1997-1998 Physics Olympiad Preparation Program

– University of Toronto –

Problem Set 5 Solutions: Electricity and Magnetism

1) The Earth as Cosmic Doorknob

The charge accumulated by the Earth during the time interval t is equal to $q = aSq_p t$, where $a = 1$ proton/cm^2.sec is the flow density; $S = 4\pi R^2$ is the area of the Earth's surface; $R = 6.4 \times 10^6$ m is the radius of the Earth; $q_p = 1.6 \times 10^{-19}$ C is the charge of proton. The critical charge q of the Earth, which will prevent protons with the kinetic energy $E < 4 \times 10^9$ eV from reaching the Earth, can be found from the following equation:

$$qq_p / (4\pi\varepsilon_0 R) = aStq_p / (4\pi\varepsilon_0 R) = E,$$

where

$$(4\pi\varepsilon_0)^{-1} = 9 \bullet 10^9 \text{ Nm}^2\text{C}^{-2}$$

From this equation we have:

$$t = (4\pi\varepsilon_0 ER)/(aSq_p^2) = (E\varepsilon_0)/(aq_p^2 R) = 10^8 / ((3.14)(0.9)(1.6)(6.4)) = 3.45 \bullet 10^6 \text{ sec} = 40 \text{ days}$$

This value is tiny compared to the age of the Earth, which is equal to 5×10^9 years. Protons from the cosmic rays continue to reach the Earth because their charge is compensated by the charge of electrons also reaching the Earth. Since their energy is extremely small, they are not considered as a component of cosmic rays.

2) C.C.? No! No!

a) At a distance of 5 cm from a large sphere, we must treat only that section of the sphere near where you are standing (your closest point to the sphere). This is for the same reason that we can treat the earth as being flat locally at its surface even though it is really a sphere.

E field of a spherical conductor with charge *Q*.

NOTE: the charge on a conductor distributes itself uniformly on the conductor's surface.

Gauss' Law

$$\int \underline{E} \bullet \underline{ds} = \frac{\text{charge enclosed}}{\varepsilon_0}$$

$$E \bullet 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\therefore E = \frac{Q}{4\pi r^2 \varepsilon_o}$$

$\therefore$ at 5 cm:

$$0.8 \times 10^3 \frac{V}{m} = \frac{Q}{4\pi\varepsilon_o (.05)^2} \Rightarrow Q = 2.2 \times 10^{-10}\, C$$

b) For the large sphere,

$$V_1 = \frac{1}{4\pi\varepsilon_o} \frac{q_1}{r_1}$$

Small sphere,

$$V_2 = \frac{1}{4\pi\varepsilon_o} \frac{q_2}{r_2}$$

But since the spheres are part of the same conductor, the potentials of the spheres must be equal:

$$V_1 = V_2 \quad \Rightarrow \quad \frac{1}{4\pi\varepsilon_o} \frac{q_1}{r_1} = \frac{1}{4\pi\varepsilon_o} \frac{q_2}{r_2}$$

$$\therefore \frac{q_1}{r_1} = \frac{q_2}{r_2}$$

The electric field at the surface of each sphere is given by

$$E_1 \propto \frac{q_1}{r_1^2},\ E_2 \propto \frac{q_2}{r_1^2}$$

With the same constant of proportionality for each.

$\therefore$The ratio of the electric fields is

$$\frac{E_1}{E_2} = \frac{q_1 / r_1^2}{q_2 / r_2^2} = \frac{\left(\frac{q_1}{r_1}\right)\frac{1}{r_1}}{\left(\frac{q_2}{r_2}\right)\frac{1}{r_2}}$$

but

$$\frac{q_1}{r_1} = \frac{q_2}{r_2}$$

$$\therefore \frac{E_1}{E_2} = \frac{r_2}{r_1}$$

So the field is higher on the surface of the smaller sphere. A more sharply curved or "pointy" surface on a conductor has a higher electric field and therefore reaches spark discharge for a lower applied charge.

I have heard that there was an ongoing disagreement between Benjamin Franklin in the U.S. and European investigators as to what the best shape for a lightning rod should be: sharp spike or small ball. The argument went like this: all this, above, is true, but the strong field around a sharp spike might cause air to ionize as the rod began to charge up. If so, there would be glob of ionized air (plasma) near the rod, and plasma is a conductor — so the *effective* conductor would have a larger radius than if a lightning rod had a ball on the top to start with!

3) Goats & Sheep – the Hall Effect in plasma

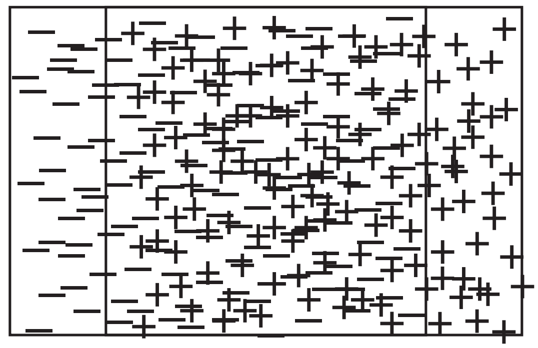
The point here is that moving charges in a magnetic field see a force — the Lorentz force:

$$\vec{F}_{Lorentz} = q\,\vec{v} \times \vec{B}$$

The Lorentz force, on each particle, will be:

$$1.6 \times 10^{-19}\ \text{C} \bullet 0.3\ \text{m s}^{-1} \bullet 0.3\ \text{T}$$

$$= 1.44 \times 10^{-19}\ \text{N} \quad \text{(opposite directions for electrons, ions)}$$



A block of plasma, with the electrons pulled aside from the ions…

So when the flowing gas becomes ionized, the electrons and ions will separate from each other in a B-field — the Lorentz force pushes opposite ways for the opposite charges.

Do they end up going in circles in the B-field? Probably not, because as the charges separate, they are still attracted to one another by the Coulomb force:

$$\vec{F}_{Coulomb} = q\,\vec{E}$$

Then the net force is going to be:

$$\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B})$$

Basically, stream of electrons will move aside from the gas stream, and the stream of positive charges will also move a little in the opposite direction. The separated charges lead to an E-field, and to an attraction between the two streams. The two will balance, so that F = 0

What will be the electric field to cancel this?

$$1.44 \times 10^{-19}\ \text{N} = \text{q E, so}$$

$$\text{E} = 0.9\ \text{V m}^{-1}$$

How much will the charges separate from each other, if we assume they keep their original density, and just separate as a stream? We can figure out the details: air at STP is 22.4 l per mole, and 1 mole = 6.022×10^{23} particles. From this we find the number of particles per unit volume:

$$n = \frac{N}{V} = \frac{6.022 \times 10^{23}}{22.4 \times 10^{3}\,\text{cm}^{3}} = 2.69 \times 10^{19}\,\text{cm}^{-3}$$

With 1% ionization, this means we have 2.69×10^{17} electrons per cm^3, or 2.69×10^{23} electrons per m^3, and the same density of positively charged ions. If the electrons are just pushed somewhat off the ions (the ions are much more massive, and move less from the forces), what we have is something roughly like a parallel-plate capacitor, with the two thin excess-charge slabs as the two charged plates of the capacitor.

For this, we can find the field — the same as between the two plates of a parallel-plate capacitor. In the middle, where there are both electrons and ions, the charges of each cancel each other out (unless you give them time to move around and redistribute themselves), so the net field in the middle is just what is produced by the thin slabs of excess (unbalanced) charge.

$E = 4\,\pi\,k_c\,\sigma$ where $k_c = 9 \times 10^9\ \mathrm{N\,m^2\,C^{-2}}$

σ = charge per unit area on each plate

The charge *per unit area* of the two slabs of excess comes from the charge density, and the distance the electrons are pushed off the ions:

$$\sigma = (n\,e\ (\Delta x) \times A)\ /\ A$$
$$= n\,e\ (\Delta x)$$

and the field between the plates is:

$$E = 4\,\pi\,k_c\,n\,e\ (\Delta x) \qquad [5]$$

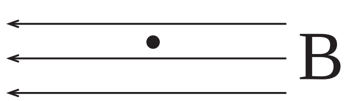
or

$$\Delta x = E/(4\,\pi\,k_c\,N\,e)$$
$$= 1.84 \times 10^{-16}\ \mathrm{m}$$

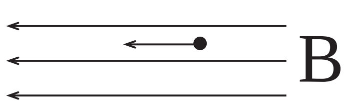
A very tiny displacement (actually nonsensically tiny, given the size of an atom!). Consider, though, that if the jet is 1mm across, the potential measured across the jet from one side to the other would be 0.9 mV.

4) Shepherding fields

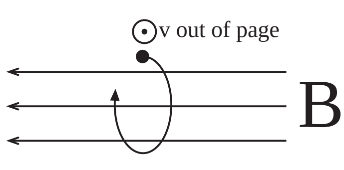
a)For the four cases suggested in the problem:



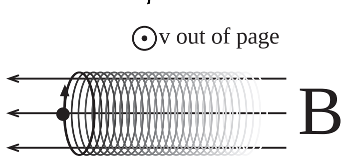
i) v=0 There is no force acting on the particle and therefore no acceleration.



ii) v_{parallel} non-zero: The cross-product is zero, therefore same result as first case.



iii) $v_{\text{perpendicular}}$ non-zero: The cross-product simplifies to a multiplication and the resulting force is always acting perpendicular to the motion. The particle moves in a circle. From the given equation and knowing that for perfect circular motion (such as a satellite in orbit) the acting force is $F = mv^2/r$ we find the radius of the circle is: $r = mv/qB$



iv) We combine the results from sections iii) and ii) and get helical motion.

Since in all cases the velocity of the particle is always orthogonal to the acting force, the work done on the particle is 0.

b) As given in the question, kinetic energy is conserved. Thus the square of the magnitude of the total velocity at any given point in time is constant:

$$v(z)^2 = v_{\text{per}}(z)^2 + v_z(z)^2 = \text{constant}$$

where v is the total velocity, v_{per} is the component perpendicular to B (i.e. in the X-Y plane) and v_z is the component parallel to B (i.e., parallel to the z axis). We want to find $v_z(z)$:

$$v_z(z)^2 = v_{\text{per}}(0)^2 + v_z(0)^2 - v_{\text{per}}(z)^2$$

We also know that flux is conserved:

$$B(0)\ A(0) = B(z)A(z)$$

where B is the magnitude of the magnetic field and A is area enclosed by the loop that is orthogonal to the field. Combining this with the result from part a):

$$v_{\text{per}}(z)^2 / B(z) = v_{\text{per}}(0)^2 / B_0$$

This gives:

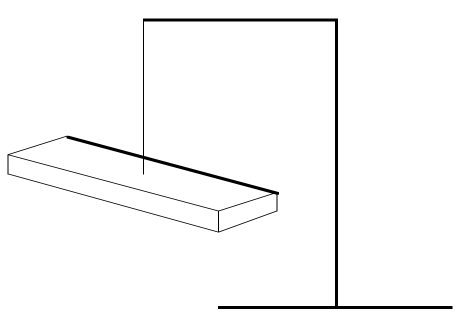
$$v_z(z)^2 = v_z(0)^2 - v_{\text{per}}(0)^2\ (b_0\ z^2)\ /\ B_0$$

c) Shades of Bob, it's simple harmonic oscillator! *KE* is transferring from parallel motion to perpendicular circular motion. The particle swings back and forth along the Z axis, in a similar way to a weight on a spring, with the added circular motion around the Z axis

d) $v_z(z)=0$ when $z^2=(B_0\ \ v_z(0)^2)\ /\ (b_0\ \ v_{\text{per}}(0)^2)$
Total *KE* is conserved. Energy from translational *KE* transfers to rotational *KE*.

5) **Science on a shoestring**

First we suspend the magnet from the stand (using the string). We tilt it by a small angle and let it oscillate.



It undergoes oscillations (much like a torsional pendulum) such that:

$$I\ddot{\theta} = -\mu_o m B_z \sin(\theta) \approx -\mu_o m B_z \theta$$

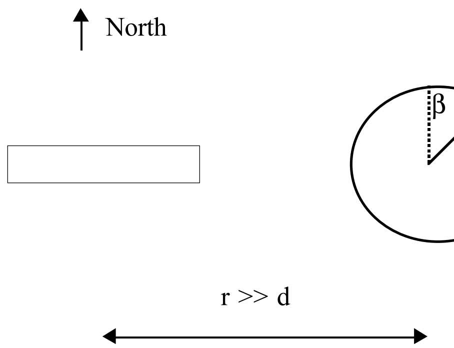
here, B_z is the horizontal component of the Earth's magnetic field, m is the magnetic moment, and I is the moment of inertia of the magnet. Treating it as a thin stick,

$$I = \frac{M d^2}{12}$$

Now, the motion is simple-harmonic with a period of

$$T = 2\pi\sqrt{\frac{I}{mB_z\mu_o}}$$

Now, we align the compass to the north. Then, we place the magnet some distance $r >> d$ away from the compass and measure the deflection angle β.



Clearly, then,

$$\tan(\beta) = \frac{B}{B_z} = \frac{2m}{4\pi r^3 B_z}$$

Combining, we get

$$B_z = \sqrt{\frac{2\pi I}{\mu_o T^2 r^3 \tan(\beta)}}$$

Ideally, one would collect a set of β*s* by varying *r*, and plot the equation (on log-log paper), and get *Bz* that way, but that is a lot of work (when done by hand).

And thus, plugging in one set of numbers will do.

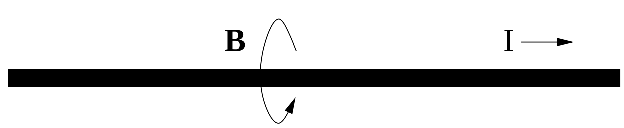
To measure *T*, it is best to time say 10 periods, and divide the result by 10 (as long as the time does not decay too much). *r* is simply measures with a ruler. Measuring β might be difficult, and this might be the biggest error here. In fact, it might be best to set β at, say 45^o, and then change *r* to obtain that deflection (to make measuring it easier) — but one has to be careful that $r >> d$!

To get the error, we note that $\delta Bz/Bz \approx \delta X/X$, where $\delta X/X$ is the biggest error found (most likely for β or *r*; δX means error in variable X).

The accepted value is about 0.15×10^{-4} Tesla, BUT it varies a lot because of interference, etc. Doing the experiment outside would thus be ideal.

6) **How much do you charge for a free ride?**

a) *B* circles around a straight wire in a direction dictated by the right-hand rule (which comes from the Biot-Savart law).



b) For a time-varying magnetic field in the *z* direction, the induced electric field circles the magnetic field direction, like the magnetic field around a wire. The direction of the induced *E* field is such that it resists the change in the magnetic field (Lenz' Law). *(figure next page)*

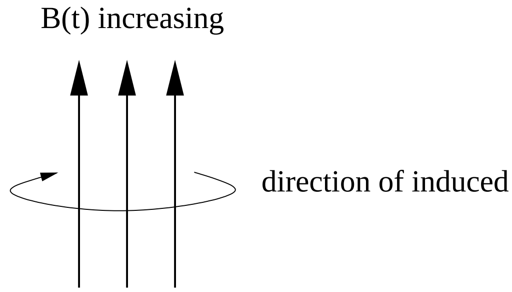
c) $\int(\boldsymbol{E}\bullet d\boldsymbol{l})$ around the loop

$= -\int ((d\boldsymbol{B}/dt)\bullet d\boldsymbol{s}$ on a surface bounded by the loop

$= -d\Phi/dt$, where $\Phi(t)$ is the magnetic flux through the loop at time t.

In our case, making our loop of radius a, centred at the centre of the wheel,

$$\int(\boldsymbol{E}\bullet d\boldsymbol{l}) = -\pi a^2 (dB(t)/dt)$$

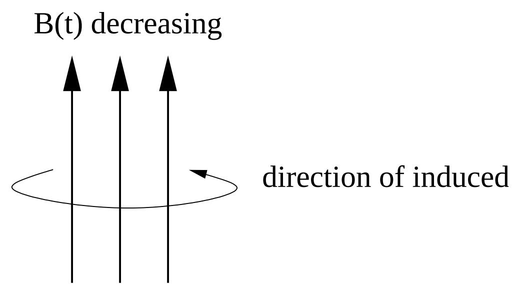


d) The total torque on the wheel about its axis:

$$N = r \times F \text{ over the whole wheel.}$$

For a piece of the wheel of length δl, r and F are perpendicular and $F = qE = (\lambda\delta l)E$. Therefore

$$\delta N = r\lambda\delta l E$$



For the total torque, we integrate around the whole wheel

$$N = r\lambda\int Edl = r\lambda\left(-\pi a^2 \frac{dB(t)}{dt}\right)$$

where we have used the result of part (c).

The total angular momentum is

$$L = \int Ndt$$

where the integral is from an initial time t_o just before the B field is changed to the final time t just after the B field has stopped changing.

$$L = -\rho\lambda\,\pi a^2 \int ((dB(t)/dt)^* dt)) \text{ (from } t_o \text{ to } t)$$

$$= -\rho\lambda\,\pi a^2 \int (dB) \quad \text{(from } B_o \text{ to } 0)$$

as initially $B=B_o$, and finally $B=0$.

so $L = \rho\lambda\,\pi a^2 B_o$

e) From part (d), we see that the value of the final angular momentum is independent of how fast the field is switched off; it depends only on the initial field value, or, more generally, it depends on the difference between the initial and final field strengths

BONUS

f) Marks were given for any interesting answer — the question is relatively sophisticated.

The torque on the Merry-Go-Round wheel comes as a result of the E-field produced as the B-field produced by the solenoid vanishes ($dB/dt \neq 0$). This induced E-field accelerates the charges into circular motion, taking the wheel to which they are bound along too.

XX: Some might argue that this new circulating current produces a B-field of its own; the charges (already moving) in the solenoid see this changing B-field from the Merry-Go-Round disk ($dB/dt \neq 0$) as it induces an E-field. However, this is a bit of a red-herring — the solenoid has equal numbers of positive and negative charges, so it doesn't pick up momentum this way. Someone might argue that the B-field pushes the electrons to one side of the wire (the Hall effect, see Q. 3) so that the angular momentum they pick up doesn't match the positive charges', but there's no hope here!

√√***:*** Basically, the wheel will accelerate when it sees the B-field changes, even if the solenoid is a long, long distance *D* away. Conservation of angular momentum can't 'hold off', or wait, until the solenoid 'learns' that it is pushing on the wheel — a time D/c at the earliest, where c is the speed of light. So the angular momentum must be in the electromagnetic field; this changing electromagnetic field must have angular momentum which is 'left' with the wheel. The changing fields the wheel's motion creates will reduce the angular momentum of the overall electromagnetic field, as the (kinetic) angular momentum of the wheel increases.