1997-1998 Physics Olympiad Preparation Program

– University of Toronto –

Problem Set 4 Solutions: Optics and Waves

1) Reflecting Lightly

a) Assume that a plane wave of wavelength λ is falling perpendicularly on the surface of a thin layer of thickness h. The path difference between the waves reflected from the lower and upper surfaces is equal to $2h$. To make the reflecting light minimal, this amount should be equal to an odd number of half-wavelengths within the layer. Within a material which has index of refraction n, the wavelength will be λ/n. Thus,

$$2h = (2k - 1)\,\lambda/2n \qquad (k = 1, 2, 3, \text{etc.})$$

$$h = (2k - 1)\,\lambda/4n.$$

Light of wavelength λ_1 = 700 nm will be minimally reflected for any thickness

$$h_1 = \lambda_1/4n,\ 3\lambda_1/4n,\ 5\lambda_1/4n, \text{etc.} = 700/4n,\ 2100/4n,\ 3500/4n, \text{etc.}$$

Light of wavelength λ_2 = 420 nm will be minimally reflected for any thickness

$$h_2 = \lambda_2/4n,\ 3\lambda_2/4n,\ 5\lambda_2/4n, \text{etc.} = 420/4n,\ 1260/4n,\ 2100/4n, \text{etc.}$$

The smallest thickness which will give a minimum for both wavelengths is

$$h = 2100/4n = (2100 \times 3)/(4 \times 4) = 394 \text{ nm}.$$

b) In this problem we want to estimate the thickness of a dielectric coating which *maximises* the reflection for a particular wavelength. Looking at a patch in the corner of a $20 bill from above, you'll notice that it is yellow (close to gold). The wavelength of reddish-yellow light is somewhere around λ_1 = 600 nm. Thus, in order to *maximise* the reflection for wavelength λ_1, we have to assume that $2h$ is equal to an *whole number* of wavelengths in the layer. Therefore,

$$hn = k_1\,\lambda_1\,/2, \text{ where } k_1 = 1, 2, 3, \text{etc.}$$

$$hn = 300 \text{ nm}, 600 \text{ nm}, 900 \text{ nm}, 1200 \text{ nm}, 1500 \text{ nm}, 1800 \text{ nm}, \text{etc.} \qquad [1]$$

If n = 1.67, the coating thickness is equal to

$$h = 180 \text{ nm}, 359 \text{ nm}, 538 \text{ nm}, 719 \text{ nm}, 898 \text{ nm}, 1078 \text{ nm}, \text{etc.} \qquad [2]$$

It isn't immediately possible to tell which of these thicknesses is correct — any will work.

Extra Points: Actually, we can estimate the value of refractive index n if we look at the patch at different angles. Let's say we observe reflected green light (i.e., peak reflection is around λ_2 = 550 nm) — at what angle 'α' will it appear?.

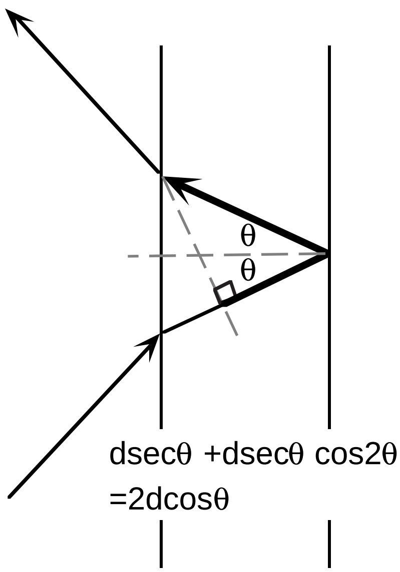
You might guess that at an angle away from normal incidence the light travels farther within the layer, so longer wavelengths will fit in, but this would be wrong — looking at the spot, as you tilt the bill away from normal incidence the colour goes from red-yellow to green to turquoise and finally quite blue. It is *shorter* wavelengths which end up reflecting resonantly! You can figure this out if you consider plane wavefronts going in, and reckon out the difference in distance between one wavefront and another, but *not* necessarily at the same point on the wavefront. Any part of a wavefront can cancel another, and you want to know the distance between wavefronts. The formula turns out to be not:

$hn \ / \ (\cos(\theta)) = k_2 \lambda_2 \ / \ 2$, where $k_2 = 1, 2, 3\ldots$ [3]

but instead:

$hn \cos(\theta) = k_2 \lambda_2 \ / \ 2$, where $k_2 = 1, 2, 3\ldots$ [3′]

(see figure at left: the bold line is the extra distance in the material which the penetrating light must travel, relative to light from the first surface)



The angle θ *within* the material can be found from Snell's law and α, the angle of incidence:

$$n \sin(\theta) = \sin(\alpha) \qquad [4]$$

Then

$$\alpha = \arcsin\left(n\sqrt{1-(\cos\theta)^2}\right) = \arcsin\left(n\sqrt{1-\left(\frac{k\lambda}{2hn}\right)^2}\right)$$

So the angle at which you see a colour λ depends on the index of refraction. Written the other way around:

$$n = \sqrt{\left(\frac{k\lambda}{2h}\right)^2 + \sin^2\alpha}$$

At normal incidence (α = 0), this gives the formula found in part (b) above.

2) A sequel - 'Twister: Part Moo'

a) Due to Doppler effect, there will be a difference between the frequency of the reflected signal and that of the radiated signal. The time interval between the n^{th} and $(n + 1)^{th}$ incoming impulses is equal to

$$1/f = 1/f_0 + T_{n+1} - T_n, \qquad (1)$$

where T_n and T_{n+1} are the times taken by the n^{th} and $(n + 1)^{th}$ impulses correspondingly to travel from the radar to the reflecting particles in tornado and back to the radar.

Let L_n and L_{n+1} be the distances between the radar and the particles in tornado at the moment when the n^{th} and $(n + 1)^{th}$ impulses are emitted. Then,

$$T_n = 2L_n/(v + c) \quad (2)$$

and

$$T_{n+1} = 2L_{n+1}/(v + c), \quad (3)$$

where v is the value we are looking for and c is the speed of light (radio waves).

The particles in tornado will cover the distance $(L_n - L_{n+1})$ during the time interval $1/f_0$:

$$L_n - L_{n+1} = v/f_0. \quad (4)$$

From (1), (2), (3), (4) we have

$$f = f_0\,(c + v)/(c - v) = \text{approx. } f_0\,(1 + 2v/c).$$

Therefore, v = approx. $c\,(f - f_0)\ /\ (2f_0) = ((3)(10^8)(5)(10^3))\ /\ ((2)(10^{10})) = 75$ m/sec.

b) The speed of sound propagation in dry air at 20° C is v = 340 m/sec. The apparent frequency increases if the source of sound is moving towards the receiver. Suppose that at some instant of time the source emits a wave crest that travels toward the receiver in the air with a velocity v. After a time interval of one period $T = 1/f_0$, the source emits the second crest. At this time, the first crest has traveled the distance vT and the source has traveled the distance v_sT, where v_s is the velocity of the source. The distance between the two crests is equal to a wavelength

$$\lambda = vT - v_sT = (v - v_s)\ /\ f_0$$

The apparent frequency to the stationary receiver is

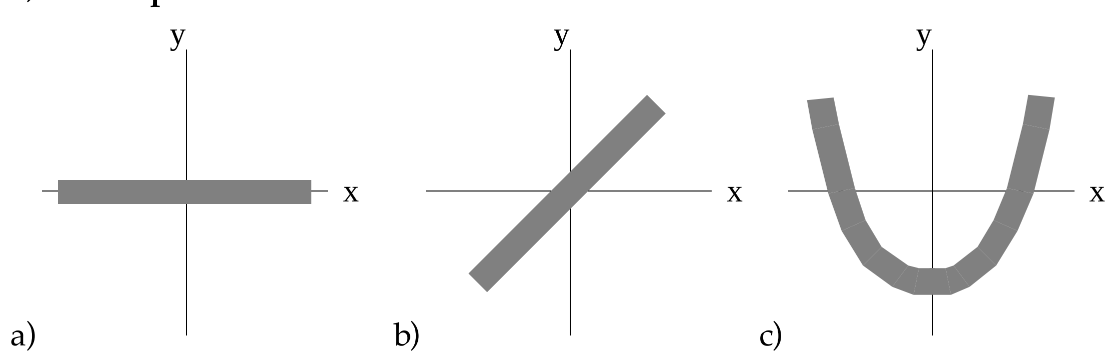
$$f = v/\lambda = vf_0\ /\ (v - v_s)$$

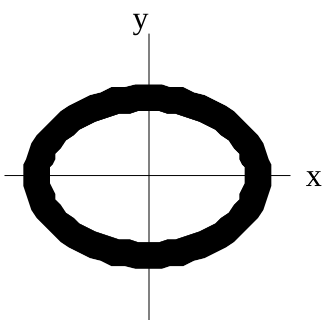
Thus, we have $(14)(10^3) = (1760)(340)/(340 - v_s)$.

$$340 - v_s = (340)(1760)/(14{,}000) = 42.7.$$

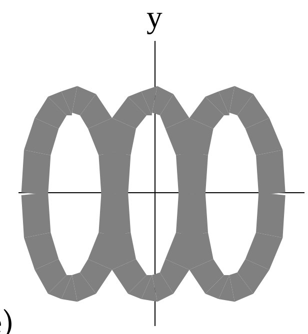
$$v_s = 297 \text{ m/sec.}$$

3) A complete Waste of Paint





d)



e)

f) This is an interesting question. Even if the paint spray is infinitely thin, you can fill space by setting $k_x{:}k_y$ to be incommensurate (i.e., their ratio is an irrational number). Start the motion at the farthest $x = y$ point and let it go. Unfortunately, this will take a long time (like forever).

4) Acousto-optic modelocking of laser modes

a) A pulse travelling at the speed of light, back and forth down the laser, must cross through the mode locker while it is briefly at zero scattering, which therefore must be the case twice each round trip. So since the light takes round trip time

$$t = \frac{2L}{c} = \frac{2 \times 80\,cm}{3 \times 10^{10}\ cm\,s^{-1}} = 5.34 \times 10^{-9}\,s = 5.34ns$$

the period of the modelocker is this number, too.

So

$$\nu = \frac{1}{T} = \frac{1}{5.34 \times 10^{-9} s} = 1.87 \times 10^{8}\,Hz, \qquad \text{i.e., 187 MHz}$$

This is pretty high, so we usually put the modelocker near one mirror, at the end of the laser cavity. Then the pulse passes in both directions in quick sucession and the modelocker needs only 93.5 MHz. Longer cavities help, too.

(NOTE: We've ignored the different speed of light in the laser medium and just used '*c*'.)

The trick is that with this periodicity of the pulse going back and forth, the standing waves that make it up must be multiples of this frequency.

b) $\nu\lambda = c \;\Rightarrow\; \lambda = c/\nu$ here 'c' $= v = 5.958 \times 10^{3}$ ms^{-1}

$\lambda = 3.19 \times 10^{-5}$ m = 31.9 µm

This is tiny — to be a standing wave, about $\frac{2cm}{31.9\mu m} = 627.7$ wavelengths fit in.

The problem is that a *standing wave* needs an *exactly integral* number of waves. There are several ways to fix this:

1. change the frequency until it is resonant with the glass block (like singing in the shower). Then the cavity length has to be adjusted a little too. We often do this.
2. heat or cool the glass a bit — at a different temperature the speed of sound is different (true in air and water, too!) and thus the block of glass can be 'tuned' to match the *existing* frequency!

c) It's a grating, so the same conditions apply

$\lambda = d sin\theta$ where d is the grating spacing

but d is really $\frac{\lambda_{us}}{2}$

$$\lambda = \frac{\lambda_{us}}{2} \sin\theta$$

$$\text{so } \theta = \arcsin\left(\frac{2\lambda}{\lambda_{us}}\right) = \arcsin\left(\frac{2.1\mu m}{31.9\mu m}\right) = 3.77^\circ$$

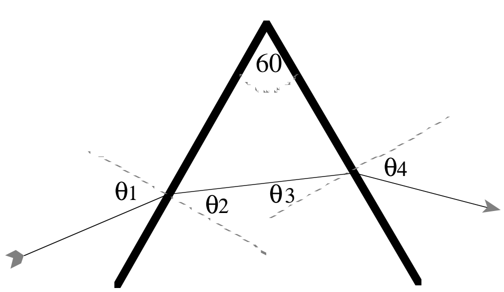
i.e., just a few degrees off the straight path. This is *1st order*: for other orders

n	θ
1	3.77°
2	7.57°
3	11.39°
etc	…

The exact amount of phase retardation and the spatial shape affects the amount of energy sent into each of these diffraction orders.

5) Halos and sun-dogs

a) Nothing helps set up a problem more than a diagram:



The normals are indicated by dotted lines.

We want the angle deviated, δ. This is:

$$\delta = (\theta_1 - \theta_2) + (\theta_4 - \theta_3) = (\theta_1 + \theta_4) - (\theta_2 + \theta_3)$$

The angle at which the two normals meet within the prism is 120°, so $\theta_2+\theta_3$ + 120° = 180°, or $\theta_2+\theta_3$ = 60°.
Thus

$$\delta=\theta_1+\theta_4-60°$$

We need to find all the θ 's in terms of θ_1. Using Snell's law (law of refraction):

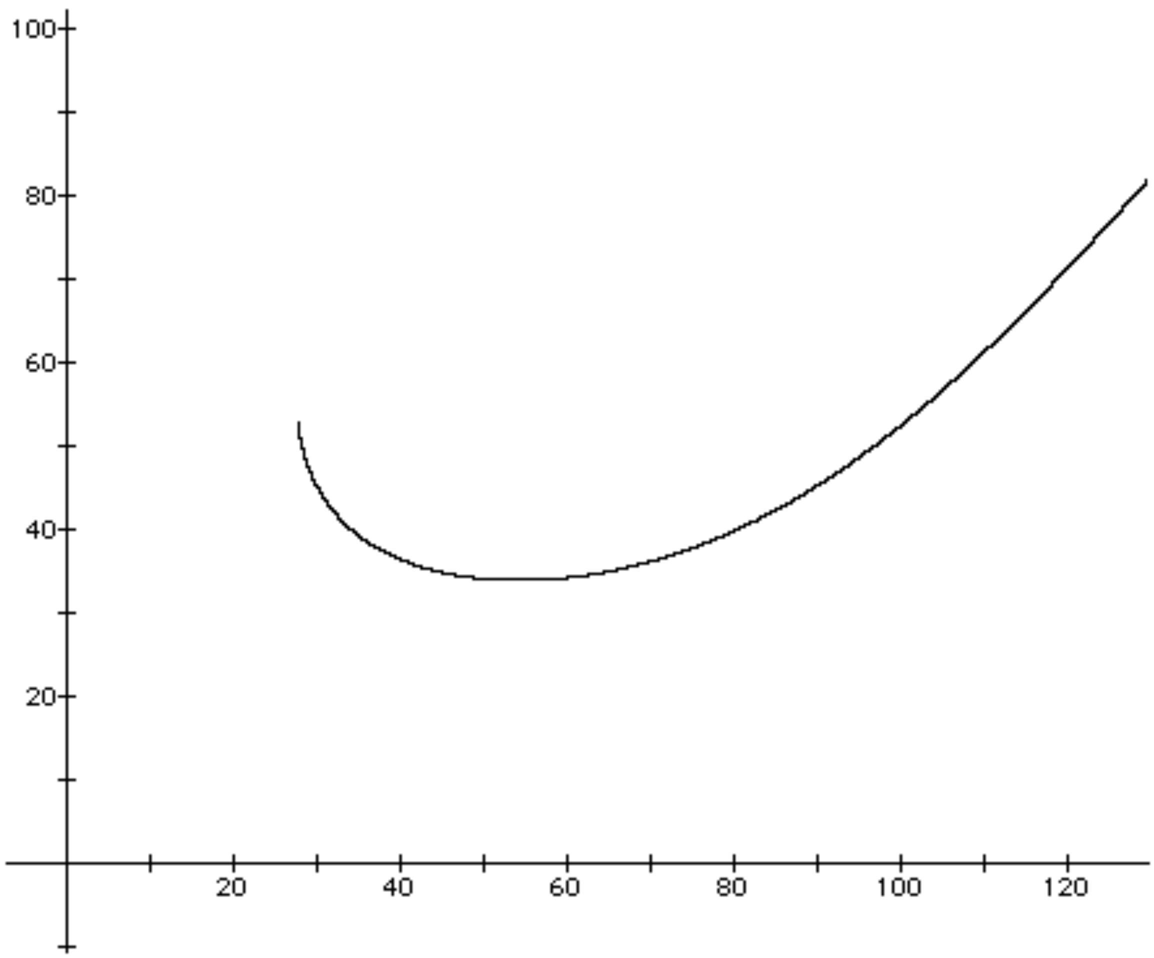
$$\sin(\theta_1)=1.5\sin(\theta_2)$$

$$1.5\sin(\theta_3)=\sin(\theta_4)$$

Put this all together we get:

$$\theta_4=\arcsin\left(\frac{3\sqrt{3}}{4}\sqrt{1-\left(\frac{2\sin\theta_1}{3}\right)^2}-\frac{\sin\theta_1}{2}\right)$$

Plug this into the above expression to find delta in terms of θ_1. What a mess!



At least it can be plotted, as shown above.

b) The brute force method is outlined in the problem. Plug in different values of θ_1 and determine δ. This is a sample of the numbers I found:

θ_1 (°)	θ_4 (°)	δ (°)
30	77.60	47.10
35	66.00	41.00
40	58.47	38.47
45	52.38	37.38
50	47.21	37.21
55	42.74	37.74
60	38.89	38.89
65	35.59	40.59
70	32.87	42.87
75	30.72	45.72

As you can see, if there is a random distribution of rays at θ_1, more rays will come out at 37° than at any other angle — the output light ends up bunching up at 37°. Due to the symmetry in the system, there are two bright areas of light: 37° and –37°.

To generalize this for an index n, notice that the minimum deviation angle occurs when $\theta_1=\theta_4$. Under this constraint, one easily finds that $\theta_2=\theta_3=30°$, independent of index. Thus:

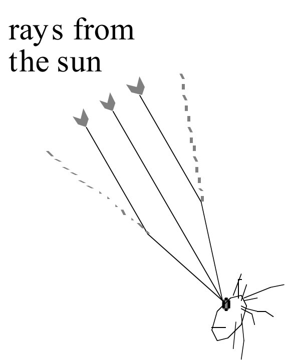
$$\sin(\theta_1)=n\sin(30°)=\sin(\theta_4)$$

and thus:

$$\delta=2*arcsin(n/2) - 60°$$

For n=1.5, δ=37.18°.

c)



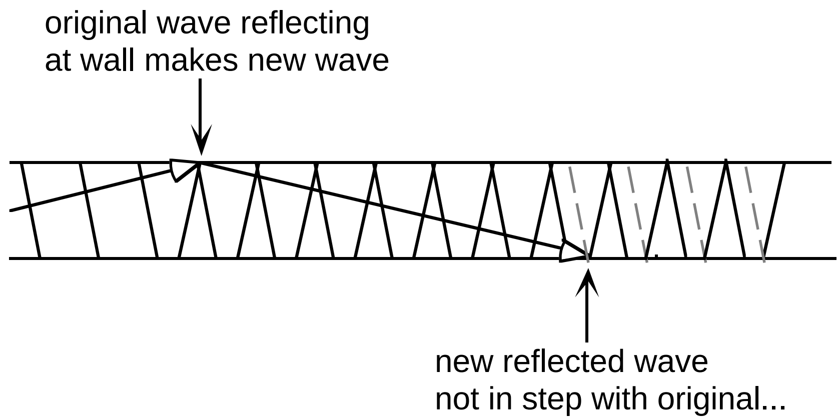
The light that enters your eye comes directly from the sun and from an arc that has a 22° inner angle. The water crystals act like the glass prisms of part (b) except they are not all stationary on a horizontal surface. The random orientation in both the horizontal and vertical planes thus causes you to see a complete circle instead of two dots. Using the expression from part (b), it is found that the material responsible for the optical effect has an index of 1.3. The index of water is 1.333.

On windless days, the crystals fall in a preferred direction: long axis parallel to the horizon. (If you don't believe this, think of the way leaves fall from a tree. Try it out yourself with a piece of paper.) This corresponds very closely to the orientation in part b), and thus the circle reduces to two bright areas, on either side of the sun, parallel to the horizon. These are called sun dogs.

A diagram & description can be found at http://www.usatoday.com/weather/askjack/wasksky.htm

6) Sound - reading the Riot Act

a) need that after 2 bounces the wave joins in constructively with the continuing wave — other combinations lead to destructive interference, i.e.,



Use θ as the angle between the **direction** and the walls. The distance l is as indicated in the drawing at right. Then,

$$\cos\theta = \frac{\lambda}{l}$$

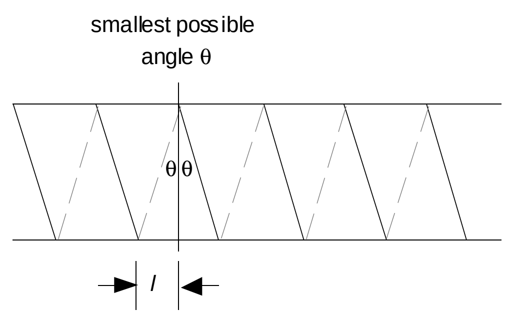
$$l = \frac{\lambda}{\cos\theta}$$

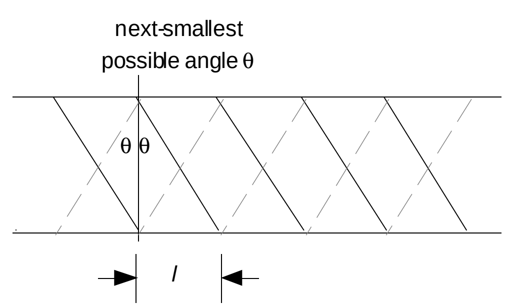
$$\frac{(l/2)}{d} = \tan\theta$$

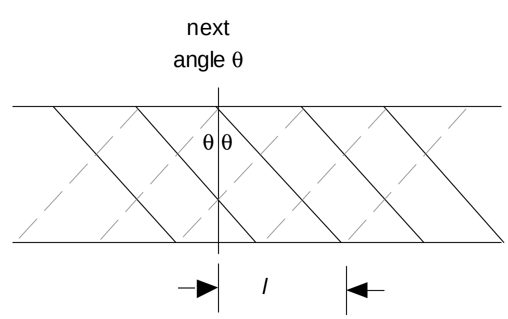
$$\frac{\lambda}{2\cos\theta} = d\ \tan\theta = d\ \frac{\sin\theta}{\cos\theta}$$

$$\frac{\lambda}{2} = d\sin\theta$$

i.e., $\lambda = 2d\ \sin\theta$







For the next higher-order mode, $\frac{\lambda}{2} = \frac{d}{2}\sin\theta$, or generally $\frac{d}{n}\sin\theta$.

This gives rise to the formula:

$$n\lambda = 2d\ \sin\theta \qquad \text{(the } \textit{Bragg} \text{ condition)}$$

Now, $\sin\theta = \frac{n\lambda}{2d} \quad \frac{\lambda}{2d}$, so $\theta > \sin^{-1}\left(\frac{\lambda}{2d}\right)$

and $1 \geq \sin\theta = \frac{n\lambda}{2d}$, so $n\lambda \leq 2d$; ie $n \leq \frac{2d}{\lambda}$

So there's a minimum angle of propagation, and there are minimum ($n = 1$) and maximum $n = \text{int}\left(\frac{2d}{\lambda}\right)$ orders of mode.

b) the speed of the new **'guided'** wave going down the waveguide depends on the original v_ϕ and on the angle of the wave

The wavelength l of the new wave, measured down the waveguide, is longer:

$$\cos\theta = \frac{\lambda}{l}$$

$$l = \frac{\lambda}{\cos\theta}$$

Then the k of the travelling trapped wave is

$$\beta = \frac{2\pi}{l} = \frac{2\pi}{\lambda}\cos\theta = k_o \cos\theta$$

we can write

$$\beta^2 = k_o^2 \cos^2\theta = k_o^2 (1 - \sin^2\theta)$$

but we know sinθ from above

$$\beta^2 = k_o^2\left(1 - \left(\frac{n\lambda}{2d}\right)^2\right)$$

$$= k_o^2 - \left(\frac{2\pi}{\lambda}\right)^2 \left(\frac{n\lambda}{2d}\right)^2$$

$$\beta^2 = k_o^2 - \frac{n^2\pi^2}{d^2} \qquad \text{or} \qquad \beta = \sqrt{k_o^2 - \frac{n^2\pi^2}{d^2}}$$

The phase speed is then

$$v_\phi = \frac{2\pi\nu_o}{\beta} = \frac{2\pi\nu_o}{c}\frac{c}{\beta} = \frac{k_o}{\beta}c$$

$$= \frac{k_o}{k_o\sqrt{1 - \left(\frac{n\lambda}{2d}\right)^2}}c \quad = \frac{c}{\sqrt{1 - \left(\frac{n\lambda}{2d}\right)^2}}$$

which is greater than the speed of sound!

c) BUT any sound-burst or *wavepacket* travels at the **group velocity**

$$v_g = \frac{d\omega(\beta)}{d\beta}$$

In free space, of course sound travels at the sound speed c.

The connection is: in free-space $\omega = kc$, $v_g = \frac{d\omega(k)}{dk} = \frac{d(kc)}{dk} = c$

But now,

$$\omega = k_o c$$

$$\Rightarrow \omega = c\sqrt{\beta^2 - \frac{n^2\pi^2}{d^2}}$$

$$v_g = \frac{d}{d\beta}\left(c\sqrt{\beta^2 - \frac{n^2\pi^2}{d^2}}\right) = c\frac{1}{2}\left(\beta^2 - \frac{n^2\pi^2}{d^2}\right)^{-1/2} \bullet 2\beta$$

$$= c\ \frac{\beta}{\sqrt{\beta^2 - \frac{n^2\pi^2}{d^2}}}$$

$$= c\ \frac{k_o\sqrt{1-\left(\frac{n\lambda}{2d}\right)^2}}{k_o} = c\ \sqrt{1-\left(\frac{n\lambda}{2d}\right)^2} \quad < c$$

So a pulse of sound travels at different speeds (all less than the free sound-speed) in different modes.

Interesting to note: $v_\phi \bullet v_g = \frac{c}{\sqrt{1-\left(\frac{n\lambda}{2d}\right)^2}} \bullet c\sqrt{1-\left(\frac{n\lambda}{2d}\right)^2} = c^2$

d) $n \le \frac{2d}{\lambda}$ and: $\lambda_1 = \frac{c}{v_1} = 0.331\,m$

$$\lambda_2 = \frac{c}{v_2} = 0.236\,m$$

$$\lambda_3 = \frac{c}{v_3} = 0.184\,m$$

$n_1 \le \frac{2 \times 0.20\,m}{0.331\,m} = 1.21$ i.e., $n_1 = 1$, only

$n_2 \le \frac{2 \times 0.20\,m}{0.236\,m} = 1.69$ i.e., $n_2 = 1$, only

$n_3 \le \frac{2 \times 0.20\,m}{0.184\,m} = 2.17$ i.e., $n_3 = 1, 2$

150 m away, in what order would we hear the notes?

$$v_g = c\sqrt{1-\left(\frac{n\lambda}{2d}\right)^2}$$

$$v_{g_1} = 0.562c = 186\,ms^{-1} \qquad t_{150\,m} = \frac{150}{v_g} = 0.81s$$

$$v_{g_2} = 0.807c = 267\,ms^{-1} \qquad t_{150\,m} = 0.56s$$

$$v_{g_3} = 0.879c\ (n=1) = 294\,ms^{-1} \qquad t_{150\,m} = 0.51s$$

$$= 0.392c\,(n=2) = 130\,ms^{-1} \qquad t_{150\,m} = 1.16s$$

At the other end we hear first 1.8 kHz, then 1.4 kHz, then 1 kHz, and then 1.8 kHz again! Descending notes, then the top one again but because of a second-order mode!

Between 1 *kHz* and 1.4 *kHz*, all modes are $n = 1$, and the v_g increases with frequency. So this 'click' has its high frequency-components get through first and its low frequency-components later — it makes a descending-tone sound like a whistle-flute with the plunger being drown out.

This actually happens in things like long smooth pipes, like large drain-pipes: if you clap your hand, what you hear at the other end is a short descending note like a bullet ricocheting in some Western movie. [see POPTOR web page comment for a picture like this].

It also happens with radio waves — radio operators in WWI heard random strange 'whistler' tones descending. It turned out to be lighting strikes in the *opposite hemisphere* of the Earth! Like the 'click' of this problem, these electrical impulses would make radio waves, which would get tied up with magnetic fields in the ionosphere. Tied up this way, these waves were *dispersive,* as in our problem, and the high frequencies in the radio wave impulse would get to the radio operators first, to be followed at later times by the lower frequency components, as the impulse spread into its frequency-parts.