1996-1997 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 6: AC Circuits and Electronics

1. **What's a gigawatt? Thirty of 'em??**

a) Power lost per meter, with P = IV (i.e., power and voltage are pre-set)

$$P_{loss} = I^2 R = (P/V_{rms})^2 R = (3 \times 10^{10} \text{ W} / 110 \text{ V})^2 \bullet (1.5 \times 10^{-7} \text{ ohm} \bullet \text{m}^{-1})$$
$$= 1.1 \times 10^{10} \text{ W} \bullet \text{m}^{-1}$$

b) length for loss: 3×10^{10} W / 1.1×10^{10} W • m^{-1} = 1.1×10^{10} W • m = 2.7 m
(a pretty pathetic distance!)

c) power loss by Stefan-Boltzmann law:

$$P = \sigma A T^4$$

trivial power gain from 20°C room temperature, i.e., likewise:

$$P = \sigma A (293 \text{ K})^4$$

so that net:

$$P = \sigma A [(T^4 - (293 \text{ K})^4] \approx \sigma A T^4$$

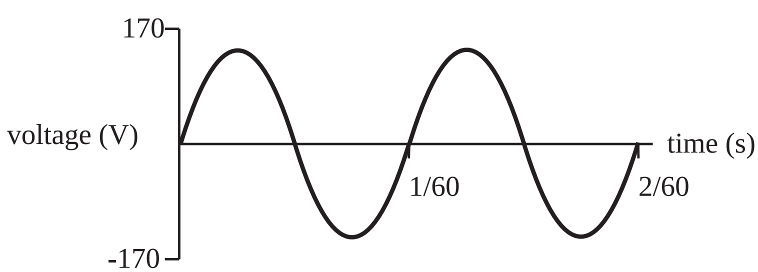
Area of surface of 2.7 m of cable is

$$A = 2\pi r L = 2 \bullet 3.14 (0.5 \text{ m}) 2.7 \text{ m} = 8.48 \text{ m}^2$$
$$T = (P/(\sigma A))^{1/4}$$
$$= (3 \times 10^{10} \text{ W} / (8.48 \text{ m}^2 \bullet 5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}))^{1/4}$$
$$= 15{,}800 \text{ K}$$

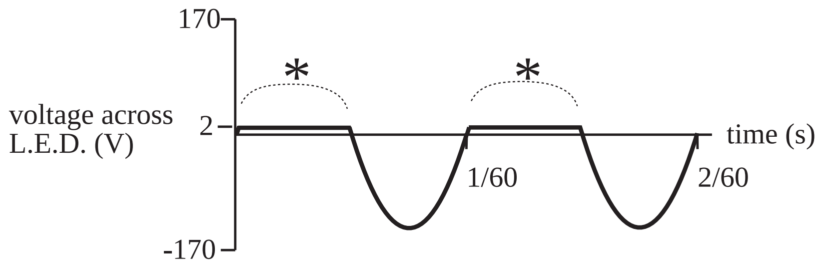
This doesn’t happen, basically because of Nikolai Tesla: with AC current we can easily manage step-up transformers, so the power is delivered at much higher voltages (like 250,000 V, I recall) with smaller losses, and then stepped-down several times, the last time quite near the delivery site. As well, the total diameter of cables is far in excess of 1m.

2. **Waffling on the issues ... the issue of waffles**

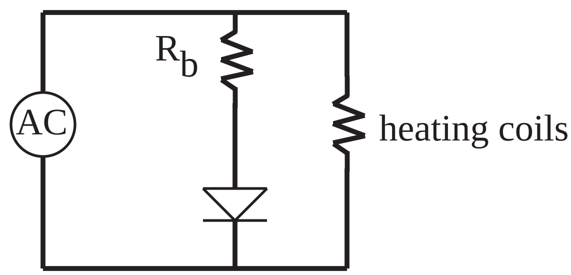
a) Normal household voltage in about 120 V (*rms*) which corresponds to 170 V peak-to-peak



b) The LED emits light only when it is forward-biased, as indicated by the asterisks in the graph below. Thus it flashes at 60 Hz.

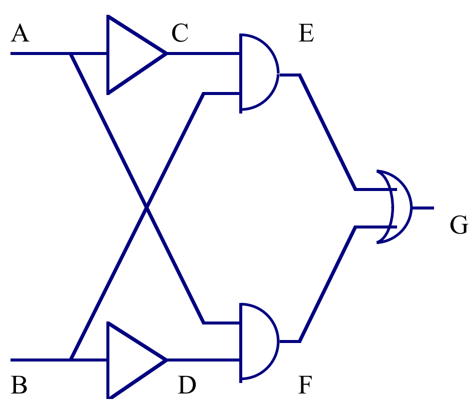


c) One possible circuit is shown at right, where
R_b is a large resistor. This reduces the amount of current that flows through the LED.



3. **2B ∨ (~2B) → Q! [... to be or not to be, that is the question!]**

The circuit of this combination is



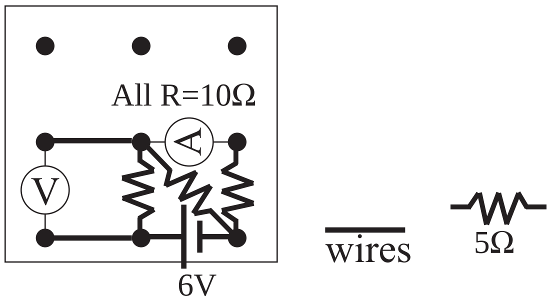
Boolean table

Input		intermediate nodes				**Output**
A	**B**	**C**	**D**	**E**	**F**	**G**
T	**T**	F	F	F	F	**F**
T	**F**	F	T	T	F	**T**
F	**T**	T	F	F	T	**T**
F	**F**	T	T	F	F	**F**

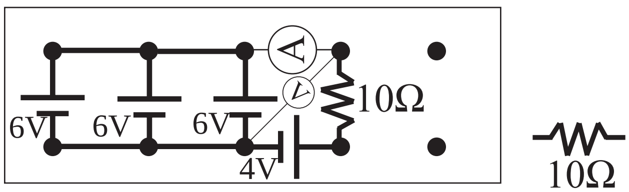
Clearly, this circuit is an exclusive OR gate (XOR).

4. **Connect the dots!**

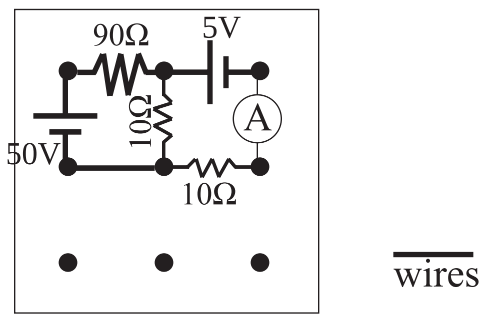
a) Set voltage=4V, I=.2A



b) Set voltage=6V, I=.2A



c) Set current I=0A
(note: short-circuiting batteries is baaaaaad!)



5. **Impeding one's reactance**

a) As they are seen by V_{IN}, the impedances are in series. Therefore, the total impedance is

$$Z_T = R + (i\omega\, C)^{-1}$$

Similarily, as seen by V_{OUT}

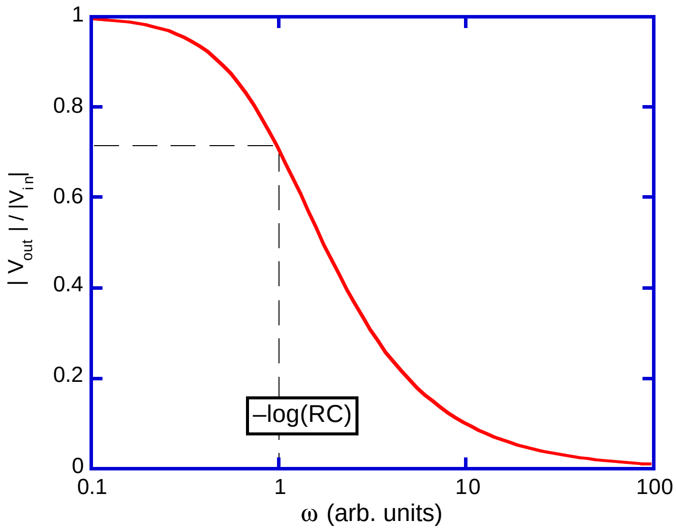
$$Z = \frac{1}{i\omega C}$$

b) The ratio of V_{OUT} to V_{IN} is given by the ratio of the impedances.

$$\frac{V_{OUT}}{V_{IN}} = \frac{Z_T}{Z} = \frac{1}{i\omega C}\left(R + \frac{1}{i\omega C}\right)^{-1} = \frac{1}{1+i\omega CR}$$

$$\frac{|V_{OUT}|}{|V_{IN}|} = \left[\frac{V_{OUT}}{V_{IN}} \times \left(\frac{V_{OUT}}{V_{IN}}\right)^*\right]^{\frac{1}{2}} = \left[\frac{1}{1+i\omega CR} \times \frac{1}{1-i\omega CR}\right]^{\frac{1}{2}} = \left[\frac{1}{1+\omega^2 C^2 R^2}\right]^{\frac{1}{2}}$$

so, $|V_{OUT}| = |V_{IN}|\left[\frac{1}{1+\omega^2 C^2 R^2}\right]^{\frac{1}{2}}$



c) for $\omega \ll \frac{1}{RC}$, all signal is passed.

for $\omega \gg \frac{1}{RC}$, all voltage is dropped

So this is a filter which allows only low frequency *AC* signals to pass. The cutoff frequency can be changed by adjusting *R* and *C*

6. Say 'cheese'! (... Zap !!)

a) Each element has a charge-dependent voltage associated across it:

resistor: $\Delta V_R = IR$

capacitor: $\Delta V_C = q/c$

inductance: $\Delta V_R = -L\,\frac{dI}{dt}$

Since $I = \frac{dq}{dt}$, and since the potential drops around the circuit must sum to zero, we can establish a relation in terms of change of *charge*. Be careful about the signs:

- choose allocation of +q/–q on capacitor
- +q plate is at higher potential, so sketch current direction (flow of +ve charge) consistent with dropping potential

Then the current carries charge **away** from +q

so $I = \frac{dq}{dt}$

$$q/c + L\,\frac{d^2q}{dt^2} + \frac{dq}{R} = 0$$

or

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$ a ‘second-order ordinary differential equation’

Then

$$\Delta V_c\ (rise) + \Delta V_L\ (drop) + \Delta V_R\ (drop) = 0$$

$$q/c - L\frac{dI}{dt} - IR = 0$$

b) assume the form $V(t) = V_o e^{-t/\tau}$

so $q(t) = CV_o e^{-t/\tau}$

and substitute into differential equation

$$\{LCV_o\left(\frac{1}{\tau}\right)^2 + RCV_o\left(-\frac{1}{\tau}\right) + V_o\}e^{-t/\tau} = 0$$

$e^{-t/\tau}$ 0 must mean

$$LCV_o\left(\frac{1}{\tau}\right)^2 - RCV_o\left(\frac{1}{\tau}\right) + V_o = 0$$

or multiplying by τ^2 both sides

$$\tau^2 - RC\,\tau + LC = 0$$

then

$$\tau = \frac{-b \pm \sqrt{b^2 - 4ac}}{2ac}$$ quadratic formula

$$= \frac{+RC \pm \sqrt{R^2C^2 - 4LC}}{2}$$

$$= RC\left(\frac{1 \pm \sqrt{1 - \frac{4L}{R^2C}}}{2}\right)$$

The solution should be a **simple exponential**, meaning that the discrimant (root here) should be non-negative (else get imaginary numbers, which lead to cosine/sine terms — oscillations)

$$1 - \frac{4L}{R^2C} > 0 \quad \text{or} \quad 4L < R^2C$$

$$L > \frac{R^2 C}{4}$$

Then there are two solutions because of ±

$$q(t) = CV_o\,(a_1 e^{-t/\tau 1} + a_2 e^{-t/\tau 2})$$

$$a_1 + a_2 = 1$$

$$\tau_1 = RC\left(\frac{1+\sqrt{1-\frac{4L}{R^2 C}}}{2}\right)$$

$$\tau_2 = RC\left(\frac{1-\sqrt{1-\frac{4L}{R^2 C}}}{2}\right)$$

One of these τ_1, τ_2 is always the longer time — the fastest decay is as $\tau_1 \rightarrow \tau_2$, i.e. as

$$1-\frac{4L}{R^2 C} \rightarrow 0^+ \qquad \textit{i.e.}\text{, from above}$$

then $\tau_1, \tau_2 \rightarrow \frac{RC}{2}$

$R = 1\Omega$, $C = 200\mu F \quad \Rightarrow \; T = 100\mu s$

c) This was rather (too?) tricky: one way to see the solution is to start with τ_1, τ_2 and the condition

$$0 = I(t) = -\frac{dq(t)}{dt} = -CV_o\left(-a_1\left(\frac{1}{\tau_1}\right)e^{-t/\tau_1} - a_2\frac{1}{\tau_2}e^{-t/\tau_2}\right)$$

so $I(t = 0) = -CV_o\left(-\frac{a_1}{\tau_1} - \frac{a_2}{\tau_2}\right)$

$a_1\tau_2 = -a_2\tau_1$

but $a_1 + a_2 = 1$,

$\Rightarrow a_1 = \frac{\tau_1}{\tau_1 - \tau_2}$

$a_2 = \frac{-\tau_2}{\tau_1 - \tau_2}$

$$I(t) = +CV_o\left(\frac{1}{\tau_1 - \tau_2} e^{-t/\tau_1} - \frac{1}{\tau_1 - \tau_2} e^{-t/\tau_2}\right)$$

$$= \frac{CV_o}{(\tau_1 - \tau_2)}(e^{-t/\tau_1} - e^{-t/\tau_2})$$

as $L \to \frac{R^2C}{4}$ and $\tau_1 \to \tau_2$ this gives a $\frac{0}{0}$ indeterminacy, which we can work with l'Hôpital's rule:

numerator $$\frac{\mathrm{d}}{\mathrm{d}\tau_1}(e^{-t/\tau_1} - e^{-t/\tau_2}) = e^{-t/\tau_1}\left(-t\left(-\frac{1}{\tau_1}^2\right)\right)$$

demominator $$\frac{\mathrm{d}}{\mathrm{d}\tau_1}(\tau_1 - \tau_2) = 1$$

So the answer

$$I(t) = CV_o \frac{t}{\tau_2} e^{-t/\tau}$$

$$= \frac{CV_o}{(RC)^2}\ t e^{-t/\tau}$$ (close enough)

$$I(t) = \frac{V_o}{R^2C}\ t e^{-\frac{t^2}{RC}}$$

$$E = \frac{1}{2}CV^2$$

$$= \frac{1}{2} 200\mu\text{F}\ (2.5 \times 10^3\ \text{V})^2$$

= 625 J of energy.