1997-1998 Physics Olympiad Preparation Program

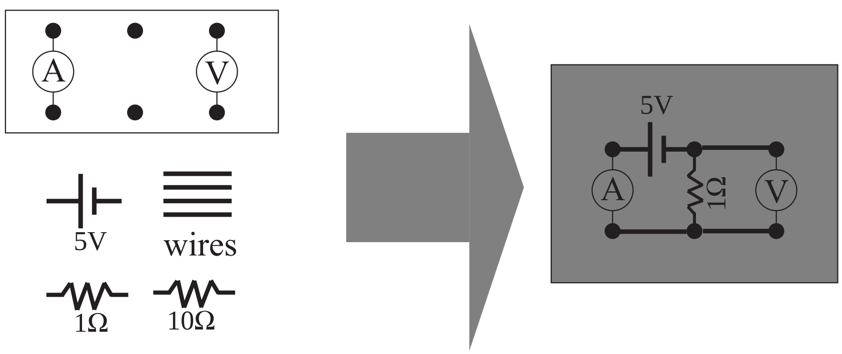
— University of Toronto —

Solution Set 1: General

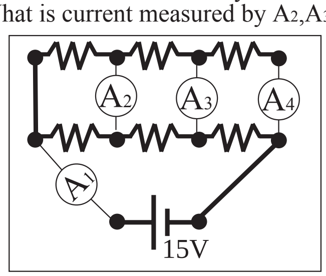
1) Connect the dots!

These solutions may not be unique:

e.g.) Set voltage=5V, current: I=5A using only the given components.

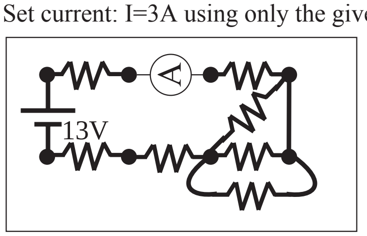


a) Set current measured by A_1 to be I=2A using only the given components.
What is current measured by A_2,A_3 and A_4?

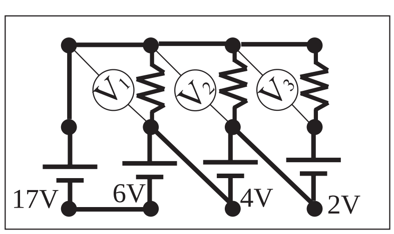


A_1=2A
A_2=A_3=0A
A_4=1A

b) Set current: I=3A using only the given components.



c) Set voltages: V_1=11V, V_2=7V, and V_3=5V using only the given components.



The voltages are 11V, 7V and 5V unless the unknown resistors actually have a lower resistance than the ideal voltage sources (not possible since they are 'ideal') or the wires themselves.

2) Out with the bad air, in with the good...

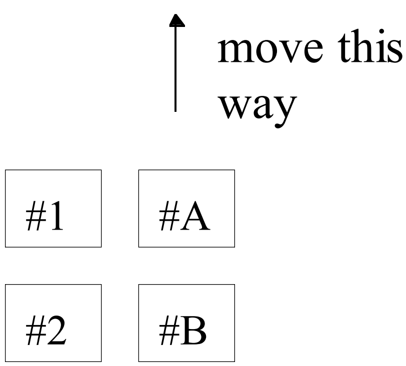
The answer is surprising (I was surprised) — it is possible to cool the hot water to almost 0°C, and heat the cold water to almost 100°C!

To do this, one builds a 'heat-exchange engine'. The following examples illustrate the concept:

1) Using 2 dishes: #1 holds warm water and #A holds cold water. Bring #1 and #A in thermal contact - because the dishes are perfect and no heat is lost, they will reach a thermal equilibrium of 50°C - i.e., the cold water will be heated to 50°C, and the hot water will be cooled to 50°C.

2) Using 4 dishes: #1 and #2 each hold 1/2L of warm water, and #A and #B hold 1/2L of cold water each. Bring #2 in contact with #A - #2 has a temperature of 50°C and #A has a temperature of 50°C. Now, bring #A in contact with #1. #1 has a temperature 50°C, and #A is up to 75°C. Now, bring #B in contact with #2. #B is at 25°C and so is #2. Bring #B in contact with #1. #B and #A are at 75/2°C. Now, mix #A with #B to obtain a temperature of (37.5+75)/2=56.25°C, and mix #1 with #2 to obtain (50+37.5)/2=43.75°C.

Thus, the cold water is heated up to 56.25°C, and the warm water is cooled to 43.75°C!



A schematic diagram, extendible to more dishes, is shown at left.

This process may be continued (i.e., more dishes may be used) and in the limit of the dish sizes going to 0, we can reach 0°C and 100°C for the warm and cold water, respectively. Of course, no dishes can be that small, and that's why I wrote "almost 0°C" and "almost 100°C" at the beginning. But what if the dish were molecule-sized?

3) Waves à la mode

In either case, the waves must have nodes at the reflecting surfaces (either the walls or the mirrors), so that half-waves fit in, so too do full waves, waves-and-a-half, etc. So, whatever wavelengths are possible, some integer times the number of half-wavelengths will equal the length between reflections. The rest depends on the speed of the wave, c — sound or light — because $\nu \lambda = c$.

a) Speed of sound: 331 m s^{-1} depending on temperature and altitude (air density). So Pavarotten's low note is 128 Hz, or a wavelength of 2.59 m; his high note is 1024 Hz, or a wavelength of 0.323 m. If half-waves of these fit into the shower dimensions, 1.5 m × 1 m × 2 m, then doubles of the dimensions are possible wavelengths, together with any integer fractions, i.e.:

In x: 3.0m (n=1) is too long for Pavarotten; 1.5m, 1m, 0.75m, 0.6m, 0.5m, 0.429m, 0.375m, 0.333m, (n=2, 3, 4, 5, 6, 7, 8, 9) are OK; 0.3m (n=10) is too short. So in that dimension, he can sing 8 different notes that will resonate.

In y: 2m, 1m, 0.666m, 0.5m, 0.4m, 0.333m, (n=1, 2, 3, 4, 5, 6) are OK; 0.286 (n=7) is too short. So, 6 resonant frequencies.

In z: 4m, (n=1) is too long; 2m, 1.33m, 1m, 0.8m, 0.666m, 0.571m, 0.5m, 0.444m, 0.4m, 0.363m, 0.333m, (n=2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) are OK; 0.307m (n=13) is too short. So, 11 resonant frequencies.

So it looks like 8 + 6 + 11 = 25 resonant frequencies, but really 0.333m has been counted three times instead of once, as has 1 m, and 0.5m. The resonant wavelengths in this picture are: 2m, 1.5m, 1.33m, 1m, 0.8m, 0.75m, 0.6m, 0.666m, 0.571m, 0.5m,0.444m, 0.429m, 0.4m, 0.363m, 0.375m, 0.333m, or 16 different frequencies resonant in x, y or z.

This question is trickier than I intended, because it is possible to make other modes, not just along x, y or z, but on different angles which combine x & y, say or y & z, or all three, and still have nodes at all walls encountered. Bonus marks for such answers.

b) A half-wave of 1 μm light is 0.5 μm long — an integral number of these must fit into the laser cavity, which is 0.8 m. Thus n ranges around the value $(0.8\text{ m})/(0.5 \times 10^{-6}\text{ m}) = 1\,600\,000$. We can use any wavelength which is resonant and which falls between (1–0.005) μm and (1+0.005) μm. From the lower limit we get $(0.8\text{ m})/((1/2) * 0.995 \times 10^{-6}\text{ m}) = 1\,608\,040.2$ which means $n \le 1\,608\,040$; From the upper limit we get $(0.8\text{ m})/((1/2) * 1.005 \times 10^{-6}\text{ m}) = 1\,592\,039.8$ which means $n \ge 1\,592\,040$. In this range are 16 001 modes, with very, very slightly different wavelengths, which all contribute to the laser output.

4) Opposites attract, but a narcissist always loves himself...

Since the particle is very close to the cylinder, compared to the cylinder radius, we can approximate it well by an infinite flat conducting sheet (just as the surface of the earth seems practically flat and infinite to us tiny particles).

Because of the external field from the orbiting particle, charges in the conductor can and will rearrange themselves at the surface of the cylinder until there is no net field inside the conductor and the surface returns to an equipotential. The charge in orbit around the wire does see this charge distribution — though the distribution creates a 'cancellation' field inside the conductor to exactly negate the field of the orbiting charge, outside the conductor the orbiting charge sees the field of the rearranged charges on the surface.

What is that field? Well, to make a flat equipotential surface in space, all it would take is a sort of 'Doppleganger' charge (not a standard term) — a charge of *exactly opposite value at exactly the same distance* on the *other* side of the plane. The electrostatic potential from one charge exactly balances the electrostatic potential from the other! This is known as the *method of images*, and it applies in fact for any shape of surface — flat, cylindrical, spherical — if you know how to find the virtual image from such a reflection. It is true that once you construct the equipotential for the surface of the conductor, then all the fields outside are exactly as the charge distribution on the conductor surface would create.

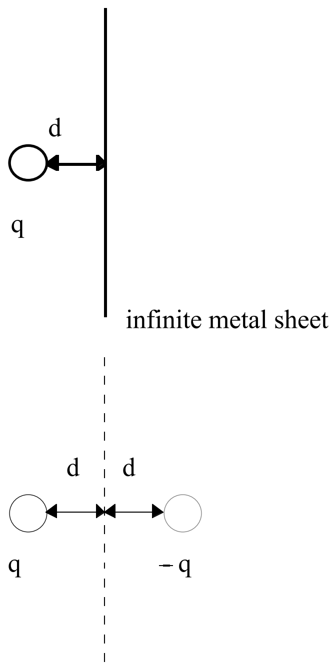


Fig. 4.1: The scenario is physically equivalent to removing the metal sheet and replacing it by a point charge -q a distance 2d away [the resulting electric potentials cancel].

From this simplified scenario above, we can calculate the force acting on the particle:

$$F = k\frac{q^2}{4d^2} \qquad (k = 9*10^9\ Nm^2/C^2)$$

This force must provide the centripetal acceleration needed for the circular motion of the orbit:

$$F = m\omega^2(r+d) \approx m\omega^2 r \quad \text{since } d << r.$$

Hence,

$$\omega^2 = \frac{kq^2}{4d^2 mr}$$

$$T = \frac{2\pi}{\omega} = \frac{4\pi d}{q}\sqrt{\frac{mr}{k}}$$

where $T = 1/\nu$ is the period

5) Fermi Questions

There are many different ways to answer these questions, usually using little bits of information you know, but the main point is how plausible your argument is. Here are some examples of solutions.

a) I start by assuming that since it's a business, for the professional tuners, they aren't just filling up 10% of their schedules. I figure that a tuner's schedule is on average at least 80% full (maybe, from the time it takes before one can come, they are more full than this. So each tuner spends, say, 32 hours per week actually travelling to houses and tuning pianos. If it takes about three hours to travel and tune (depending on where the customer is, and whether the tune-up is minor or major — not the keys, the amount of work!), they do about 10 jobs per week, or 500 per year, give or take 10–15%. Now how many pianos get tuned each year in Toronto? Take a population of roughly 3 million people in Metro. Averaging families with two kids (on average) together with single-living folk, maybe there are 3.3 people per household, at a guess, or something like 900,000 households. Not all households have pianos, of course; let's guess that 10% do, so something like 90,000 pianos overall. Now, music-lovers may have their pianos tuned regularly, but many might go 3 or more years between tuning. So, in a year, about 30,000 pianos might get tuned. If each tuner is doing 500, this would be enough business for about 60 tuners altogether in Metro.

In the Yellow Pages, between the 'tuning' section, and the sales ads that mention tuning services, I count a little over 30 firms that will tune your piano. Perhaps some of these employ more than one tuner, but in an any event, for Fermi-type questions a factor of two is pretty good.

b) Well, this would depend on how close two tires on wheels have exactly the same diameter. I'd be surprised if they were within 2 mm of the same diameter after mounting on the wheel rims, just given manufacturing tolerances — then their circumferences would differ by $2\pi\Delta r$, or about 1.26 cm. Most racing-style bikes have wheels about 27-inch diameters, so about 215 cm circumference. To get half a revolution (107.5 cm) out in steps of 1.26 cm per revolution would take about 85 revolutions. At 215 cm per revolution, this would then be about 183 m.

Another interesting way to reckon: about 60% of a rider's weight is on the back wheel, and 40% on the front. So each tire will compress to make a 'footprint' on the road, but the rear tire will have to push further down. If a tire has 90 psi pressure inside, then you can figure out how big each 'footprint' is, because the pressure in the tire times the area of the footprint has to give the force on each wheel (bike tires are hardly at all stiff — air pressure give virtually all the support. From the cross-section size of a tire you can figure out roughly how much each tire pushes down, and therefore the difference, for each tire, for the distance between the axle and the ground, then proceed as above.

c) Fifty spots for shoppers, and shoppers spend an average amount of time in Eaton's. If it is just the store, a shopper may spend half an hour, on average, in the store. If it is part of a mall, they might spend two hours average. So with fifty drivers, the average time between people leaving would be (30 minutes)/50 = about 40 seconds between times *someone* leaves. In a mall, it might be more like (120 minutes)/50 = 2.4 minutes between times that someone leaves.

Now, it may turn out that at the other end of the lot, Michael Schumacher is waiting for a spot too — you know how that turns out!

d) Hmm, let's say that in Canada smokers are about 35% of the population (it might be a little higher), and that an average smoker smokes just less than a pack (15, out of 20 cigarettes/pack) a day (a wild guess). So, with nearly 30 million people, 10.5 million smokers, 210 million cigarettes per day (big money!). Let's assume that smokers indoors always use an ashtray to discard their butts, but that this rarely happens when outside. What fraction of cigarettes are smoked indoors? Fewer, these days, judging by the lineup in the cold outside this no-smoking building. Let's say that smokers are outside, smoking 1/4 of their cigarettes, and that there is an ash-bin there half of the time then. So about 1/8 of all cigarette-butts might go on the ground, which is 26 million butts per day, or about 9 billion butts per year. I don't know the weight of a cigarette butt, but it would float on water (1 g cm^{-3}) with only about 20% submerged, so let's say 0.2 g cm^{-3} in density, and 1 cm diameter $\times$ 3 cm long when discarded, so 2.4 cm^3, or about one-half gram in weight (sounds low!). So, 4.5 billion grams, or 4.5 million kg of butts per year, give or take a factor of two or so!

e) Let's say that the half-time action all gets started in the first five minutes: fridge-doors open for snacks and drinks (that little 40W light-bulb, plus the cold air spilling out & the fridge turning on), kettles getting plugged in for hot water, water running for tea & coffee, toilets flushing.

A kettle coming on is about a kilowatt; a fridge cooling-compressor running is probably the same; toilets flush a few gallons each and take a few minutes to fill. Just from these things alone we might calculate the increased load.

If something like 25% of all households with TVs turned on watch the Grey Cup in Canada, and 50% of all households have TVs turned on at that time on the weekend, then with the same 3.3 people per household, from (a) and nearly every household with a TV, and 30 million people in the country, then that's 1.14 million households watching the Grey Cup. With, say, 2 people per household watching, then perhaps one goes to the kitchen and one to the bathroom, or 1.14 million events of electricity or water use, each, in those five minutes.

For water, this means about 3 gallons/3 minutes, or 1 gallon/minute extra load per use, or 1.14 million gallons per minute. For electrical power, this means, for a little while, about 1 kW extra per household, for a total of 1.14 GW of extra load — roughly comparable to the output of another whole generating station.

In New York — where the living is dense — on Superbowl Sunday this can be a real problem!

6) 'Stacking the Deck — Friction in the House of Cards,' by Kitty Kelly

a) We could set up a whole bunch of equations, but that's too much work, so let's try to argue our way through this.

Note that there are 2 reasons why the cards could be unstable:

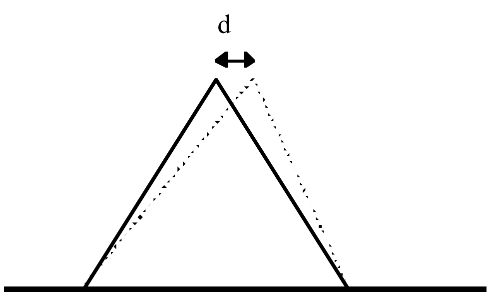
1) their bottoms slip

2) their tops slip

Now note that at a very small θ (the cards are almost vertical) the normal force to the ground is at its highest, the lateral component of that force is least, and the cards don't slip on the ground.

Now consider a large θ: the cards are almost horizontal — here the situation is opposite - the cards don't slip against each other, but do against the ground.

Assume that F_{min} is horizontal. In any case, when you're stacking cards you're most likely to push them over from the side, rather than "squishing" them [F_{min} vertical].



Now, note that the system is in unstable equilibrium — an unbalanced force in the horizontal direction will cause the cards to fall. So, the structure must withstand a sideways force of F_{min} without moving. For greatest instability, F_{min} will be applied at the tips of the cards. At that point, the frictional forces will compensate for F_{min}. Note that the greater θ, the greater the normal force acting on each card, and hence the frictional force between them is greater. Thus, F_{min} is greater. At most, θ is the angle when the bottoms of the cards start slipping.

We get:

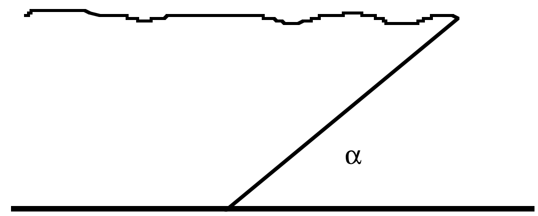
$$\mu_1 = 1/2\text{arccot}((180-\theta)/2), \text{ or } \theta = 180 - 2\cot(2\mu_1).$$

[see next answer].

PART II –

b) I am sure that there are millions of ways of doing this, but here's a fairly accurate one:

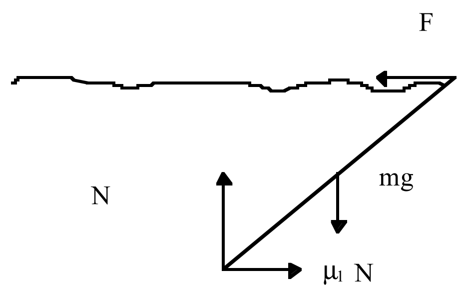
Attach a piece of string to the top of the card. Now tilt it at an angle - hold onto the string and keep lowering the card until it is about to slip. Note that angle (between card and ground - call it α).



The physics of the measurement is shown in the next figure, below.

From force balance:

$$N = mg$$
$$F = \mu_1 N$$



Taking torques around point of contact of card and ground:

$$mg\cos(\alpha) = F\sin(\alpha)$$
$$\mu_1 = \frac{1}{2}\operatorname{arc}\cot(\alpha)$$

One must repeat the experiment a number of times (at least 10) to get a reasonable answer - random errors are dominant here. The error can be found by taking the standard deviation (easiest with a calculator).

My answer: $\alpha = (55 \pm 5)^\circ \Rightarrow \theta = (70 \pm 6)^\circ$ [for the card and the table]

$\mu = 0.350\ldots$, but what's the error? There exist error propagation formulas, but they are complicated. It is always true that the percent error is approximately equal - i.e., $\Delta\theta/\theta = 6/70 = \Delta\mu/\mu$, from which $\Delta\mu = 0.03$. In standard form, $\mu = (0.35 \pm 0.03)$. [note that we only keep one significant digit for the error and round the answer to that decimal place!]

The answer (θ) seems reasonable, as a quick experiment shows.