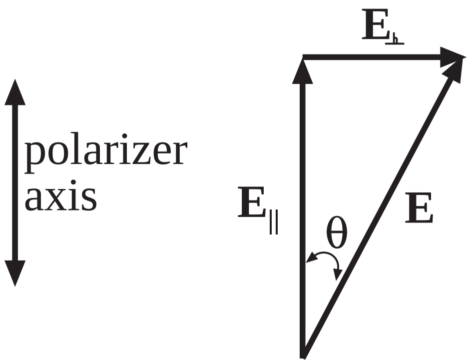
$$\text{but } I_o = < \left|\vec{E}_o\right|^2 >$$

$$\therefore I_T = I_o \cos^2\theta$$

b) Required assumption: current on photodetector ∝ intensity of light.

A flashlight is an unpolarized source. This means the polarization of the EM (light) field does not have a preferred direction. A really rough answer to this would be that since there is no preferred direction, 1/2 of the intensity is polarized parallel to the polarizer, and 1/2 is orthogonal.
∴ she measured 0.5 mA.





Let's do this a little more rigourously. We can model the flashlight as a very large number of linearly polarized sources, all with the same amplitude, and with their polarizations pointing in all directions.

What does Kimberley measure without the polarizer?

$$I_{TOT} = <|\sum_{n=0}^{N} \vec{E}_n|^2 >$$

where N is very large.

This can be written as:

$$I_{TOT} = <|\sum_{n=0}^{N} \vec{E}_{n||} + \sum_{n=0}^{N} \vec{E}_{n\perp}|^2 > \tag{1}$$

where $\vec{E}_{n||}$: is a parallel component of the nth source.
N: a very very big number (I'm working hard to avoid calculus here. If using calculus, N → ∞ and all the sums become integrals).

With the polarizer she oberves

$$I_{pol} = < | \sum_{n=0}^{N} \vec{E}_{n||} |^2 > \tag{2}$$

We need $\frac{I_{pol}}{I_{TOT}}$

Take (1) ⇒

$$I_{TOT} = <| \sum_{n=0}^{N} \vec{E}_{n||} |^2 + | \sum_{n=0}^{N} \vec{E}_{n\perp}|^2 + 2|(\sum_{n=0}^{N} \vec{E}_{n||})\cdot(\sum_{n=0}^{N} \vec{E}_{n\perp})|>$$

Due to the absolute value brackets, we can write.

$$= < | \sum_{n=0}^{N} \vec{E}_{n||} |^2 > + <| \sum_{n=0}^{N} \vec{E}_{n\perp}|^2 > + 2 < |(\sum_{n=0}^{N} \vec{E}_{n||}) \cdot (\sum_{n=0}^{N} \vec{E}_{n\perp})|>$$

We know that the flashlight polarization had no preferred direction. Therefore no matter how Kimberly placed the polarizer, she would get the same result. Thus ‘parallel’ and ‘orthogonal’ are arbitrary and the first two terms must be equal.

To simplify the third term, consider a case with only two sources. $\vec{A}, \vec{B}$
The third term looks like

$$<|(\vec{A}_{||} + \vec{B}_{||})\cdot(\vec{A}_{\perp} + \vec{B}_{\perp})|>$$

$$= 0 \qquad \text{since} \quad \begin{aligned} \vec{A}_{||} \cdot \vec{A}_{\perp} &= 0 \\ \vec{A}_{||} \cdot B_{\perp} &= 0 \\ \vec{B}_{||} \cdot \vec{A}_{\perp} &= 0 \\ \vec{B}_{||} \cdot B_{\perp} &= 0 \end{aligned}$$

Therefore (1) ⇒

$$I_{TOT} = 2 < | \sum_{n=0}^{N} \vec{E}_{n||} |^2 > \text{ (since first two terms are equal).}$$

$$\therefore I_{pol} = \frac{I_{TOT}}{2}$$

She observes 0.5 mA of current.

c) Only linearly polarized light is transmitted by the first polarizer. Therefore the second polarizer transmits

$$I_2 = I_1 \cos^2\theta$$

For minimum I_2, chose $\theta = \frac{\pi}{2}$, $\therefore I_2 = 0$ and current = 0.0 mA. The angle between the axes is 90°.

d) Let I_1 be intensity of light passing through first polarizer. Let I_2 be intensity of light passing through second polarizer.

$$I_2 = I_1 \cos^2 90° = 0$$

And the final polarizer is aligned at 90° – 45° = 45° to the second one:

$$I_3 = I_2 \cos^2 45° = 0$$

∴ she measured 0 mA.

After exchanging:

$$I_2 = I_1 \cos^2 45° = \frac{I_1}{2}$$

$$I_3 = I_2 \cos^2 (90° - 45°) = \frac{I_1}{4}$$

∴ she measured 0.13 mA.

e) Output intensity is

$$I_{final} = I_1 \cos^2(30°) \cos^2(30°) \cos^2(30°)$$

$$= I_1\left(\frac{\sqrt{3}}{2}\right)^6$$

$$\approx 0.42\ I_1$$

Therefore she measures 0.21 mA. By adding more polarizers, she is outputting more light!
For *N* polarizers:

$$I_{FINAL} = I_1\left(\cos^2\left(\frac{90^\circ}{(N-1)}\right)\right)^N$$

By ‘subbing’ in increasing values for N, we see this coefficient is approaching 1.
Slightly more rigourously

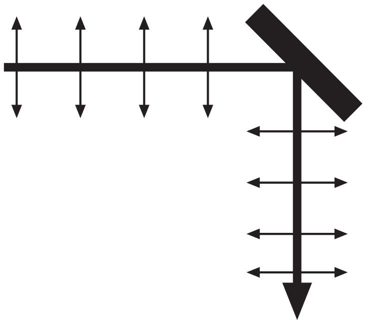
$$\lim_{N\to\infty} \frac{90}{N-1} = 0$$

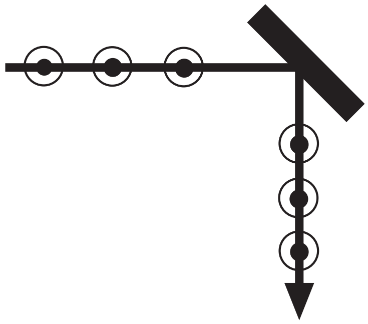
$$\lim_{N\to\infty} \cos^{2N} 0 = 1$$

Therefore she measures 0.5 mA. (Recall I_1 is intensity of light passed by one polarizer).

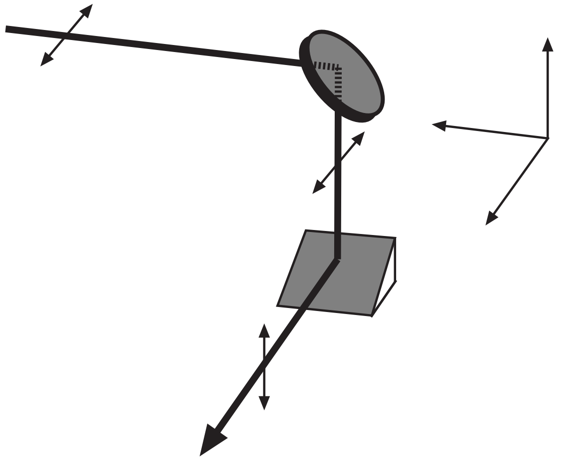
f) Real polarizers are not perfect. They still absorb some of the EM wave polarized parallel to their axes. Thus very little light would make it through N polarizers, where N is a large number.

To change the polarization direction using two standard mirrors, recall what happens to a polarized beam in reflection:

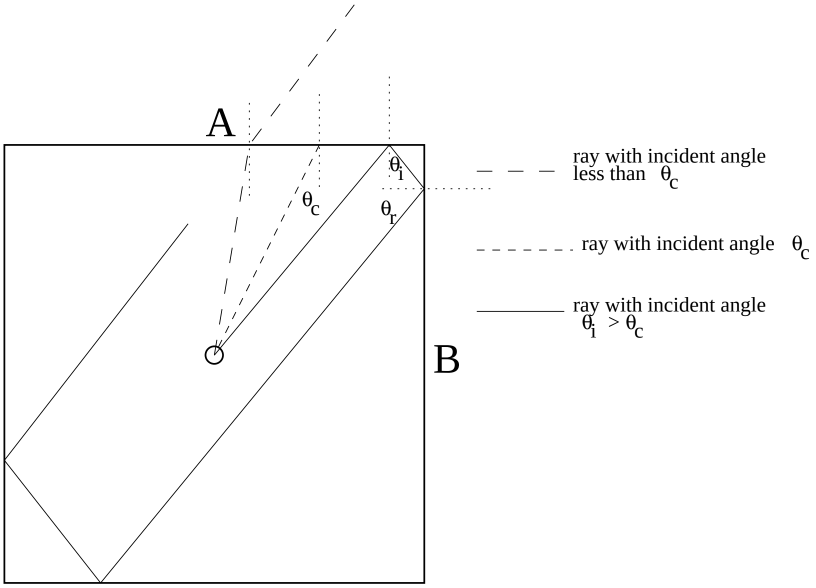




So arrange two mirrors as follows:



3. People see through Claire



The light-ray from the pellet will be refracted out if the angle of incidence (θ_I), measured from the normal, is less than the critical angle (θ_c). Due the geometry of the cube (shown as a square in the figure at left), θ_c has to be less than 45°. Otherwise the whole surface has to be covered. Any direct rays fall with $\theta_i > 45°$ in one side will fall on the adjacent one.

If $\theta_c < \theta_i < 45°$, the ray will be totally reflected and it will be reflected forever! To see this effect refer to the figure. Coming from the source, the light will be reflected by side A with θ_i. The second reflection will be $\theta_r = 90° - \theta_i > 45°$ which is certainly greater than θ_c, and the third will be θ_i, and so on alternately. In this way a cube is a good container for hiding an object.

The light rays can also run in a direction not parallel to the edges of the faces of the cube: the extreme case is one in which the rays are directed along the planar diagonal of the cube. In this case θ_i can be as large as 54.73° (i.e., $\tan^{-1}$ (2)), with $\theta_r = 35.26°$. To contain this ray also, we need $\theta_c < \theta_r$, i.e., $\theta_c < 35.26°$.

a) A point source will emit light spherically. So Claire must paint in a circular shape on each side. The area she covers must be such that $\theta_i < \theta_c$ cannot penetrate. The radius of each circle will be determined from θ_c.

$$\theta_c = \sin^{-1} (n_{fluid} / n_{cube})$$

For air $n_{fluid} = 1.00$, thus $\theta_c = 34.85°$ and the radius of the circle $r = \frac{1}{2} d \tan \theta_c = 0.349\, d$, where d is the vertex of the cube. So the area she must cover is

$$R = \pi r^2 / d^2 = \frac{1}{4} \pi \tan^2 \theta_c = 0.3808$$

i.e., 38.08% of the total surface.

b) First we have to check whether in water, $\theta_c < 35.26°$. Using the same formula, $\theta_c = 49.46°$ so Claire has to paint it all!

4. Nothin' but blue sky ...

a) Assume that the radiation is emitted symmetrically for 2 radii, R_1 and R_2. We must have same radiation flux through surfaces $4\pi R_1^2$ and $4\pi R_2^2$, so

$$I \propto \frac{1}{R^2}$$

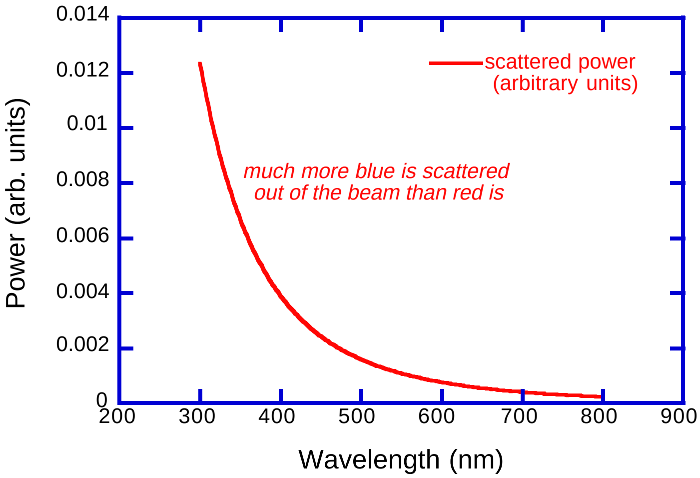
b)
$$\frac{I_{scatt}}{I_o} \propto p^2 \propto (a^3)^2$$

and

$$\frac{I_{scatt}}{I_o} \propto \frac{1}{R}$$

So for $\frac{I_{scatt}}{I_o}$ to be dimensionless requires a $\frac{1}{(length)^4}$ dependence. As ∞ is the only remaining length–dependent variable, we have

$$\boxed{\frac{I_{scatt}}{I_o} \propto \frac{1}{\lambda^4}}.$$

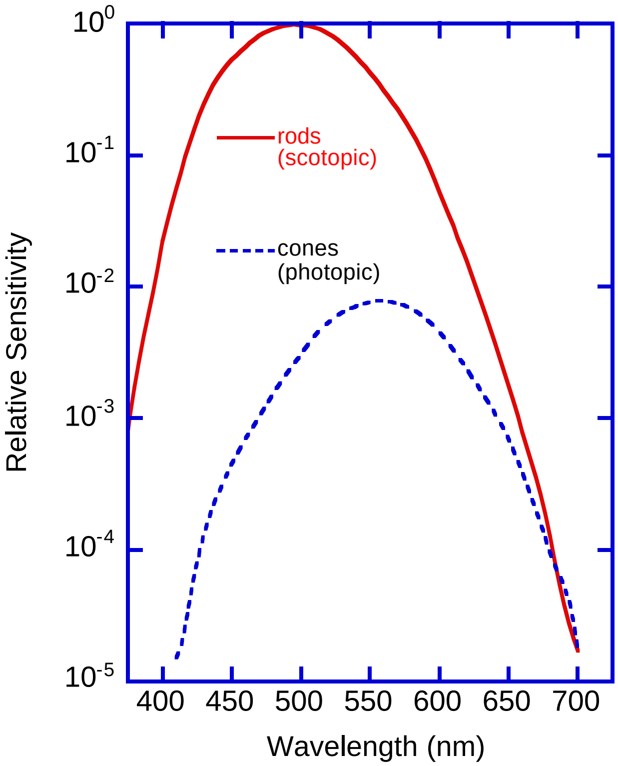


This $\frac{1}{\lambda^4}$ dependence is known as 'Rayleigh scattering'.

c) From part (b), longer wavelengths experience smaller scattering. Thus especially at sunset (when the sunlight passes through the longest stretch of atmosphere), looking at the sun, the blue end of spectrum is preferentially scattered away by water molecules and the sun appears reddish or red-orange. The shorter-wavelength, bluer light — sent sideways from the transmitted beam of sunset light — has showered sideways to become someone else's blue sky.

Our eyes' sensitivity peaks near green and sensitivity falls off towards red and violet.

So, superposing the eye's sensitivity and the Rayleigh scattering law, we see a peak near the blues.



5. Just tweezing ...

As the beam enters the sphere, its direction deviates, due to refraction at the interface

$$\sin\theta_1 = n \sin\theta_2 \qquad \text{(Snell's Law)}$$

The amount the beam **deviates** is then $\Delta\theta$

$$\alpha = \theta_2 + \Delta\theta = \theta_1 \qquad \text{(opp. } \angle \text{ th'm)}$$

$$\Rightarrow \qquad \Delta\theta = \theta_1 - \theta_2$$

when the beam leaves the sphere the angle of incidence at the exit is again θ_2 and then θ_1 at the exit, and again the angular deviation is $\Delta\theta$.

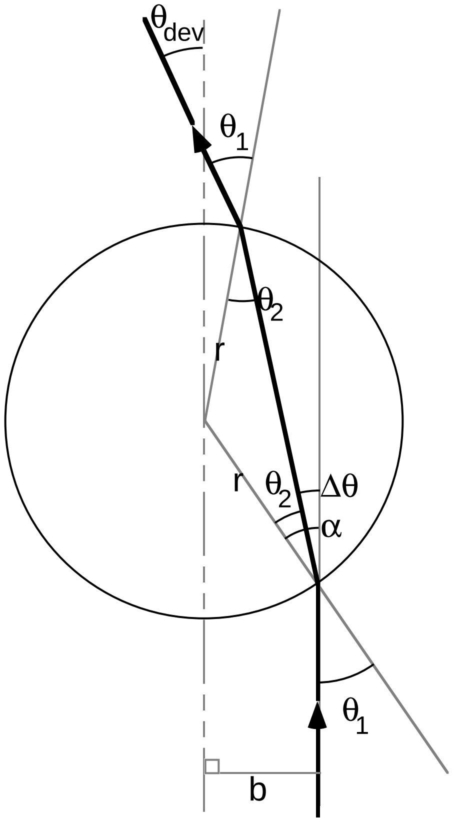
Thus the overall deviation in angle is

$$\theta_{dev} = 2\Delta\theta = 2(\theta_1 - \theta_2)$$

$$\sin\theta_1 = \sin\alpha = \frac{b}{r} \rightarrow \theta_1 = \sin^{-1}\left(\frac{b}{r}\right)$$

$$\sin\theta_2 = \frac{1}{n}\sin\theta_1 = \frac{b}{nr} \rightarrow \theta_2 = \sin^{-1}\left(\frac{b}{nr}\right)$$

Thus $\theta_{dev} = 2\left(\sin^{-1}\left(\frac{b}{r}\right) - \sin^{-1}\left(\frac{b}{nr}\right)\right)$



$$\left.\begin{array}{l} b = 6\mu\text{m} \\ n = 1.4 \end{array}\right\} \quad \Rightarrow \quad \theta_{dev} = 2\left(\sin^{-1}\left(\frac{6\mu\text{m}}{25\mu\text{m}}\right) - \sin\left(\frac{6\mu\text{m}}{1.4\;\;25\mu\text{m}}\right)\right)$$

$$= 2\,(13.88° – 9.87°)$$
$$= 8.03°$$

This new direction means photons have a new x-component of momentum. If the photon momentum is p_γ originally, then

$$p_x = p_\gamma \sin(\theta_{dev})$$
$$= p_\gamma \sin\left(2\left\{\sin^{-1}\left(\frac{b}{r}\right) - \sin^{-1}\left(\frac{b}{nr}\right)\right\}\right)$$

here

$$p_x = p_\gamma \sin(8.03°)$$
$$= p_\gamma\,(0.140)$$

For a photon: $p = \frac{h\nu}{c} = \frac{h}{\lambda}$, where h = 6.62×10^{-34} J • s. Here λ = 530 nm = 5.30×10^{-7} m:

$$p = \frac{6.62 \times 10^{-34}\text{ J s}}{5.30 \times 10^{-7}\text{m}} \qquad [\text{but J: kg•m}^2\text{•s}^{-2}]$$
$$= 1.25 \times 10^{-27}\text{ kg• m•s}^{-1}$$

so $\Delta p_x = 1.75 \times 10^{-28}$ kg•m•s^{-1}

b) Laser 1W = 1 J/s

1 photon has energy $h\nu = \frac{hc}{\lambda} = \frac{(6.62 \times 10^{-34}\text{Js})\;3 \times 10^8\text{ms}^{-1}}{5.3 \times 10^{-7}\text{m}}$

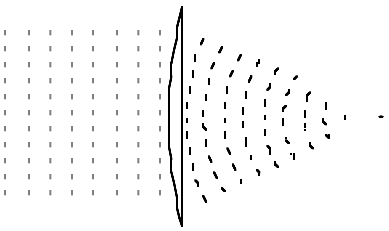
$$= 3.75 \times 10^{-19}\text{J}$$

so 1 J = 2.67×10^{18} photons per second

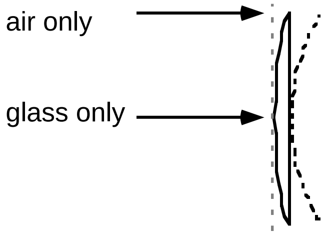
$$F_{TOT} = \frac{dp_x}{dt} = N_x \bullet \Delta p_x = 2.67 \times 10^{18} \cdot 1.75 \times 10^{-28}\text{ kg• m•s}^{-1}$$
$$= 4.67 \times 10^{-10}\text{ kg•m•s}^{-2}$$
$$= 4.67 \times 10^{-10}\text{ N}$$

6. Photons get the bends

a)



$v_\phi = c / n$



phase fronts move slower in glass where n = 1.66

- time through 5 mm *glass* t = d / v_ϕ = $\dfrac{nd}{c} = \dfrac{1.66\ \ 5\times10^{-3}\text{m}}{3\times10^{8}\text{ms}^{-1}}$

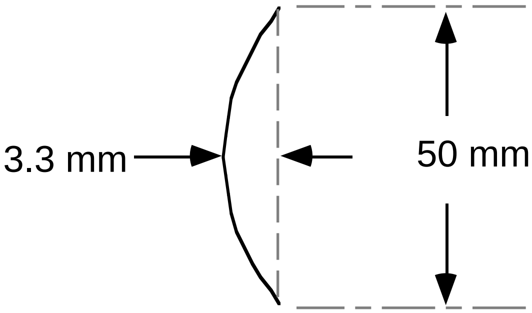
$$= 2.77 \times 10^{11}\ \text{s}$$
$$= 27.7\ \text{ps } (\textit{picoseconds})$$

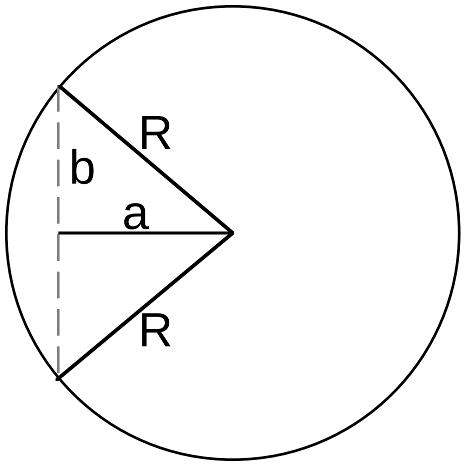
- time through 5 mm *air* t = $\dfrac{d}{v_\phi} = \dfrac{nd}{c} = \dfrac{1.00\ \ 5\times10^{-3}\text{mm}}{3\times10^{8}\text{ms}^{-1}}$

$$= 1.67 \times 10^{-11}\ \text{s}$$
$$= 16.7\ \text{ps}$$

⇒ time *difference* is (27.7 – 16.7) ps = 11 ps

b) In one way of describing it, lenses focus by putting a spherical curvature on phase fronts, then Huyghen's Principle shows the wavefronts converge to a point — the focus
Right after lens, the 11 ps relative delay means distance ct = $3 \times 10^{8}\ \text{ms}^{-1} \cdot 11 \times 10^{-12}\ \text{s} = 3.3$ mm behind edges:



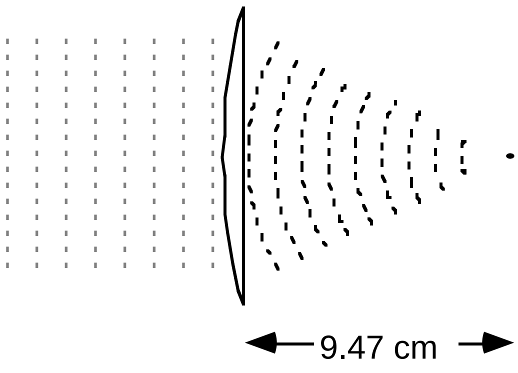


This is a chord of a circle. What is the radius of the circle?

$$\left.\begin{array}{l} b = 25\ \text{mm} \\ a = R - 3.3\ \text{mm} \end{array}\right\} a^2 + b^2 = R^2$$

$$\text{so } R^2 = (R - 3.3\ \text{mm})^2 + (25\ \text{mm})^2$$
$$R^2 = R^2 - 6.6\ \text{mm}\cdot R + 625\ \text{mm}^2$$
$$R = \frac{625\ \text{mm}^2}{6.6\ \text{mm}} = 94.7\ \text{mm}$$
$$= 9.47\ \text{cm}$$

So the radius of curvature is 9.47 cm, and this is where the beam comes to a focus 9.47 cm after the lens!



c) The form of I looks complicated. That is because it is pre-cooked to give an easier answer!

$$n = n_o + n_2 I$$

$\Delta n = n_2 I$ across the beam, making wavefronts curved following intensity changes across beam. Let's find wavefront delay according to x

$$\Delta t = \frac{n(x)d}{c}$$

$n(x) = n_o + n_2 I(x)$
$d = 10 \text{ cm} = 0.1 \text{ m}$
$c = 3 \times 10^8 \text{ ms}^{-1}$

$$\Delta t = \frac{n_o d}{c} + \frac{n_2 I(x) d}{c}$$

constant for all ↑ ↑ causes curvatures in wavefronts

and the physical displacement of the wavefronts is:

$$c\Delta t = n_o d + n_2 I(x) d$$

constant (0.17 m) ↑ ↑ depends on x

So all the curvature in the beam comes from the second term:

$$\Delta z = n_2 I(x)\, d = 5 \times 10^{-15} \bullet 2 \times 10^{13} \text{ W} \bullet \text{cm}^{-2} \bullet (-b + \sqrt{10^6 - x^2})$$
$$= 0.01\,(-b + \sqrt{10^6 - x^2})$$
$$= 0.01\,(\sqrt{10^6 - x^2} - b)$$

Note the units of n_2: cm^2W^{-1} — there was an error in the question! (*eek!*)

Then $(100\,\Delta z + b) = \sqrt{10^6 - x^2}$
$(100\,\Delta z + b)^2 = 10^6 - x^2$

Take $y = 100\,\Delta z$, then

$$x^2 + (y + b)^2 = 10^6 \text{ cm}^2$$

the equation of a circle, offset by $-b$ along y, with $R = 10^3$

What is the y-displacement?

$y = y_o$ where $x = 0$
$(y + b)^2 = 10^6$
$y + b = 10^3$
$y = 10^3 - b$
$= 1000 - 999.996875$
$y = 3.13 \times 10^{-3}$

so

$$\Delta z = y/100 = 3.13 \times 10^{-5} \text{ m}$$

Use the same method as in (a) to get the curvature

$a^2 + b^2 = R^2$

$a = 2.5$ cm
$b = 999.996875$ cm

$R^2 = 10^6$ cm^2
$R = 10^3$ cm = 10 m

The beam focuses in 10 m.