1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 5: Electricity and Magnetism

1. **Gauss visits Planesville ...**

a) for a pt. charge in 2-D:

GAUSS' LAW: $E \bullet$ (circumference of circle) = $\dfrac{Q}{4\pi\varepsilon_o R^2}\, 2\pi R = \dfrac{Q}{2\varepsilon_o}\cdot\dfrac{1}{R} \neq \dfrac{Q}{\varepsilon_o}$

which is what one would expect from Gauss's law

b) to 'fix' this, make Coulomb's law $\propto \dfrac{1}{R}$

, i.e.,

$$E = \frac{Q}{2\pi\varepsilon_o R}\cdot\frac{1}{R} \qquad \text{(point charge)}$$

then $\quad E\cdot(\text{circumference of circle}) = \dfrac{Q}{2\pi\varepsilon_o}\cdot\dfrac{1}{R}\cdot 2\pi R = \dfrac{Q}{\varepsilon_o}$ (satisfies Gauss's law)

c) <u>inside disc</u>:

$$E \bullet 2\pi \bullet R = \sigma \bullet \pi R^2$$

where $\sigma = 0.5\ \mathrm{mC{\bullet}m^{-2}}$

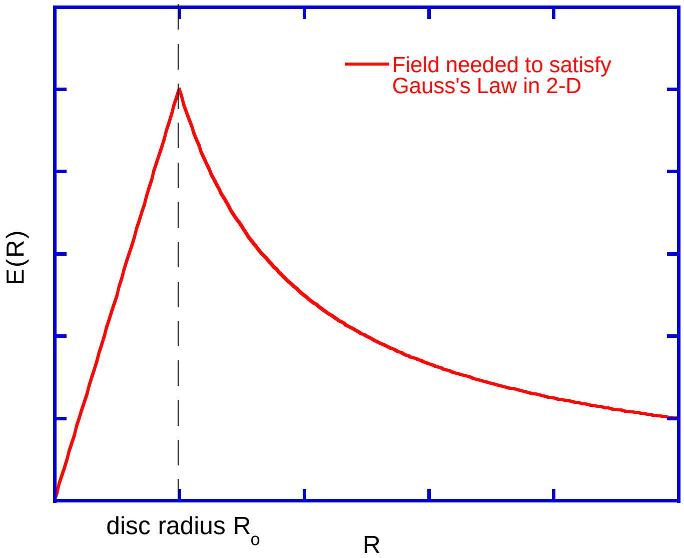
$$\Rightarrow E = \frac{\sigma}{2}R, \quad R < R_o, \quad (R_o = 10\ \mathrm{cm})$$

<u>outside disc</u>:

$$E \bullet 2\pi R = \sigma \bullet \pi R_o^2$$

where $\sigma = 0.5\ \mathrm{mC{\bullet}m^{-2}}$

$$\Rightarrow E = \frac{\sigma}{2}\,\frac{R_o^2}{R}$$



2. **It's not just a good idea ... it's Ohm's law!**

a) $F = ma = QE, \quad \therefore\ a = \dfrac{qE}{m}$ $\quad\therefore$ charge experiences constant acceleration

$J \propto V_{avg} = \dfrac{qEt}{m}$ $\quad\therefore$ current should increase linearly with time! ('non-Ohmic')

b) $$d = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2d}{a}} \propto \sqrt{\frac{2d}{E}}$$

$$\therefore\ v_{avg} = \frac{1}{2}at = \frac{1}{2}a\sqrt{\frac{2d}{a}} \propto \sqrt{a} \propto \sqrt{E}$$

$\therefore$ it is non-ohmic

c) if $v_{\text{thermal}} >> v_{\text{due to field}}$, $t = \frac{d}{v_{thermal}}$

then, $v_{\text{avg}} = \frac{1}{2}\text{at} = \frac{1}{2}\text{a}\frac{\text{d}}{v_{\text{thermal}}} \propto E$

$\therefore$ it is ohmic

d) $\sigma \propto \frac{1}{v_{\text{thermal}}}$ from part (c)

so, as T increases, v_{thermal} increases so the resistance is greater. If T decreases, v_{thermal} decreases, so the resistance decreases.

3. **Total Recoil**

a) The voltage increase causes an increasing current in the wire of the solenoid. This increasing current creates an increasing magnetic field through the solenoid. This increasing magnetic field acts back on the solenoid wires to create an 'induced' voltage which *opposes* the original voltage change, and will continue until the system reaches steady-state. In brief, the effect is the 'induction' of a 'reverse' electric field (and current) in the solenoid.

$$B = \frac{\mu NI}{2\ell} = \frac{\mu N}{2\ell}(1 - e^{-t/T})$$

b) $$= \frac{500\mu_o}{2(0.5)}(2000)(1 - e^{-t/0.04})$$

$$= 1.3(1 - e^{-t/0.04})\ \text{Tesla}$$

c) Faraday's law of induction

$$E = -\frac{d\Phi}{dt}$$

where ε is the induced electric field

M is the flux = $\vec{B}\cdot\vec{A}$

$\vec{B}$ is the magnetic field

$\vec{A}$ is orthogonal to the loop area

$|\vec{A}|$ = area of loop

Since the $\vec{B}$ direction is effectively orthogonal to the loop

$$\Phi = \vec{B}\cdot\vec{A} = A\left|B\right|$$

so

$$\varepsilon = -A\,\frac{\mathrm{d}B}{\mathrm{dt}}$$

$$= -A\ \frac{\mu N}{2\ell}\,\frac{\mathrm{d}}{\mathrm{dt}}(1 - \mathrm{e}^{-\mathrm{t}/T}),\ \ \mathrm{a} = \left(\frac{0.01}{2}\right)^2 \pi\,\mathrm{m}^2$$

$$= 2.5 \times 10^{-3}\ \mathrm{e}^{-\mathrm{t}/T}\ V$$

$$.\,.\,\mathrm{i} = \frac{\varepsilon}{R}$$

$$= 2.5\ \mathrm{e}^{-\mathrm{t}/-0.4} \bullet A$$

(in opposite direction to current in solenoid.)

(We can ignore back-induction from the ring since it is very small.)

d) $F = \mathrm{q}\vec{\mathrm{v}} \times \vec{B}$

Let n be the charged particle density per unit length. Thus q = –neR

$$F = -\,\mathrm{ne}\ell\vec{\mathrm{v}} \times \vec{B}$$

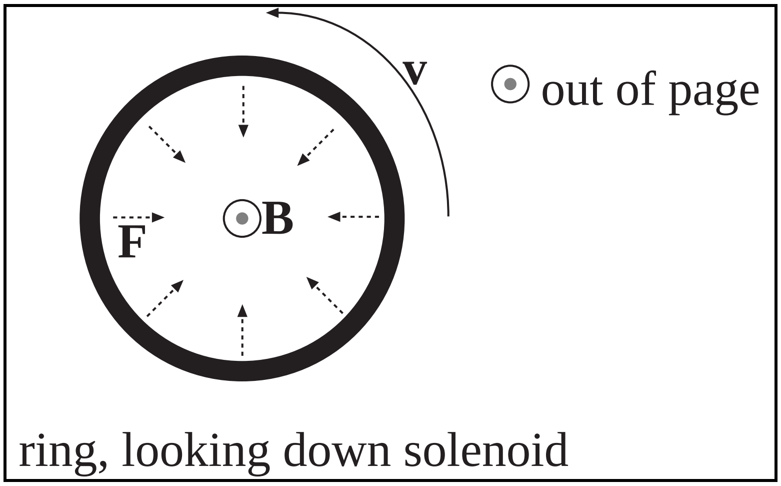
$$\frac{\vec{F}}{\ell} = -\,\mathrm{ne}\vec{\mathrm{v}} \times \vec{B}$$

But within this model $\vec{B}$ is orthogonal to $\vec{\mathrm{v}}$ and $-\mathrm{ne}\,|\mathrm{v}| = \mathrm{i}$ (current).

$$\frac{\left|\vec{F}\right|}{\ell} = \mathrm{i}\,\left|\vec{B}\right|$$

$$= 2.5\ \mathrm{e}^{-\mathrm{t}/0.04}\ \frac{\mu N}{2\ell}(1 - \mathrm{e}^{-\mathrm{t}/0.04})$$

$$= 3.3\ \mathrm{e}^{-\mathrm{t}/0.04}\,(1 - \mathrm{e}^{-\mathrm{t}/0.04})$$

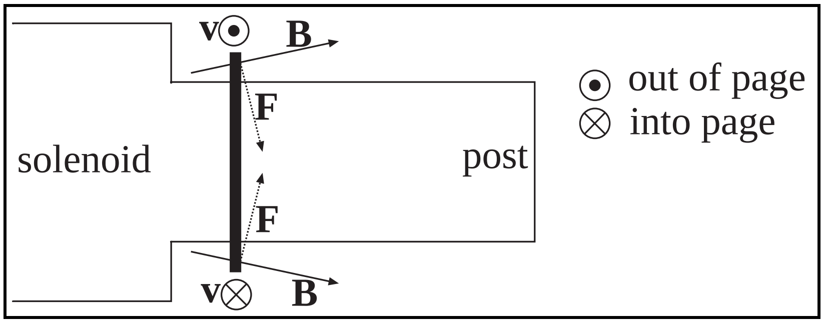


Direction of $\vec{F}$ is parallel to $\vec{v}\times\vec{B}$; $\vec{\mathrm{v}}$ is parallel to the ring; $\vec{B}$ is orthogonal to the ring. Thus $\vec{F}$ is on the surface enclosed by the ring, pointing towards the centre of the ring.

This results in **NO** motion (unless the ring implodes).

e) Even in this case $\vec{v}$ is still orthogonal to $\vec{B}$, but now $\vec{F}$ is not on the surface enclosed by the ring.

The vertical components of $\vec{F}$ cancel, and we are left with a horizontal component facing to the right.



$$F_{\text{NET}} = \left|\frac{\vec{F}}{\ell}\right| \ell \sin$$

$$= 3.3^{-t/.04}\,(1 - e^{-t/.04})\,(.01)\,\pi \sin 15^{\circ}$$

$$= 2.7 \times 10^{-2}\, e^{-t/.04}\,(1 - e^{-t/.04})\ \text{N}$$

f) $F(t = 0.04\ \text{s}) = 2.7 \times 10^{-2}\, e^{-1}\,(1 - e^{-1})$

$$= 6.3 \times 10^{-3}\ \text{N}$$

but $F = ma$

$$\therefore a = \frac{F}{m}$$

$$= 13\ \text{m/s}^2$$

For a constant acceleration:

$$v_2^{\,2} - v_1^{\,2} = 2ad \qquad \text{but } v_1 = 0$$

$$\Rightarrow \qquad v_2 = \sqrt{2ad}$$

$$= 1.1\ \text{m/s}$$

The ring would leave the post going 1.1 m/s.

g) Noah forgot about the conservation of linear momentum.

Consider m_1 to be the mass of the ring
m_2 to be the mass of the gun and Noah
v_1 to be the final velocity of the ring
v_2 to be the final velocity of the gun and Noah

Since the system starts at rest:

$$m_1 v_1 + m_2 v_2 = 0$$

$$v_1 = -v_2 \frac{m_2}{m_1}$$

Guess that Noah and gun mass is on the order of = 100 kg. Their final velocity is

$$
\begin{aligned}
v_1 &= -3 \times 10^6 \ \frac{0.0005}{100} \\
&= 15\, m/s \\
&= 54\, km/hr
\end{aligned}
$$

Being accelerated to > 50 km/hr in such a short time could hurt quite a bit. Even if Noah wasn't hurt in the acceleration, the deceleration would probably be quite disastrous.

4. The Lambda particle: outstanding in its (B) field

We can determine the speed of proton from the radius of its circular path

$R_p = p_p/e\,B = M_p\, v_p / e\,B$ (for non-relativistic case).

So, its speed

$$
\begin{aligned}
v_p &= R_p\, e\, B / M_p = (3 \times 10^{-2}\ \text{m})\ e\ (1\ \text{T}) / (938 \bullet 10^6\ \text{eV} \bullet c^{-2}) \\
&= 2.88 \times 10^6\ \text{m} \bullet \text{s}^{-1} = 0.00959\ c. \qquad (4.1)
\end{aligned}
$$

The assumption of non-relativistic proton is applicable here.

We apply the conservation law of energy and momentum:

$$
\begin{aligned}
E_\Lambda &= E_p + E_\pi \\
p_\Lambda &= p_p + p_\pi
\end{aligned}
$$

which can be written non-relativistically as

$$M_\Lambda c^2 + 1/2\, M_\Lambda v_\Lambda{}^2 = M_p c^2 + 1/2\, M_p v_p{}^2 + M_\pi c^2 + 1/2\, M_\pi v_\pi{}^2, \qquad (4.2)$$

$$M_\Lambda v_\Lambda = M_p v_p + M_\pi v_\pi. \qquad (4.3)$$

v_Λ can be eliminated by taking the square of (4.3) and substitute the term in (4.2)

$$M_\Lambda c^2 + 1/2\,(M_\Lambda)\,(M_p v_p + M_\pi v_\pi)^2 = M_p c^2 + 1/2\, M_p v_p{}^2 + M_\pi c^2 + 1/2\, M_\pi v_\pi{}^2$$

$$
\begin{aligned}
&(1115 - 938 - 140)\ \text{MeV} + (938 \times 0.00959 + 140\ v_\pi/c)^2/(2 \times 1115)\ \text{MeV} - \\
&\qquad 938/2\ \text{MeV}\ (0.00959)^2 - 140/2\ \text{MeV}\ (v_\pi/c)^2 = 0
\end{aligned}
$$

$$61.21\ (v_\pi / c)^2 - 1.13\ (v_\pi / c) - 36.99 = 0$$

So we can obtain $v_\pi = 0.787\ c = 2.36 \times 10^6\ \text{m} \bullet \text{s}^{-1}$. This is a relativistic problem and the assumption of non-relativistic pion is incorrect. Nevertheless, let us find out the radius of the pion path.

$$
\begin{aligned}
R_\pi = p_\pi / e\,B = M_\pi v_\pi / e\,B &= (140 \times 10^6\ \text{eV} \bullet c^{-1} \bullet 0.787) / e\ (1\ \text{T}) \\
&= 0.367\ \text{m} = 36.7\ \text{cm}
\end{aligned}
$$

The full relativistic treatment (especially for pion) gives the radius of 40.0 cm.

Use the following formulae:

$M_\Lambda c^2 + 1/2\, M_\Lambda v_\Lambda{}^2 = M_p c^2 + 1/2\, M_p v_p{}^2 + E_\pi$,

$M_\Lambda v_\Lambda = M_p v_p + p_\pi$,

$E_\pi{}^2 = M_\pi c^4 + p_\pi c^2$,

to solve for p_π and $R_\pi = p_\pi / e B$.]

5. Bohring after the truth

a) Applying Coulomb’s law:

$$F = k\, e^2/r^2 = (9.00 \times 10^9) \cdot (1.602 \times 10^{-19})^2 / (0.053 \times 10^{-9})^2$$
$$= 8.22 \times 10^{-8}\ \mathrm{N}$$

b) The centripetal force $F = m v^2 / r$. Hence the speed of the electron is

$$v = \sqrt{(F r / m)} = \{(8.22 \times 10^{-8}) \cdot (0.053 \times 10^{-9}) / (9.31 \times 10^{-31})\}^{1/2}$$
$$= 2.19 \times 10^6\ \mathrm{m \cdot s^{-1}} = 1/137\ c$$

6. Plasma, plasma, on the wall

a) Consider a slab of plasma, as in the question, which measures $x \times y \times z$. If the negative electrons are pushed off the positive protons to one side, by a tiny amount Δx, then there will be a thin slab of excess charge on either side — one positively charged and one negatively charged. The volume of each little excess slab will be:

$$(\Delta x) \times y \times z \qquad [1]$$

If the density of electrons is N [cm^{-3}], then the *amount* of excess charge will be

$$N\, e\ (\Delta x) \times y \times z \ = \ N\, e\ (\Delta x) \times A \qquad [2]$$

where A is the area of the side of the slab

In other words, what we have is something like a parallel-plate capacitor, with the two thin excess-charge slabs as the two charged plates of the capacitor.

b) For this, we can find the field between the two plates of a parallel-plate capacitor. In the middle, where there are both electrons and ions, the charges of each cancel each other out (unless you give them time to move around and redistribute themselves), so the net field in the middle is just what is produced by the thin slabs of excess (unbalanced) charge.

$$E = 4\, \pi\, k_c\, \sigma \qquad \text{where} \quad k_c = 9 \times 10^{-9}\ \mathrm{N\, m^2\, C^{-2}} \qquad [3]$$

σ = charge per unit area on each plate

[The easiest way to see this is to use Gauss's Law (if you know it), drawing little rectangular boxes with sides parallel to the plates, and having one side between the plates and the opposing side outside the plates. The field is perpendicular to the plates, by the symmetry of the situation, and it is quick to find the contribution each parallel plate makes to the overall field.]

Then from the second part of [2], we can find the charge *per unit area* of the two slabs of excess:

$$\sigma = (N\,e\,(\Delta x) \times A) / A = N\,e\,(\Delta x) \qquad [4]$$

and the field between the plates is:

$$E = 4\,\pi\,k_c\,N\,e\,(\Delta x) \qquad [5]$$

Now the charges in the middle see the field produced by the excess charges, and experience a force:

$$F = q\,E \qquad \text{which depends on q, +ve or –ve} \qquad [6]$$

This then gives us the force on each charge within the thin slab:

$$F = q\,E = q\,4\,\pi\,k_c\,\sigma = e\,4\,\pi\,k_c\,\sigma = e\,4\,\pi\,k_c\,(N\,e\,(\Delta x)) \qquad [9]$$

$$= 4\pi\,k_c\,N\,e^2\,(\Delta x)$$

c) The whole block of electrons in the diagram of the question is free to move (the ions, being more massive, have a certain 'right of weigh') in the electric field they see. Most of them see the electric field between the two plates, and this pulls them back onto the ion background. So the force on an electron between the charge-excess thin slabs is a *restoring force* against the separation of charge.

$$F_r = -\,k\,(\Delta x) \qquad \text{where } k = 4\pi\,k_c\,N\,e^2 \qquad [10]$$

This is the restoring force of a simple harmonic oscillator (SHO) — it means the electrons mostly will slosh back and forth past the ions, barring collisions, once they are released from their initial displacement. Since the ions are relatively massive, they hardly move, but the electrons oscillate sinusoidally, as does a SHO, with a frequency which depends on the electron mass:

$$\omega_p = \qquad \text{where } \varepsilon_o = \frac{1}{4\pi k_c} \qquad [11]$$

Then $\nu_p = \omega_p / 2\pi$ to get the result shown. This is called an *electron plasma wave*.

d) When an electromagnetic wave is incident on a plasma, the E-field can begin to drive electrons back and forth, causing something like the excess charge in the model above. The electrons then oscillate at their own frequency, given just above in [11].

If the EM wave is at exactly the same frequency, it is in resonance with the electron plasma wave, and oscillations can grow very large. This means that a large AC current flows back and forth in the plasma, and that current can radiate EM waves itself, cancelling the light that comes in and sending it back as a reflection. At what frequency does this happen for shortwave radio at λ = 30m?

$$c = \lambda \nu \;\; \Rightarrow \;\; \nu = c / \lambda = 3 \times 10^{8} \text{ cm} \bullet \text{s}^{-1} / 30 \text{ m} = 1 \times 10^{7} \text{ s}^{-1} \qquad [12]$$

$$\mathrm{v} = 1 \times 10^{7} = \frac{1}{2\pi}\sqrt{\frac{Ne^{2}}{\varepsilon_{o}\, m_{e}}} \qquad\qquad \mathrm{v}_{p} = \frac{1}{2\pi}\omega_{p}\frac{1}{2\pi}\sqrt{\frac{Ne^{2}}{\varepsilon_{o}\, m_{e}}}$$

$$\Rightarrow N = \frac{(2\pi \mathrm{v})^{2}\, \varepsilon_{o}\, m_{e}}{e^{2}} = 1.24 \times 10^{12} \text{ m}^{-3} = 1.24 \times 10^{6} \text{ cm}^{-3}$$

Compare this to gases at sea-level: 1 mole of ideal gas occupies 22.4 *l* @ STP,

$$22.4\, l = 22.4 \times 10^{3} \text{ cm}^{3}$$

$$\frac{6.022 \times 10^{23}}{22.4 \times 10^{3}} \text{cm}^{-3} = 2.69 \times 10^{19} \text{ cm}^{-3}$$

Much less dense than this — shortwave radio is usually reflected not from fully ionized gas at atmospheric pressure, but from a layer of *partially ionized highly rarefied* gas high up, in the ionosphere.