1998-1999 Physics Olympiad Preparation Program

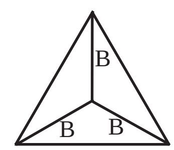
— University of Toronto —

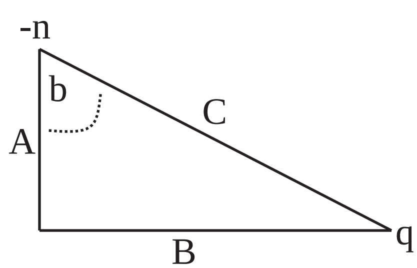
Solution Set 5: Electricity and Magnetism

1) Bully for you!

i) The symmetry gives it away. Put the negative ball in the middle, and place the other two to balance the Coulomb force. No matter how far apart are the positive charges, the net force on each of the three balls will be zero.

ii) This is a classic lateral thinking problem. Trying to balance three balls in a triangle, with two balls sounds impossible, UNLESS you think in 3D. Place each ball above and below the plane holding the three balls. Symmetry says that each negative ball must be an equal distance from each +q ball. Now balance the Coulomb forces. Define the equilateral triangle to have sides of length 1 unit. From the top this looks like the figure at right, from which we can find that $B = \frac{1}{\sqrt{3}}$.





From the side, this looks like the figure at left, where angle b is opposite side B. We find that we have to satisfy two equations:

$$\frac{n^2}{4A^2} = \frac{3nq\cos(b)}{C^2}$$ (vertical forces acting on ball –n)

$$\frac{2nq}{C^2}\sin(b) = \sqrt{3}\ q^2$$ (horizontal on ball q)

Since A = C * cos(b), the first equation simplifies to:

$$\frac{n}{q} = 12\left(\frac{A}{C}\right)^3$$

The second equation simplfies to

$$\frac{n}{q} = \frac{3}{2}C^3$$

So $2A = C^2$

Recall that $A^2 + B^2 = C^2$. These two equations then give:

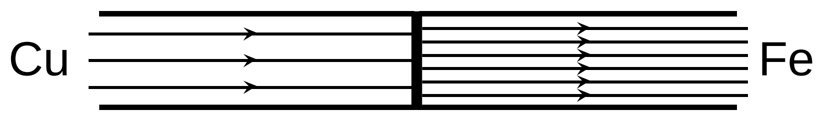
$$A = 1 \pm \sqrt{\frac{2}{3}}$$

and the value for n=10.4q or 0.33q. You win! *[James]*

2) Charged with resisting…

Let I denote the current flowing in the wire, A the cross-section of the wire and ρ_1 and ρ_2 the resistivities of the materials. Ohm's law for a wire of length ℓ gives $U = I\,\rho\ell\,/A$ which yields $E = U\,/\,\ell = \rho I/A$ for the electric field strength in the wire.

The resistivity of copper is lower than that of iron. Therefore to give the same current in each part, the electric field strength has to be smaller in the copper than in the iron.



According to Gauss's law, the difference in the electric field strengths implies an accumulation of charge at the boundary of the two metals (see figure). The total charge accumulated at the interface is

$$Q=\varepsilon_0 EA=\varepsilon_0 I(\rho_{Fe}-\rho_{Cu})$$

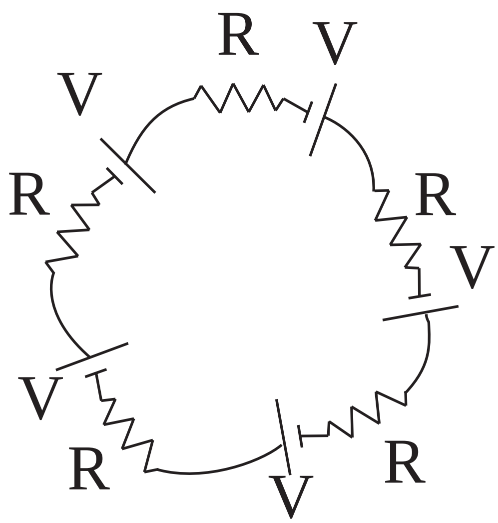
It is interesting that this quantity depends purely on the current and material constants, but not on the cross-sectional area of the wire (i.e., of the interface).

Substituting the known data, the charge is found to be $Q \cong 5.10^{-21}$ C, which is only 1/30th of an elementary charge! Though a measurable macroscopic current flows through the wire, the accumulated charge is only a small proportion of the microscopic elementary charge. This strange result shows that classical electrodynamics (imagining charge carriers as small balls) cannot always correctly describe electrical phenomena. Only application of the more sophisticated laws of quantum theory and statistical physics can give a more accurate description. [You might be interested to note that the 1998 Nobel Prize in Physics was shared by Robert Laughlin for his theory of the fractional quantum Hall effect, and fractional charge excitations.]   *[Gnädig/Honyek]*

3) 'Ascending and Descending Voltages': a circular argument

a) Obviously, I did not miss my true calling as a screenwriter. The total induced electric field, if the given equation is correct, is simply $0.001B_o$, over a distance of $2\sqrt{0.001\pi}$, so the total voltage (E•d) is: $B_o \cdot 0.001 \cdot 2\sqrt{0.001\pi}$, and the total current is this value divided by 500 ohm. The voltage across one bulb is 1/5 of the total.

b) The picture you are mumbling about is called "Ascending and descending" (note hint in the title of the question). M.C. Escher makes use of a lovely illusion to draw a staircase with no beginning or end, and with people on it constantly descending or ascending. Actually, my colleague Julian (who is a fine violinist I might add) comments that there is an additional Escher print on the same theme, with a water fall linked in a similar way, endlessly producing power.

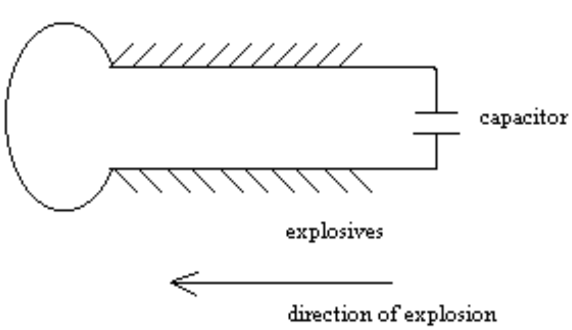


Of course, we have no such 'tricks' in physics, so what is going on in the circuit? The voltage "source" is distributed through every bit of wire, so the voltage drop across each resistor matches the voltage increase across 1/5 of the length of the wire. Thus you start out at the same voltage you began with. This is similar to the circuit at left.

c) Consider the right-hand rule in comparison to the equation your new friend is trying to sell you (another hint that I included in the equation). The induced current will create a magnetic field that will increase the flux in the loop, this will increase the induced current that will create more flux, you have a runaway effect! Any small perturbations in any small magnetic field will cause it to increase dramatically and without end. Of course this is not realistic so there must be a problem with your buddy's model. The real equation has a negative sign. The induced current acts *against* changes in the magnetic field. Mother Nature is truly a conservative force (no political insinuations intended) *[James]*

4) Flux compression — putting the squeeze on a B-field

This experiment is apparently done at Los Alamos National Laboratory, in New Mexico, USA. The two sides of the device have explosives attached to them, so that they start touching each other at the far end and finally end up being connected at the low end. The area of the loop is actually much smaller than that of the sides. The current is caused by a capacitor discharging.



As hinted, the Maxwell equation we will use is that for induced emf. We have:

$$\frac{d\Phi_B}{dt} = -Emf$$

$$\Phi_B = \oint \vec{B} \bullet \vec{dA}$$

So the change in the magnetic flux causes and induced emf. In our case, the initial flux is just B•A, since the field is uniform and perpendicular to the plane of the device. As the area of the device is shrinking, the flux is decreasing. However, we see from above that this change induces an emf in the circuit, which in turn produces a current. Which direction is this current? From the equation we see that it will oppose the decrease in the flux – i.e., the current will flow in a direction to oppose the decrease in flux and will thus increase it. How much does the flux grow by? It will keep growing until it equals the original flux, because then $\Delta B = 0$, emf induced = 0, and we have reached a steady state.

Thus, we see that flux will be conserved in this process and we will get

$$\Phi_B = const = BA = B'A'$$

$$B' = B\frac{A}{A'}$$

Field of solenoid is $B = \mu_0 In$ using Ampere's Law. A direct application of i) gives $B' = B\left(\frac{r}{r'}\right)^2 = \mu_0 In\left(\frac{r}{r'}\right)^2$ which gives 1.26 T in this case. Note that the field grows as the square of the radius, which means that we can achieve very large fields (~ 1000 T) with this technique. *[Peter and Bryan]*

5) Scratch and dent sale on capacitors

i) Capacitance C is defined such that $Q = C\ V$, where Q is the charge on the capacitor and V its electrostatic potential. Thus we can write $C = Q/V$. For a spherical distribution of charge, the field and potential outside the sphere is exactly as it would be if the entire charge were concentrated at a point at the centre of the sphere. Thus at the sphere's surface or a vanishingly small distance beyond, $V = k\ Q/R$, where R is the sphere's radius. From this it follows that the capacitance of an isolated sphere is:

$$C_{sphere} = Q/V = Q/\ (k\ Q/R,) = R/k$$

(the 'grounded' capacitor plate is at infinite distance, where the electrostatic potential approaches zero)

The earth's radius is 6.378×10^6 m, and $k = 8.9875 \times 10^9$ N m^2 C^{-2}; from this it follows that the capacitance of the earth is 7.1×10^{-4} C^2 N^{-1} m^{-1}, or 710 μF. Seems kind of small for such a big planet!

ii) The energy of a capacitor of charge Q and capacitance C is $Q^2/(2C)$. If the change in energy of the capacitor can be found the change in its capacitance can also be calculated.

The energy of the capacitor is higher when it is indented, since the surface charges move in a direction opposite to the force acting on them. Also an electrostatic field of field strength **E** has an energy $\varepsilon E^2/2$ per unit volume and when the capacitor is indented the electric field penetrates a volume where it was not previously present.

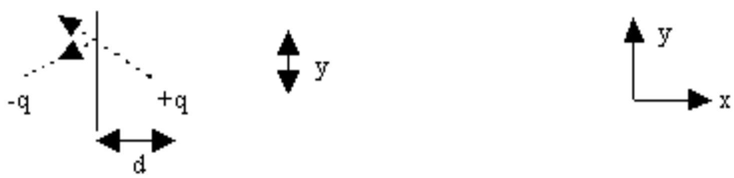
If the surface of the capacitor is only changed a little the electric field can be considered as identical to the original one near the surface. Thus, the change in energy depends purely on the change in volume and not on the actual shape of the indent.

Imagine that the capacitor is hammered so that its volume decreases by 3%, but its shape remains spherical. Its radius is therefore reduced by 1% (as the volume of the sphere is proportional to the cube of its radius). The ratio of the energy of such a capacitor to the energy of the original one is the same as the ratio of the energy of the indented capacitor of the problem to that of the original one. Thus, the relative change in their capacitance is identical as well.

Further, the capacitance of a spherical capacitor is proportional to its radius. The capacitance of the new capacitor (of reduced size) is therefore 1% smaller, and the capacitance of the capacitor of the original question decreases by the same amount. *[Robin & Gnädig/Honyek]*

6) Mocking mirrors

i) We have:

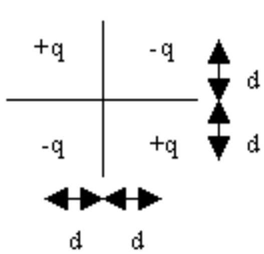


So that the field at a point (0, y) is given by:

$$\vec{E}(0,y) = -\frac{2\cos(\theta)q}{4\pi\varepsilon_o R^2}\hat{x} = -\frac{2dq}{4\pi\varepsilon_o\left(y^2+d^2\right)^{3/2}}\hat{x}$$

And it has no y-component. Hence, the potential, which is just the integral of the field (up to a constant) gives zero (since $\int 0dy = 0$), so the potential is indeed constant in the planes.

ii) By comparing this to i), or noticing that the image charges correspond to images of object as produced by a mirror, a self-consistent solution is:

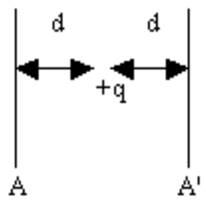


The field at any point (0, y) is

$$\vec{E}(0,y) = -\frac{2dq}{4\pi\varepsilon_o\left((y+d)^2+d^2\right)^{3/2}}\hat{x} + \frac{2dq}{4\pi\varepsilon_o\left((y-d)^2+d^2\right)^{3/2}}\hat{x}$$

which is also the field at a point (–x, 0), by symmetry.

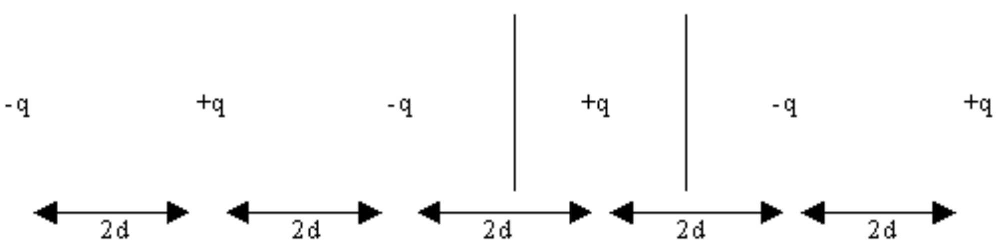
iii)



Consider this reasoning: to make A equipotential we have to put a charge –q a distance d left of it. To make A' equipotential, we put –q a distance d to the right of A'. But now the two –q charges cause A and A' not to be equipotential again. So we put a charge +q a distance 3d left / right of A / A' (respectively). And so on, ad infinitum.

This is equivalent to the forming of infinitely many images when you stand between two mirrors — the 1st seems a distance d away, the 2nd, a distance 3d = d + the distance of the light ray bouncing from the opposite mirror.

We have:



[Peter]