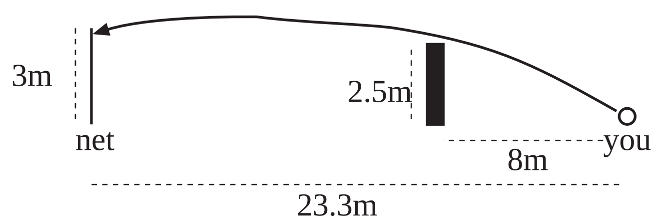
1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 2: Mechanics

1) Football fizix

Your best bet is to go for the corner of the net farthest from Plante (see figure given in question). We can reduce this to a two-dimensional problem by looking at the cross-section along the ball's trajectory. Since there are no air effects, the ball heads straight from your foot, over the heads of the defenders and into the upper corner of the net, as in the figure at right. (Yay! Cheers!)



If you kick it at too high an angle, it will easily clear the heads of the defenders but will take longer than necessary to make it into the net. This gives Plante more diving time to intercept the ball. To minimize the flight time, you want the ball to have the maximum horizontal speed (covering the distance to the net the fastest), but a corresponding vertical speed that will allow it to barely miss the defenders and still go into the top corner of the net. As always in these projectile motion problems: *separate the problem into horizontal and vertical components*.

<u>Constraints:</u>

1) At $x = 8$ m, $y > 2.5$ m
2) At $x = 23.3$ m, $y < 3$ m.
3) At $t = 0$ s, you kick the ball from $x = 0$ m, $y = 0$ m.
4) Max speed $= 30$ m s^{-1}

The equations of motion are horizontal at constant speed, and vertical motion with gravity:

$v_x(t) = v_x(t{=}0) = v_x$ (note how I am defining v_x and v_y)
$v_y(t) = v_y(t{=}0) - g * t = v_y - g * t$

So the ball's path is:

$x(t) = v_x * t$
$y(t) = (v_y - (g/2) * t) * t$

Constraint 3 is satisfied by how I set up the equations.

Constraint 1:

Let $t = t_1$ when $x = 8$ m.

$8 = v_x * t_1$.

$y(t_1) = (v_y - (g/2) * (8/v_x)) * (8/v_x)$ but $y(t_1) > 2.5$

Thus $2.5 / (8/v_x) < v_y - (g/2) * (8/v_x)$

$v_y > 2.5 / (8/v_x) + (g/2) * (8/v_x)$

$v_y > v_x * (2.5/8) + (4*g) / v_x$ **<=Eqn 1**

Constraint 2:

Let $t = t_2$ when $x = 23.3$.

$23.3 = v_x * t_2$.

$y(t_2) = (v_y - (g/2) * (23.3/v_x)) * (23.3/v_x)$ but $y(t_2)<3$

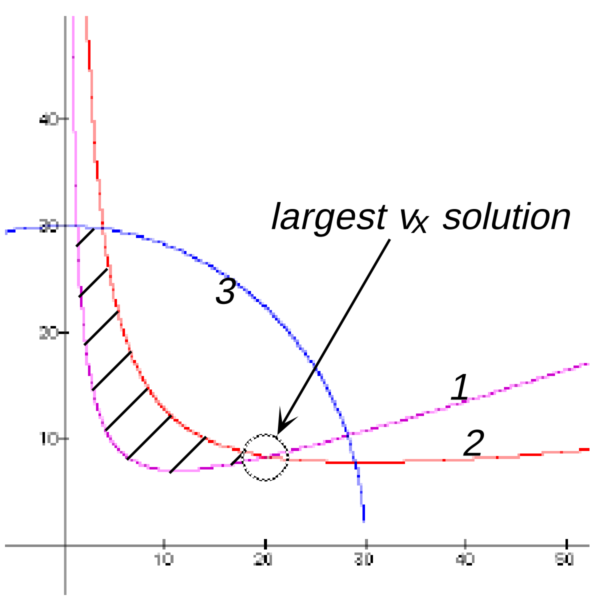
Thus $3 / (23.3/v_x) > v_y - (g/2) * (23.3/v_x)$

$v_y < 3 / (23.3/v_x) + (g/2) * (23.3/v_x)$

$v_y < v_x * (3/23.3) + (23.3/2) * g / v_x$ **<=Eqn 2**

Constraint 4:

$v_x^2 + v_y^2 \le 30^2$ **<=Eqn 3**



Graph the three inequalities, to get a clear picture of how they can be satisfied, in the region above curve '1', below curve '2' *and* below curve '3': see that the maximum horizontal speed v_x occurs very close to the intersection of eqn 1 and eqn 2.

The intersection is at:

$$v_x * (3/23.3) + (23.3/2) * g / v_x = v_x * (2.5/8) + (4*g) / v_x$$

or

$$v_x = 20.2 \text{ m s}^{-1}$$

This corresponds to $v_y = 8.25$ m s^{-1}.

To make sure that you are not hitting the crossbar, reduce v_x slightly to 20.1 m s^{-1}, say. This will still meet the constraints imposed by equations 1 and 2.

Therefore you must kick it with an initial speed of 21.7 m s^{-1} at an angle of 22 degrees above the horizontal. (Soccer is a science!). (I am using g = 9.8 m s^{-2} throughout this problem.)

BONUS:

Air effects can help you. By giving the ball the proper spin, you can cause the ball to deflect downward from its normal parabolic path. This will allow you to increase the initial horizontal velocity but still have the ball go in the net. If you are really fancy, you might be able to curve the ball enough so you go around the defenders instead of over ('curl' or 'swing' the ball left or right). *[James]*

2) Chow, baby

Find acceleration *a:*

– elevator has constant acceleration over two floors,	$d = 2 \bullet 4\text{m} = 8$ m
– final velocity is one floor per 3 seconds,	$v = 4\text{m}/3\text{s} = 1.33$ m s^{-1}
– initial velocity is zero,	$u = 0$

$$v^2 = u^2 + 2ad$$
$$(1.33)^2 = (0)^2 + 2a(8\text{ m})$$
$$a = 0.111\text{ m s}^{-2}$$

Adding this to the acceleration due to gravity, the *total* acceleration will be:

$$g' = 9.80\text{ m s}^{-2} + 0.11\text{ m s}^{-2}$$
$$= 9.91\text{ m s}^{-2}$$

The ratio of the acceleration in the elevator to that standing on the ground is therefore: 9.91/9.80 = 1.01. The acceleration is only 1% greater.

Find weight of cans with this *additional* acceleration:

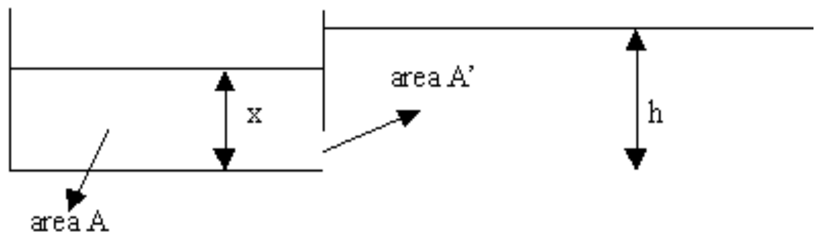
Then 15 cans of cat food accelerated in the elevator would weigh on the bag the way that 15 • (1.01) = 15.15 cans would at rest. This is less than 16.

Does it mean the bag would not break? No! We only know that 16 *will* break the bag, but we *don't know* whether a little bit over 15 would. That's because cans come in units of 1 can — there was no way to test whether 15.15 cans would break the bag. Until now: Kit can be the first person to test out the bags at this weight, without having to cut up a can of cat food in the band-saw.

Recommendations? Double bag, or else carry only 14 cans of cat food. The 14 would weigh in the accelerating elevator the same as 14.14 would on the ground.

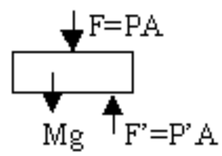
And any weight 15.00 or under in the elevator is definitely safe, because 15 cans under *normal* acceleration was! *[Robin]*

3) Sink, sank, sunk



Consider a pill box of water of area A and height h.

For equilibrium,



$$PA + mg = P'A$$
$$PA + Ah\rho g = P'A$$
$$P' = h\rho g + P$$

Now extend the pill box to be a tall cylinder of height h (depth h to water level). Ignoring air pressure, P = 0, and we get:

$$h = \frac{P}{\rho g}$$

Here, h ≈ 12600 feet [roughly] for 5500 punds/inch^2, so there's good agreement.

As the system starts without any water inside, the water must be accelerated (that's why you cannot use Bernoulli's equation **inside** the ship, which only applies to steady flows). Later, it reaches steady state.

Let us use the hint and use a force approach:

First note that outside of the hole the water will hardly be moving — the water comes in from all directions, so that the velocity at any one point is almost 0 (but because the ocean is so big this adds up to give a large flow in the hole).

Now, we will consider the system in a small time interval Δt. Since we are in equilibrium, the water inside is stationary at first. For a small time Δt an external force (due to external pressure) and a negative force (due to pressure of water in the ship) acts at the hole. The net force will force in a small amount of water Δm. Additionally, a part of the force will push some of the water upwards, but as Δt –> 0, Δx –> 0 and this is negligible.

Δm is hence pushed an amount

$$\Delta s = 0 + \frac{1}{2} a(\Delta t)^2$$

$$\Delta s = \frac{1}{2} \frac{(\rho g h - \rho g x)A'}{\Delta m} (\Delta t)^2$$

Since the pressure due to a height z of water is ρgz (from i))

$$\Delta s = \frac{1}{2} \frac{\rho g A'(h - x)}{\rho \Delta s A'} (\Delta t)^2 = \frac{1}{2} \frac{g(h - x)}{\Delta s} (\Delta t)^2$$

From conservation of mass, and amount Δm of water at the hole must be equal to the amount of water that causes a rise of the level in the ship, i.e., $\Delta m = \rho \Delta s A' = \rho \Delta x A$

Using this gives:

$$(\Delta s)^2 = \left(\frac{A}{A'}\right)^2 (\Delta x)^2 = \frac{1}{2} g(h - x)(\Delta t)^2$$

(in the limit)

$$\frac{\Delta x}{\Delta t} = \frac{A'}{A} \sqrt{\frac{1}{2} g(h - x)} = \frac{dx}{dt}$$

As the water flows in, the system comes to equilibrium. To compensate for an increase Δm in mass the ship must have moved (down) an amount

$$\Delta h = -\Delta x$$

Note that this does not imply that h = x! This simply says that x and h change by the same amount. In fact, at time = 0,

x = 0

And h = M/(ρA)

(M is the mass of the ship — 40, 000 tons; this is just Archimedes's principle)

And so

h(t) = x(t) + M/(ρA)

Plugging this into the expression for dx/dt yields:

$$\frac{dx}{dt} = \frac{A'}{A} \sqrt{\frac{1}{2} \frac{gM}{\rho A}}$$

So the water level rises at a constant rate.

From this we find:

$$\frac{dh}{dt} = \frac{A'}{A}\sqrt{\frac{1}{2}\frac{gM}{\rho A}}$$

$$h(t) = \left(\frac{A'}{A}\sqrt{\frac{1}{2}\frac{gM}{\rho A}}\right)t + \frac{M}{\rho A}$$

Putting h = 80 m, we find t ≈ 40 minutes. We didn't catch them this time…

———

This answer is pretty good — only the "1/2" factor above may have to change [I did an experiment sinking a shoe box, and it looked more like it should be a "1", which is fairly close]. Using Bernoulli's equation gives nonsense for the time it takes the ship to sink (something on the order of 1 minute).

Note that you can use Bernoulli's equation, if you set it up the following way: take a streamline from the top of the water to the hole in the ship — this flow is steady and you get something like:

$P = \frac{Mg}{A} = \frac{1}{2}\rho V^2$ (where V is the speed of the water coming in, P the pressure at the hole's depth).

Using conservation of mass we now get:

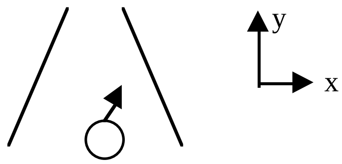
$$\frac{dx}{dt} = \frac{A'}{A}\sqrt{2\frac{gM}{\rho A}}$$

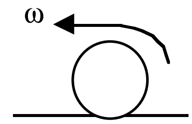
which is the same as the equation above, except for the factor of 2 — which is just as good because we used approximations in either solution. Note also that the flow of the water is different while the hole is not completely covered with water — that's why the height of the hole (0.5 m) is so small, to make this effect negligible.

Speaking of Bernoulli's equation, the proof may be found in virtually any textbook (which is too bad) — see e.g. Halliday & Resnick. *[Peter]*

4) When bowling, always signal your lane-change

The math here isn't too hard, but one has to think about what's going on or it's very easy to be off be a minus sign (the International Olympiad judges kill (well, almost) for errors like this). First, since we know there is friction, we see that it will oppose the ball's motion. Now we resolve the motion to two directions — x and y — and two motions — translational and rotational.





Y:

Translational: (friction slows motion)

$$M a = -\mu_k (Normal) = -\mu_k Mg$$
$$a = -\mu_k g$$
$$V = V_y - \mu_k g t$$

Rotational: (friction causes ball to spin faster, until

$$I \alpha = \tau$$
$$\frac{2}{5} M R^2 \alpha = \mu_k (Normal) R$$
$$\alpha = \frac{5\mu_k g}{2R}$$
$$\varpi = 0 + \frac{5\mu_k g}{2R} t$$

The ball will continue to skid until it's rotating fast enough so that it can roll. Then, V = ωR. This will occur at a time given by:

$$V_y - \mu_k g t^* = \frac{5\mu_k g}{2} t^*$$
$$t^* = \frac{2V_y}{7\mu_k g}$$

The motion thus consists of two phases — with skidding and without:

$$y(t) = V_y t - \frac{1}{2}\mu_k g t^2, \qquad t \le t^*$$
$$y'(t) = y(t^*) + V(t^*)t = \frac{12 V_y{}^2}{49\mu_k g} + \frac{5V_y}{7} t, \qquad t \ge t^*$$

X:

This is very similar:

$$V = V_x - \mu_k g t$$
$$\varpi_y = \varpi - \frac{5\mu_k g}{2R} t$$

When skidding stops – V = ωR. But we have to be careful! In this case, V changes sign, and so the above should actually read $|V| = |\omega| R$!

[How do we know it changes sign? Well, you can do what I did — go through the whole calculation carrying the wrong sign. Then do the plot I asked for and get nonsense. Or you can be a lot smarter — assume that V changes sign (or not), and then go back and *see* if your assumption holds]

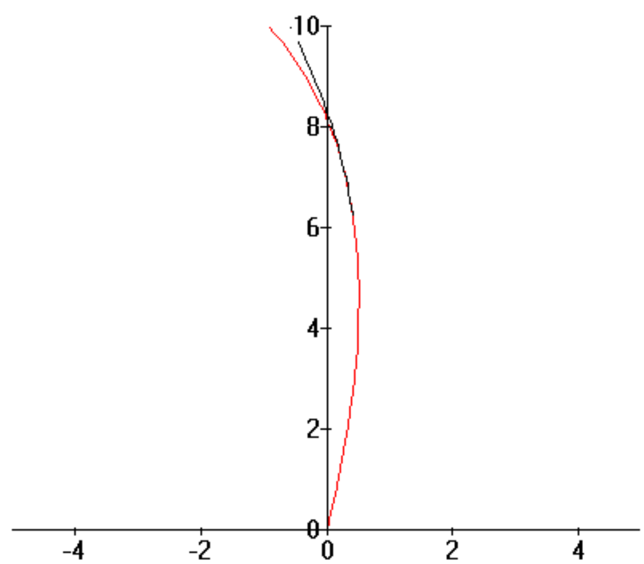
$$t_2{}^* = \frac{2(\varpi R + V_x)}{7\mu_k g}$$

$$x(t) = V_x t - \frac{1}{2}\mu_k g t^2, \qquad t \le t_2{}^*$$

$$x'(t) = x(t_2{}^*) + V(t_2{}^*)t = \frac{(\varpi R + V_x)}{49\mu_k g}\left(12V_x - 2\varpi R\right) + \frac{5V_x - 2\varpi R}{7}t, \quad t \ge t_2{}^*$$

Now we can finally proceed to plot this. We have to calculate t* and t_2* to know which motion occurs when. It turns out that the ball stops slipping after about 3 seconds in the x-direction, and after about 8 in the y, but by that time the ball has long left the bowling alley…

Plotting yields the figure at right.

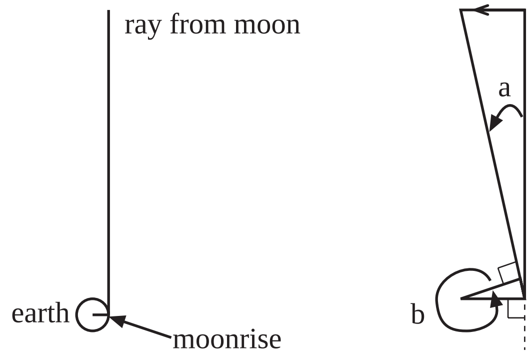


[Peter]

5) A matter of some gravity

i) From the information given we know that if the earth was rotating more slowly, the lunar cycle would be *shorter*. This means the direction of rotation of the earth is the *same* as the direction of the moon around the earth — it has to 'chase' it instead of meeting it 'head-on'. Thus, the moon rises later every day. By how much?

In the time the moon has moved by angle a, the Earth has moved by angle b. The two triangles are set so that the moon rays are parallel to the surface of the earth. They are similar triangles so $a = b - 360°$



Also (where t is measured in hours):

$$a = t / (29*24) *360°$$

$$b = t / 24 *360°.$$

Thus the time between moon rises is: 24.86 hours. Thus if the moon rose today at 7PM, tomorrow it will rise at 7:52PM.

ii) This is a standard two-particle collision problem. The fact that the interaction is through gravitational forces does not change how one solves the problem. Both energy and momentum of the entire system must be conserved. We consider the system before the interaction (far from the earth) and after the interaction (far from the earth).

Variables: v_e, v_e+dv_e: velocity of the earth before and after the interaction
v_s,v_s+dv_s: velocity of the ship before and after interaction
m_e: mass of earth
m_s: mass of the ship

Conservation of energy gives:

$$m_e * v_e^2 + m_s * v_s^2 = m_e * (v_e + dv_e)^2 + m_s * (v_s + dv_s)^2$$

For maximum increase in speed for the ship, the initial velocities should be parallel. We can drop the vector notation:

$$m_e * v_e + m_s * v_s = m_e * (v_e + dv_e) + m_s * (v_s + dv_s)$$

$$\Longrightarrow - m_e * dv_e = m_s * dv_s$$

Substituting this into the above gives and solving for dv_s gives:

$$dv_s = 2 * (v_e - v_s) / ((m_s/m_e) +1)$$

We need to know the speed of the earth relative to the sun. We can write:

$$Fg = m_e * v_e^2 / r = G * m_{sun} * m_e / (r^2)$$

This gives $v_e = 3 \times 10^4$ m s^{-1}. Thus the final speed of the ship is: 5.5×10^4 m s^{-1}.

There are 3.1×10^7 s in each year. For this to change by 1 s, then we must change the momentum of the earth (i.e., its speed) by $1/(3.1 \times 10^7)$ of its total. One slingshot maneuver changes the momentum by $m_s * dv_s$. The total initial momentum is $m_e * v_e$. Thus the total number of sling-shots, N, would have to be:

$$N * m_s * dv_s = 1/(3.1\times 10^7) * m_e * v_e$$

$N \cong 1\times 10^{14}$ Huge. The earth is not in peril from this, yet! *[James]*

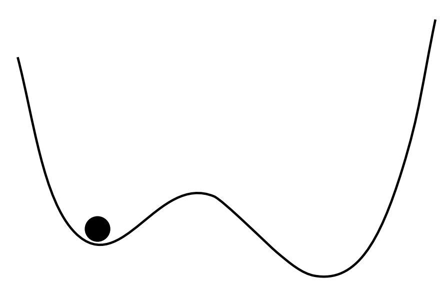
6) Balloon tug-o-war experiment

This experiment is great — it shows all kinds of different physics relationships, even though it is pretty simple. It is even a little bit easier to do than to describe!

What did you think would happen between the two balloons fitted to opposite ends of a straw? The first time I thought of this, I thought perhaps the balloons would end up being the same size, because the balloons are identical and used the same way, so I figured they should do the same thing — and that means they would keep the same amount of air in them. *Wrong!*

Instead, what happens is that one balloon will be bigger and the other smaller. And if you squeeze the bigger one, and make it a bit smaller than the other one, it will almost collapse, and become the smaller one while the other is the bigger one. So, they each do the same thing, but not at the same time! The balloon you squeeze to be the smaller one always expells most of its air. So there are two 'stable' or fixed arrangements: the left balloon small, or the right balloon small. These will stay that way, but you can switch between them. This is technically called 'bistability', and it also describes the kind of room-light switch that you can snap on or snap off — it can be one or the other, but not much in-between, and it won't just change by itself.

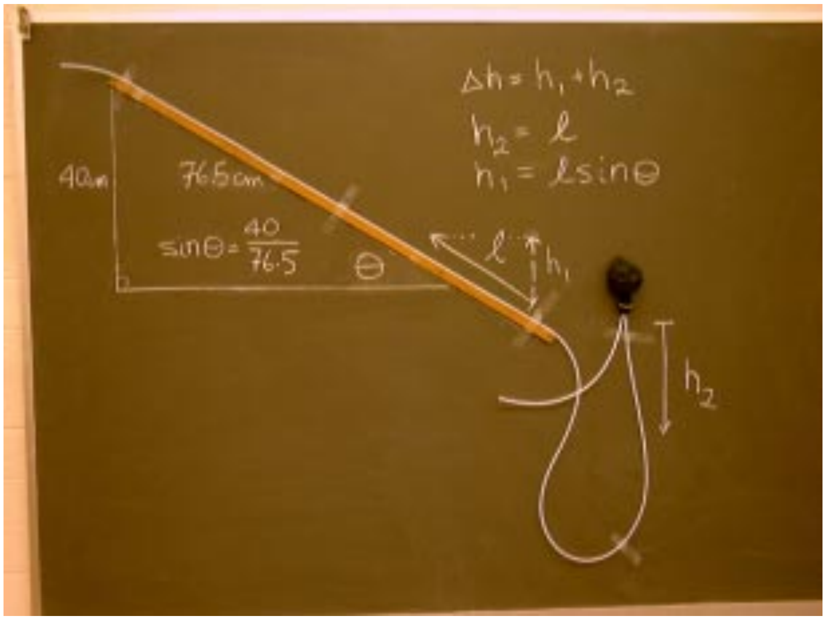
Bistability can be described with a potential-energy diagram that has two local minima: there is equilibrium in one minimum or in the other, but it will cost a little energy to switch between them. That's where squeezing the balloon comes in!



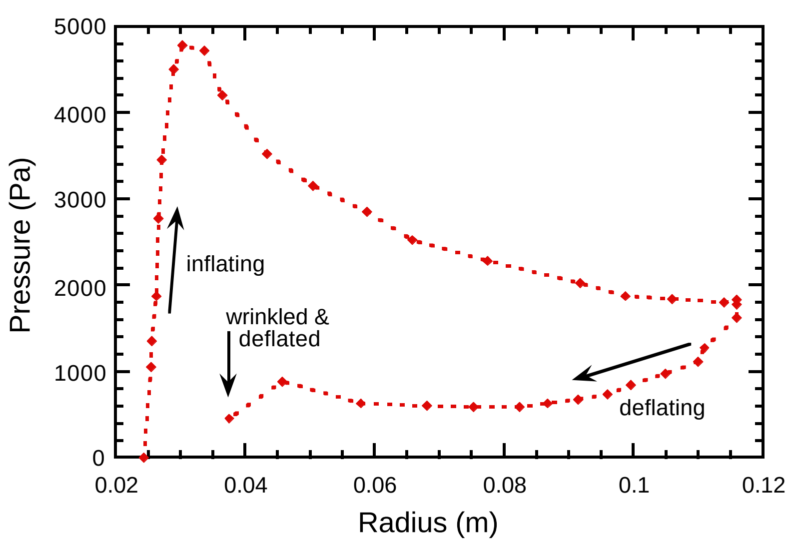
A ball can come to rest at either of two places on a curve like this. Therefore it is 'bistable' (bi = two). It is the same for any system with a similar potential energy curve — like the two balloons on a straw.

The rest of the experiment is about showing why there are two minima, or maybe even showing what the potential energy curve looks like for two balloons on a straw.

The picture below shows my own setup for this experiment. I guessed it would be a bit hard to read the fairly small pressure changes of a balloon, so I tilted the manometer arm, taped it to a meter stick, and measured the surface of the water as it moved along. Since the height difference gives me the pressure in the balloon, over atmospheric pressure, tilting the tube made the water move farther along, ΔL, for a given height change Δy (i.e., $\Delta L/\Delta y = 1/\sin\theta$).



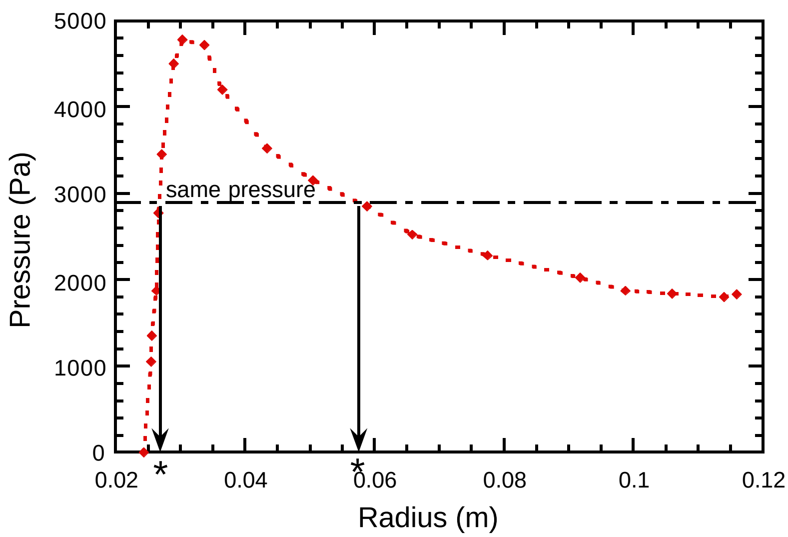
I blew up the balloon, through the extra smaller tube, and measured the difference in height Δh of the water surfaces (meniscus). Actually, I measured ΔL, and converted it to a height raised, adding it to the height ΔL by which the water level in the other arm dropped. This gave me the pressure inside the balloon. To measure balloon size, I tried two methods: deflating the baloon by letting air out into a beaker turned upside down underwater, then measuring its volume; also I used a tape measure to measure the balloon's circumference. The second method was good — it was sensitive, it was pretty easy, I could use it as I inflated the balloon or deflated the balloon, either way, and I didn't get as wet. Below is the curve I graphed from the data I took. Each symbol marked on the graph is a measurement I took.



It's interesting that the curve for the balloon as it is being inflated is not identical to the curve for the same balloon as it is being deflated. This is because the balloon is not perfectly elastic — it has a sort of 'memory' because it takes a little while to recover it's shape after being stretched. So the deflated balloon ends up wrinkly, since it has not gone back to its original shape. This means that even though it returns to zero 'gauge pressure' (pressure above atmospheric), the balloon is larger after being inflated. The balloon may recover its shape after a half-hour or so.

This kind of general effect — having a curve which is different going up than it is coming down, for the same values of radius — is called *hysteresis*. You may see it in the future in things like magnetism (say, a nail used in an electromagnet, which stays a bit magnetized afterward), and it comes up in many places in physics.

The balloon tug-of-war doesn't depend on hysteresis though — it is due mostly to the curve of the inflating balloon, which shows a *maximum pressure* as the balloon is being inflated.



When connected together by the straw, the two balloons must have the same pressure. Draw a line across the graph for this pressure, and you'll find that *two different sizes* of balloon can have that same pressure — these are the sizes of the two balloons. When you squeeze the larger balloon, the pressure of both balloons increases (they are still connected together!), so the line of the shared pressure rises, on the graph above. The large balloon is squeezed smaller, and the small balloon inflates because there is a (nearly) constant volume of air for them to share — they approach each other in size. With a perfect setup, when the line touches the maximum, the balloons have the same size, and can switch which one will be smaller, then the balloon collapses under your hand to become the small balloon. Since the balloon becomes small, you aren't squeezing it much anymore. The pressure drops back to what it was originally — only now the large balloon has become the small one, and the small one the large one!

<u>More for the very interested:</u>

You can also describe all of this in terms of the potential energy of the two-balloon system, which is to be *minimimized*. This potential energy has the same type of curve as shown above for *bistability*, and the system rests in one of the local minima of the curve. You can see how the size-changeover works, in the following way:

The work done when squeezing, to compress a balloon is force × distance moved ($W = F \bullet \Delta r$), and the force is the pressure × area ($F = P \bullet A$). So the potential energy *change*, for the balloon as it changes size, is the *work done on it*:

$$\Delta U = -W = -F \bullet \Delta r = -P \bullet A \bullet \Delta r = -P\,\Delta V,$$

since $A \bullet \Delta r$ is the change in volume of the balloon, for a small change Δr. This potential energy change is positive, because $\Delta V < 0$.

You do work, in squeezing the balloon and increasing the pressure, so the potential energy of the system increases. You need to increase it to get it over the hump in the potential energy, and then the system of two balloons can go over to the other minimum. Because each local minimum is 'stable', and there are two of them, the system is *bistable* — it can be at equilibrium in two different states.

[Robin]