1997-1998 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 2: Mechanics

1) The Pluto Problem

Kepler's Second Law results from the conservation of angular momentum of the planet about the sun. According to this law, the straight line joining the sun and a given planet sweeps out equal areas in equal intervals of time. Thus, the ratio of the time interval t, during which the Pluto's elliptical orbit is situated inside the Neptune's circular orbit, to the Pluto's orbital period T is equal to

$$t/T = S/(\pi ab)$$

The denominator of the right side is the area of the ellipse representing the Pluto's orbit, and the numerator S is the area of the sector of subtended angle 50 degrees. We can estimate the value of S as the area of a circular sector $(50/360)\pi R^2$, where for more accuracy we can substitute R by $(r_{min} + R)/2$. Therefore,

$$t = (50/360) \times (((r_{min} + R)/2)^2 \times T)/(ab) = (5/36) \times ((4.4 + 4.25)/2)^2 \times$$

$$(248)/(5.9 \times 5.73) = (5/36) \times (18.71 \times 248)/(5.9 \times 5.73) = \text{approx. } 20 \text{ years.}$$

It turns out that the question was in error, and Pluto became closer to the sun in **1979**, not 1969. Therefore, Pluto will be again the ninth planet from the sun in 1999.

[http://seds.lpl.arizona.edu/nineplanets/nineplanets/pluto.html]

2) Free falling, at $4.50 a throw

a) Let l be the length of a non-stretched rope ($l + h < H$), k is its elasticity constant, m the mass of a person who is jumping down and v is a value we are looking for. It is obvious that at the equilibrium height h your velocity had the maximum value. We can write the following three equations:

1) the equilibrium condition at the height h :

$$mg = k\,(H - l - h) \qquad [2.1]$$

2) the energy conservation law at the height h :

$$mgH = mgh + (k/2)(H - l - h)^2 + (1/2)mv^2 \qquad [2.2]$$

3) the energy conservation law on the ground level :

$$mgH = (1/2)\,k\,(H - l)^2 \qquad [2.3]$$

Divide [3] by [1] and obtain

$$l = \sqrt{H\,(H - 2h)} \qquad [2.4]$$

Then rewrite [2] using [1] in the following form :

$$mg\,(H - h) = (mg/2)\,(H - l - h) + (mv^2/2)$$

and get

$$v = \sqrt{g\,(H - h + l)} \qquad [2.5]$$

or using [4],

$$v = \sqrt{g\,(H - h + \sqrt{H\,(H - 2h)})} \qquad [2.6]$$

If $H = 25$ m, from [6] we have $v = 16.2$ m/sec or 58 km/hour. If $H = 50$ m, then $v = 28$ m/sec or 101 km/hour.

b) Only the third equation will be different compared to case a) :

$$mg\,(H - h_1) = (1/2)\,k\,(H - l - h_1)^2 \qquad [2.3']$$

Hence,

$$l = \sqrt{H\,(H - 2h) + h_1\,(2h - h_1)} \qquad [2.4']$$

Use [4′] in [5] and get the new value for v. If $H = 25$ m, then $v = 17.1$ m/sec or 61.4 km/hour. If $H = 50$ m, then $v = 28.2$ m/sec of 101.6 km/hour.

c) Elasticity constant can be found from [1] : $k = (mg)/(H - l - h)$.

For $H = 25$ m, $h = 10$ m, $m = 100$ kg and $l = 11.18$ m (from [4]), we have $k = 261.8$ H/m. For $H = 50$ m, $h = 10$ m, $m = 100$ kg and $l = 38.73$ m (from [4]), we have $k = 787.4$ H/m (the rope must be 3 times stronger for the height $H = 50$ m compared to the one for the height $H = 25$ m).

3) 'Bob's your uncle', or sometimes he's a *simple harmonic oscillator (SHO)*

a)

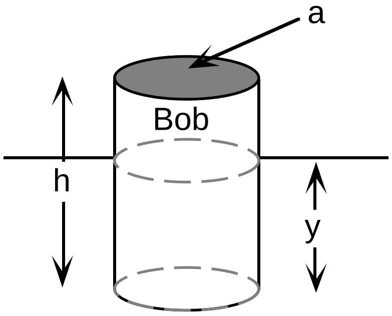
submerged length of the cylinder	Y
force of gravity on the cylinder:	$F=m_{cyl}g$
volume of water displaced by the cylinder	Ya

So the force of gravity on the cylinder = $Ya\rho_{water}g$

At equilibrium, the two forces are equal, so:

$$m_{cyl}g = Ya\rho_{water}\,g$$

$$Y = \frac{m_{cyl}}{\rho_{water}\pi a}$$



b) For a small displacement from equilibrium, y, there is a small difference between the buoyant force and gravity. Assume we push the cylinder slightly deeper into the water than the equilibrium point. Then the *increase* in the buoyant force is an excess force, in the amount:

$$F = -mg = -ay\rho_{water}\,g$$

and likewise if we pull the cylinder up, the excess is in the opposite direction.

c) (Note: in this section, A is acceleration, while a is the cross sectional area of Bob)

$$F = m_{cyl}A = m_{cyl}\frac{d^2y}{dt^2} = -\rho_{water}ayg \qquad [3.1]$$

so,

$$\frac{d^2y}{dt^2} = -\frac{\rho_{water}ag}{m_{cyl}}y. \qquad [3.2]$$

d) Just by substituting, it is easy to show that $y(t) = y_0 \cos(\omega t+\phi)$ is a solution of *any* equation of the form

$m(d^2y/dt^2) = -ky$,

as in part (c). We will use this same form in future POPTOR problems!

The maximum/minimum value that y can take occurs when $\cos(\omega t+\phi) = \pm 1$. So, y_0 is the maximum amplitude of Bob, measured from the equilibrium point.

If we let go of Bob at an amplitude y_0, at time t=0, then this should be his maximum amplitude. Then

$$y(t=0) = y_0\cos\phi = y_0 \qquad [3.3]$$

holds, whether we start off by lifting Bob or by pushing him down a little. This requires $\cos(\phi) = 1$, which is true as long as we make $\phi = n\,2\pi$, where n is any integer.

For convenience, we simply set $\phi = 0$. (If however t=0 is chosen at some other point in Bob's oscillation, we will need ϕ to take on some other value.)

$$\frac{dy}{dt} = -\omega y_0 \sin(\omega t + \phi) \qquad [3.4]$$

To determine ω for Bob, note that,

$$\frac{d^2y}{dt^2} = -\omega^2 y_0 \cos(\omega t + \phi) = -\omega^2 y = -\frac{k}{m} y \qquad [3.5]$$

so, $\omega = (k/m)^{1/2}$. From part (c),

$$\frac{k}{m_{cyl}} = \frac{\rho_{water} a g}{m_{cyl}} \qquad [3.6]$$

therefore,

$$\omega = \sqrt{\frac{\rho_{water} a g}{m_{cyl}}} \qquad [3.7]$$

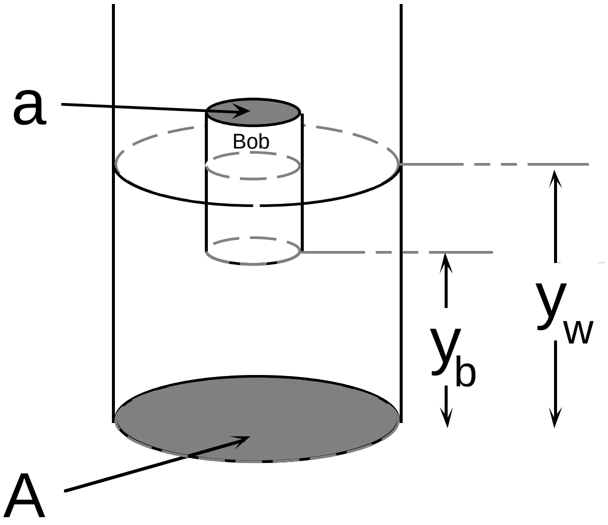
e) F = buoyancy + gravity

$= \rho_w g a(y_w - y_B) - m_B g$

at equilibrium F = 0, so

$0 = \rho_w g a(y_{wo} - y_{Bo}) - m_B g$

$m_B g = \rho_w g a(y_{wo} - y_{Bo})$



where y_{wo} and y_{Bo} are equilibrium (rest) values.

In general,

$m_B \bullet$ accel'n = F,

so,

$m_B \ddot{y}_B = \rho_w g a(y_w - y_B) - m_B g$, but we substitute for $m_B g$ the value from equilibrium:

$$\begin{aligned} m_B \ddot{y}_B &= \rho_w g a(y_w - y_B) - \rho_w g a(y_{wo} - y_{Bo}) \\ &= \rho_w g a((y_w - y_{wo}) - (y_B - y_{Bo})) \\ &= \rho_w g a(\Delta y_w - \Delta y_B); \qquad \Delta y_w \equiv y_w - y_{wo} \end{aligned}$$

since $(\ddot{y}_{Bo}) = 0$, we have

$\ddot{y}_B = (\Delta\ddot{y}_B)$, (where by $(\Delta\ddot{y}_B)$ I mean the second derivative in time of (Δy_B)).

Thus,

$$m_B(\Delta\ddot{y}_B) = \rho_w g a(\Delta y_w - \Delta y_B) \qquad [3.9]$$

The overall volume of water in the tank is constant, and this is the same value at equilibrium:

$$V = A\cdot y_B + (A - a)\,(y_w - y_B) = \text{const}$$

$$= A\,y_{Bo} + (A - a)\,(y_{wo} - y_{Bo}) \qquad [3.10]$$

thus

$$A(\Delta y_B) + (A - a)\,(\Delta y_w - \Delta y_B) = 0$$

or

$$(A - a)\,(\Delta y_w) + a(\Delta y_B) = 0 \qquad [3.11]$$

Substituting this into $(A - a) \times$ I we get

$$m_B(A - a)\,(\Delta\ddot{y}_B) = \rho_w a g[(A - a)\,\Delta y_w - (A - a)\,\Delta y_B]$$

$$= \rho_w a g[-a\Delta y_B - (A - a)\,\Delta y_B]$$

$$= \rho_w a g[-A\Delta y_B\,]$$

So

$$m_B(\Delta y_B) + \frac{aA}{(A - a)} g\rho_w \Delta y_B = 0 \qquad [3.12]$$

Thus

$$\omega_o^2 = \frac{aA}{(A - a)}\frac{g\rho_w}{m_B} \qquad [3.13]$$

Note: as $A \to \infty$, i.e., as the cylinder becomes huge, $\omega_o^2 \to \frac{a g \rho_w}{m_B}$ as above!

$$T = 2\pi\sqrt{\frac{(A - a)}{aA}\frac{m_B}{g\rho_w}} \qquad [3.14]$$

i.e., the period is smaller if the water level also rises and falls.

4) Full of fury, and signifying nothing...

a) The system will annihilate. There is a net force of

$$\frac{3kee}{2a^2} \text{ [towards centre of square]} \qquad (k = \frac{1}{4\pi\varepsilon_o}).$$

on each particle

To see this, consider any particle — the net force is, by symmetry, directed towards the centre of the square. It has a magnitude of:

$$F_{NET} = \frac{kee}{a^2} + \frac{kee}{a^2} - \frac{kee}{2a^2} = \frac{3kee}{2a^2}$$

Since this result holds for any a, and the masses of each particle are equal, and they start from rest, they will accelerate toward the centre of the square at the same rate. Since they all start from rest, their velocities along the diagonals will be equal, and hence their positions will be equal. So the formation will be preserved, and they will keep accelerating inwards.

Their final velocities [same number] could be found from:

$$potential = \frac{1}{2}\sum_{i\neq j}\frac{kq_iq_j}{r_{ij}} = \frac{4kee}{a} + \frac{2kee}{\sqrt{2}a}$$

Hence,

$$\frac{4kee}{b} + \frac{2kee}{\sqrt{2}b} - \frac{4kee}{a} - \frac{2kee}{\sqrt{2}a} = 4\left(\frac{m_eV^2}{2}\right)$$

where b is the final size of the box. Note that this gives infinity for b = 0.

b) This is not as easy as it looks.

The particles will obviously be repelled to some large distance, where their interactions will be virtually zero. By symmetry, the final velocity of each of the positrons will be equal and opposite; the same is true for the protons.

Conserving energy

$$E = \frac{kqq}{\sqrt{2}a} + \frac{kqq}{\sqrt{2}a} + 4\frac{kqq}{a} = 2\frac{mV^2}{2} + 2\frac{MU^2}{2}$$

where V is the final speed of the positrons, U is the final speed of the protons and q is the magnitude of the charge of an electron ($k = 1/(4\pi\varepsilon_0)$). The terms on the left are due to: the energy of the positron-positron pair, the proton-proton pair, and the positrons' with their neighbouring protons (respectively).

Conserving momentum yields no new information — it is conserved because of the symmetry of the system.

Note, however, that because M >> m, the accelerations of the protons = 1/2000 of accelerations of positrons — thus, the positrons will escape before the protons will have moved. I.e., the protons are virtually stationary. This gives

$$E = \frac{kqq}{\sqrt{2}a} + \frac{kqq}{\sqrt{2}a} + 4\frac{kqq}{a} = 2\frac{mV^2}{2} + \frac{kqq}{\sqrt{2}a}$$

Solving,

$$mV^2 = \frac{kqq}{\sqrt{2}a} + 4\frac{kqq}{a} \qquad [1]$$

Now, that the positrons have escaped, we can consider the protons. We get:

$$MU^2 = \frac{kqq}{\sqrt{2}a} \qquad [2]$$

Dividing [1] by [2] gives

$$\frac{mV^2}{MU^2} + 1 = 4\sqrt{2} + 2$$

$$\frac{V}{U} = \sqrt{\frac{M}{m}\left(4\sqrt{2}+1\right)}$$

Or, 115 plugging in M/m = 2000.

5) Sikorsky meets Newton

From Newton's 2nd law, the force is dp/dt.

Now, we'll put ourselves in a frame rotating with the blades.

At a distance r from the origin, an oncoming air particle has a horizontal velocity of r*ω m/s.

We'll assume that the collision is partially elastic with a parameter k. I.e., after the collision the air particle has a velocity k*r*ω (for perfect elasticity, k = 1).

After a collision, the particles will move downward with some speed U and backward with some speed V. Note that the magnitude of their velocity is $kr\varpi = \sqrt{U^2 + V^2}$

By Newton’s second law there must be equal an opposite forces acting on the blade — thus there is a vertical component (lift) mU, and a horizontal one mV, slowing the rotation of the blade. This would make the problem complicated, but luckily we assumed that the rotation rate is constant (the engine compensates). (m = mass of one air particle)

Now, after rotating through an angle dα, the area swept out by a blade is:

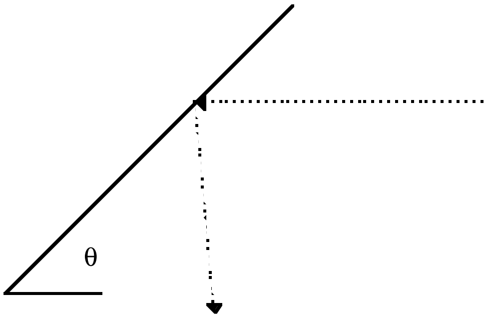
$$area = 1/2 * d\alpha * a^2$$

The volume thus swept out is:

$$volume = 1/2 * d\alpha * a^2 * b\sin(\theta)$$

And the mass hit is:

$$mass = \rho * 1/2 * d\alpha * a^2 * b\sin(\theta)$$



The downward component of k*r*ω (U) is: $kr\varpi\cos(\theta)\sin(\theta)$

Note that this varies this r — technically speaking, we should integrate here, but we'll make life easy and use the average r — that is, R/2.

Thus, for 1 blade:

$$F = \left(\rho * 1/2 * d\alpha/dt * a^2 * b\sin(\theta)\right) * \left(kR/2\varpi * \cos(\theta) * \sin(\theta)\right)$$

For N blades:

$$F = N\left(\rho * 1/2 * \varpi * a^2 * b\sin(\theta)\right) * \left(kR/2\varpi * \cos(\theta) * \sin(\theta)\right)$$

BONUS:

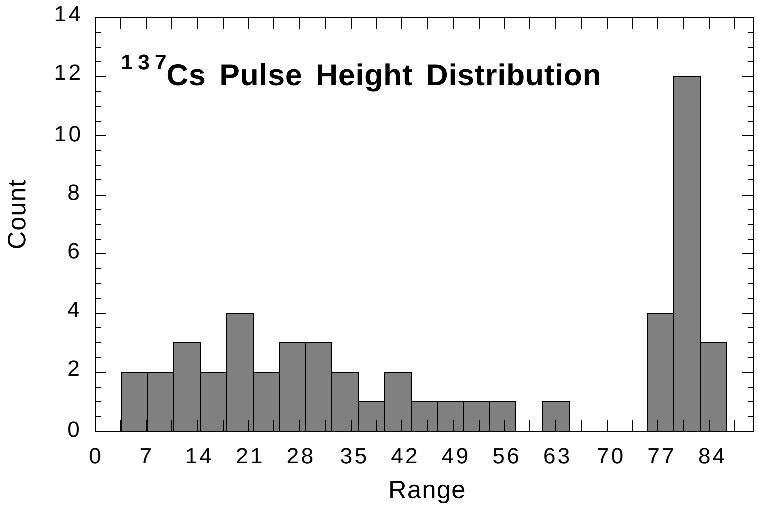
Well, tiliting the rotor axis seems like a good idea (so part of the lift is sideways) but I suppose too hard to build. What is done, instead, is something much more interesting — during a part of the cycle (say, when passing over the front) each blade's pitch (θ) is changed — this adds an extra force, as necessary.

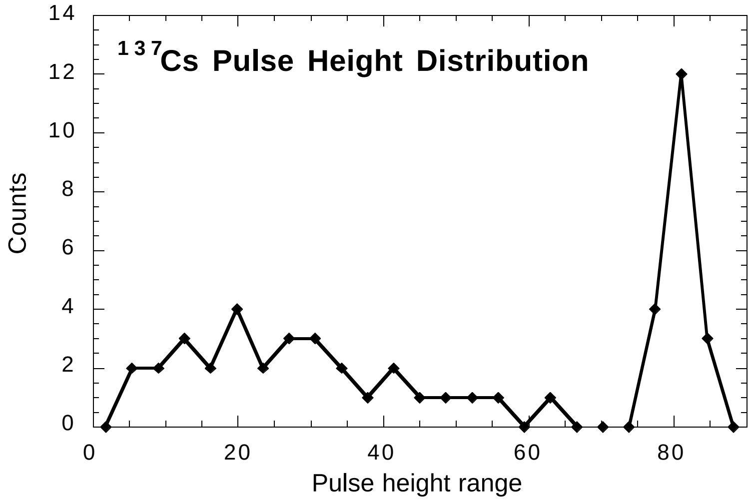
Electronic circuits control that ω is steady and the helicopter does not spin around.

6) Pulse-height analysis

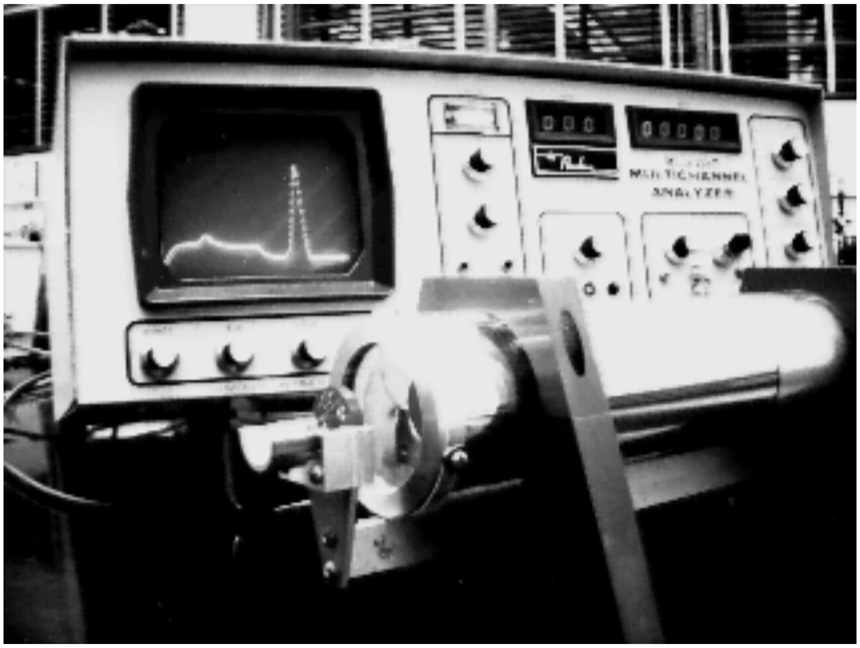
a) The data in the table can be plotted as a histogram — a range of values of the output, and then a count of how many times the values fell into that range. Here is such a table, and two different ways of plotting it:

X_o	N
1.80	0
5.40	2
9.00	2
12.6	3
16.2	2
19.8	4
23.4	2
27.0	3
30.6	3
34.2	2
37.8	1
41.4	2
45.0	1
48.6	1
52.2	1
55.8	1
59.4	0
63.0	1
66.6	0
70.2	0
73.8	0
77.4	4
81.0	12
84.6	3
88.2	0





The first plot better shows how there are bins, and that the columns marked indicate how many count data-points fell into the range of each box of pulse-heights.



For comparison, here is how a 'pulse-height analyzer' package does the job:

In the foreground is the Cs gamma-radioactive source, and the package which holds the NaI scintillator and the photomultiplier tube. In the background is the electronic apparatus which sorts pulses as they happen, and adds them to bins much as you did by hand.

b) The raindrops will have a *terminal velocity*, i.e., some maximum constant velocity. If the velocity is terminal (constant) there must be no net force on the droplet. So we can find the terminal velocity by figuring out the balance of forces: gravity and wind resistance.

Working from:

g	acceleration due to gravity:	980 cm s^{-1}
ρ_w	density of water:	1 g cm^{-3}
ρ_f	density of air:	1.2928×10^{-3} g cm^{-3} at NTP
r	droplet radius	

we need to subtract the buoyant force of the air — a small correction given the density of water compared to air.

$$F_g = m_{drop}g - m_{air}g = (\rho_w - \rho_f)Vg = (\rho_w - \rho_f)\frac{4\pi}{3}r^3 g \qquad [6.1]$$

For wind resistance, with density of air ρ_f, drag coefficient C_D, speed v and cross-sectional area S:

$$F_d = \frac{1}{2}\rho_f C_D v^2 S = \frac{1}{2}\rho_f C_D v_{term}{}^2 \pi r^2 \qquad [6.2]$$

Thus we start with $F_g = F_d$ so that:

$$\frac{1}{2}\rho_f C_D v_{term}{}^2 \pi r^2 = (\rho_w - \rho_f)\frac{4\pi}{3} r^3 g \quad \textit{or, cancelling,}$$

$$\frac{1}{2}\rho_f C_D v_{term}{}^2 = (\rho_w - \rho_f)\frac{4\pi}{3} g r \qquad [6.3]$$

If we assume that we have a sphere, we can consider our two limiting cases for C_D: we assume each case to start, and then see from the results what it will take to justify the assumption.

Case I: $R_e < 80$; $C_D \sim 24/R_e$.

First we find v_{term}:

$$\frac{1}{2}\rho_f \frac{24}{R_e} v_{term}{}^2 = (\rho_w - \rho_f)\frac{4}{3} g r \qquad \textit{and substituting for } R_e$$

$$\frac{6\eta v_{term}}{r} = (\rho_w - \rho_f)\frac{4}{3} g r \qquad \textit{thus, solving for } v_{term}, \qquad [6.4]$$

$$v_{term} = \frac{2(\rho_w - \rho_f)g}{9\eta} r^2$$

Then this goes into the formula for the Reynolds number:

$$R_e = \frac{\rho_f v_{term} 2r}{\eta}$$

$$= \rho_f \left[\frac{(\rho_w - \rho_f) 2 g r^2}{9\eta}\right] \cdot \frac{2r}{\eta} \qquad [6.5]$$

$$= \frac{\rho_f (\rho_w - \rho_f) 4 g r^3}{9\eta^2}$$

Now, using this we require that $\boldsymbol{R_e < 80}$, letting us find the corresponding condition on r:

$$80 > R_e = \frac{\rho_f (\rho_w - \rho_f) 4 g r^3}{9\eta^2} \quad \textit{then solving for } r,$$

$$r^3 < 80 \bullet \frac{9\eta^2}{\rho_f (\rho_w - \rho_f) 4 g r^3} = \frac{180\eta^2}{\rho_f (\rho_w - \rho_f) g} \qquad [6.6]$$

$$r < \sqrt[3]{\frac{180\eta^2}{\rho_f (\rho_w - \rho_f) g}}$$

$$\boxed{\Rightarrow \quad r < 0.0166\,\text{cm}; \quad v_{\text{term}} = 333.8\ \text{cm}\,\text{s}^{-1}}$$

So this approach works for droplets smaller than 0.17 mm, for which the terminal velocity will be around 3.3 m s^{-1}.

<u>Case II:</u> $R_e > 1000; \quad C_D \cong 0.4$

First we find v_{term}:

$$\frac{1}{2}\rho_f\, 0.4\, v_{term}{}^2 = (\rho_w - \rho_f)\frac{4}{3}g r \qquad \textit{thus, solving directly for } v_{term},$$

$$v_{term} = \sqrt{\frac{(\rho_w - \rho_f)\frac{4}{3}g r}{\frac{1}{2}\rho_f\, 0.4}} = \sqrt{\frac{(\rho_w - \rho_f)}{\rho_f}\frac{20}{3}g r} \qquad [6.7]$$

Then this goes into the formula for the Reynolds number:

$$\begin{aligned} R_e &= \frac{\rho_f\, v_{term}\, 2r}{\eta} \\ &= \sqrt{\left(\frac{\rho_f\, v_{term}\, 2r}{\eta}\right)^2 \frac{(\rho_w - \rho_f)}{\rho_f}\frac{20}{3}g r} \qquad [6.8] \\ &= \sqrt{\frac{\rho_f\ (\rho_w - \rho_f)}{\eta^2}\frac{80}{3}g r^3} \end{aligned}$$

Now from this we require that $R_e > 1000$, letting us find the corresponding r:

$$R_e^2 > 10^6$$

$$\frac{\rho_f\ (\rho_w - \rho_f)}{\eta^2}\frac{80}{3}g r^3 > 10^6 \qquad \textit{then solving for } r,$$

$$r^3 > \frac{3\eta^2 10^6}{80\, g\, \rho_f(\rho_w - \rho_f)} \qquad [6.9]$$

$$r > \sqrt[3]{\frac{3\eta^2 10^6}{80\, g\, \rho_f(\rho_w - \rho_f)}}$$

$$\boxed{\Rightarrow \quad r > 0.098\,\text{cm}; \quad v_{\text{term}} = 703.4\ \text{cm}\,\text{s}^{-1}}$$

So this approach works for droplets larger than 0.98 mm, for which the terminal velocity will be around 7 m s^{-1}.

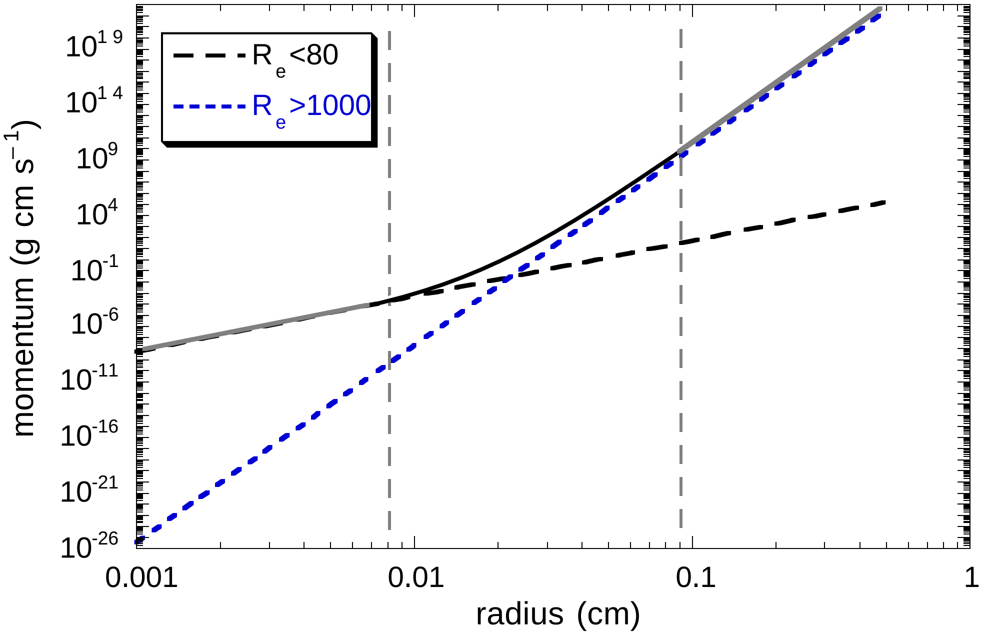
Momentum

The microphone will approximately record the impulse, or momentum deposited by the droplets, so it is the scaling between terminal *momentum* $m_{drop} \bullet v_{term}$ and droplet radius which is needed:

$$\Delta p = m_{drop}\, v_{term} = \rho_w V v_{term} = \rho_w \frac{4\pi}{3} r^3 v_{term}$$

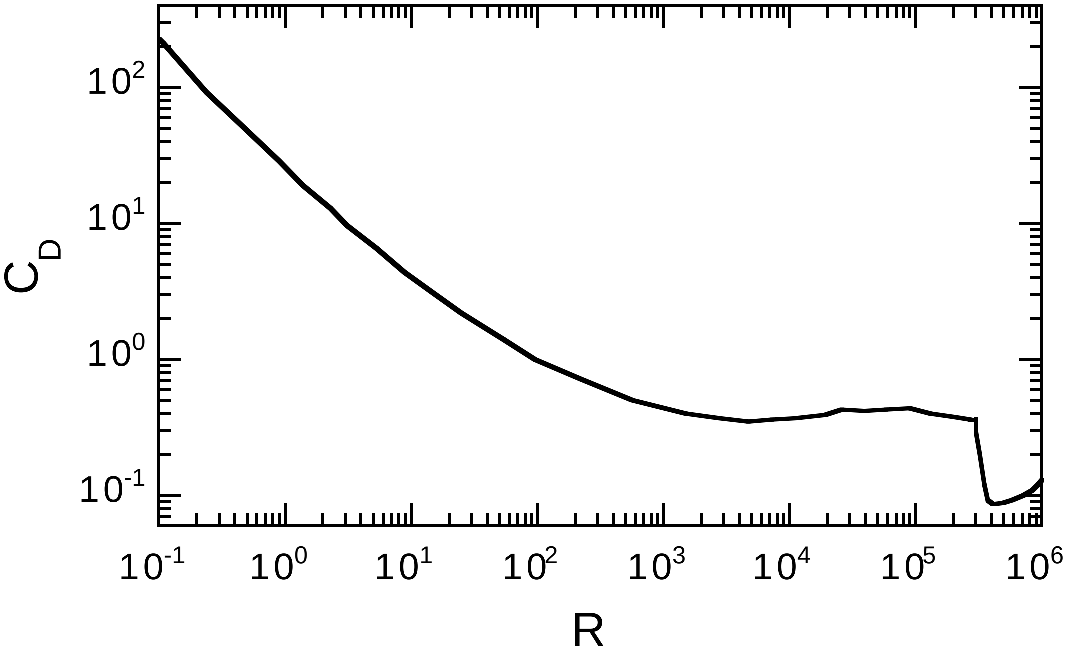
$$= \begin{cases} \rho_w \dfrac{4\pi}{3} r^3 \bullet \dfrac{2(\rho_w-\rho_f)g}{9\eta} r^2 = \dfrac{8\pi}{27\eta}\rho_w(\rho_w-\rho_f) g r^5 \\ \qquad = 5.07\times10^6\, r^5 \quad (r<0.016\text{ cm}) \\ \rho_w \dfrac{4\pi}{3} r^3 \bullet \sqrt{\dfrac{(\rho_w-\rho_f)}{\rho_f}\dfrac{20}{3} g r} = \dfrac{4\pi}{3}\bullet\sqrt{\dfrac{\rho_w(\rho_w-\rho_f)}{\rho_f}\dfrac{20}{3}g}\;\; r^{7/2} \\ \qquad = 9.39\times10^3\, r^{3.5} \quad (r>0.98\text{ cm})\end{cases} \qquad [6.10]$$

We can plot this; for points in-between, there is a plausible smooth curve to sketch the transition between the two regimes.



In order to find *radius* from *experimental momentum* data, we start on the y-axis (ordinate) and look up the radius on the x-axis (abscissa).

For your interest, a fuller set of data for C_Ds for spheres — found by experiments — looks like this:



adapted from: A Physicist's Desk Reference, H.L. Anderson, ed., American Institute of Physics, New York (1989).