1998-1999 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 6: AC Circuits and Electronics

1) Thief of time

i) Let's use these symbols for the various bits:

$R = 33\text{k}\Omega$, the resistor after the battery
$R_B = 220\text{k}\Omega$, the normal resistance of the bulb
$C = 30\ \mu\text{F}$, the capacitance of the capacitor in parallel with the bulb

Our basic equations are:

$V_C = V_B$ (in parallel)
$V_R + V_B = 100\,V$ (the whole voltage drop around a closed loop)

$I_R = I_B + I_C$ (the current is conserved as it splits over the bulb and capacitor paths)

$V_R = I_R \cdot R$ (Ohm's law)
$V_B = I_B \cdot R_B$

$Q_C = C\ V_C$ (charge on a capacitor), which gives:

$$\Rightarrow I_C \equiv Q_C = C\dot{V}_C \text{ (just taking the derivative)}$$
$$= C\dot{V}_B$$

Then this can go into Ohm's law to give a voltage,

$$V_R = I_R \cdot R = (I_B + I_C) \cdot R$$
$$= (I_B + C\dot{V}_B) \cdot R$$
$$= \left(\frac{V_B}{R_B} + C\dot{V}_B\right) \cdot R$$

and then this can be used in the voltage-sum expression

$V_R + V_B = 100\,V$, to reduce just to an expression for the bulb-voltage:

$$\left(\frac{V_B}{R_B} + C\dot{V}_B\right) \cdot R + V_B = 100\,V$$

$$\dot{V}_B + V_B \frac{1}{C}\left(\frac{1}{R_B} + \frac{1}{R}\right) - \frac{100\,V}{RC} = 0$$

which we can write in a simpler form for the constants,

$$\dot{V}_B + V_B = b + c = 0$$

$$b = \frac{1}{C}\left(\frac{1}{R_B} + \frac{1}{R}\right)$$

$$c = -\frac{100\,V}{RC}$$

We can solve this several ways – the most straightforward is just to integrate, using $\dot{V}_B = \frac{dV_B}{dt}$, so that you get an expression $d\,V_B$ on one side and dt on the other, to integrate.

A neater answer is to think about what this means — what is going on. After a long time, we can guess that if the bulb hasn't broken down, the capacitor will be all charged up and that nothing will be changing (*i.e.*, will have a 'steady state'). Then there could be no changes in time, so $\dot{V}_B(t) = 0$, and therefore:

$$V_B(\infty) \cdot b + c = 0$$

$$\Rightarrow V_B(\infty) = -\frac{c}{b} \;=\; (100\ V) \cdot \frac{R_B}{(R + R_B)}$$

Thinking about the start, when the battery is connected but no charge has yet flowed through R, we need to have $V_B(0) = 0$. So the right solution starts at zero and asymptotically (over a long time) goes to a constant as a limit. If you already know RC circuits, or if you know 1st order differential equations, or if you integrate the equations (or if you're lucky!), you might guess that the change is an exponential relaxation, which if we start at zero and go to $V_B(\infty)$ would have the form

$$V_B(t) = V_B(\infty) \cdot \left(1 - e^{-at}\right)$$

If you put this into the equation above, you'll see easily that it actually works:

$$(-a) V_B(\infty) e^{-at} + V_B(\infty)\left(1 - e^{-at}\right) \cdot b + c = 0$$

$$\left\{(a - b) \cdot V_B(\infty) e^{-at}\right\} + \left\{V_B(\infty) \cdot b + c\right\} = 0$$

This has to be true for *all* values of t, which means *each* parenthesis term must be zero independently. Taking the second term to be zero gives $V_B(\infty)$, if we didn't already guess it above. The first term to be zero gives us a=b, so the solution is:

$$V_B(t) = 100\,V \cdot \frac{R_B}{R + R_B} \cdot \left(1 - e^{-\frac{1}{C}\left(\frac{1}{R_B} + \frac{1}{R}\right)t}\right)$$

or with our exact values put in:

$$V_B(t) = 87.0\,V \cdot \left(1 - e^{-1.16t}\right)$$

$V_B(t)$ reaches breakdown for the bulb (80V) when $80 = 87 \bullet \left(1 - e^{-1.16t}\right) \quad \Rightarrow t = 2.18\text{s}$

ii) At breakdown, the capacitor discharges through the bulb, until the bulb 'recovers'.

$$V_C = V_B$$

$$I_B + I_C = 0$$

$$Q_C = C V_C$$

$$\Rightarrow I_C = C \dot{V}_C$$

and with

$$V_B = I_B R_B$$

and also (with current left-to-right taken positive)

$$0 = I_B + I_C$$

This then gives

$$\frac{V_B}{R_B} + C \dot{V}_B = 0$$

$$\Rightarrow \dot{V}_B + \frac{1}{RC} V_B = 0$$

which has a solution $V_B(t) = V_B(0) \cdot e^{-\frac{1}{RC}t}$

So a good characteristic time of the bulb's flash is $RC = 0.5\Omega \cdot 30\,\mu\text{F} = 15\,\mu\text{s}$

iii) The energy stored in a capacitor charged to voltage V is: $\frac{1}{2}CV^2 = \frac{1}{2}\,30\mu\text{F} \cdot (80\,V)^2 = 96\,\text{mJ}$ (not enough to make the bulb explode, I hope!) It is all dumped through the neon lamp.

iv) going back to (ii) we can make the flash last longer by adding a resistor R' series with the bulb. Then $\tau = RC = (0.5\Omega + R') \cdot 30\,\mu\text{F}$ which you can control by the value of R' and we can make the bulb light up for longer. But since

$$I_B = I_C = -C\dot{V}_C = -C\dot{V}_B$$

and $\dot{V}_B(t)$ will be smaller with R' added, then the current through the bulb will be smaller and it won't be as bright. *[Robin]*

2) Logic rules (!)

i) This is an OR gate. When all of A, B, C are low (i.e., voltage is, say, 0 V), the output is naturally low. But when either A or B or C are high, so will the output be.

Problem: when either A or B or C conduct, some current flows through the circuit. But when more than one input is high, the currents actually add. If we have something fairly sensitive connected to the output, we could fry it if the current is too large. What's worse, in a computer many gates are connected together – but if we connect a few of these gates to the inputs of another gate, the currents will add (if we have n inputs, final current is nI. If we feed n of such gates into another similar gate, the current is now n^2I) – this is clearly a problem…

ii) a) This is a NOT gate. When A is low (ca. 0 V), the base and the emitter are both at 0 Volts. The transistor is thus cut-off – no current flows from the collector to the emitter (it acts like an open circuit). If we now connect a device with a large resistance (>> *R*), the voltage drop across *R* is negligible and almost all voltage (Vo) goes through the device. So the output is high.

[Voltage across *R* would be *R*I and that across the device rI, but since r >> *R* the first may be ignored. Note that the current flowing through both is the same]

When A is high, the transistor conducts (with almost no resistance) – thus, all the voltage drop will be across *R* [since it is smaller than r – the resistance of the output device]. Hence, the output is low.

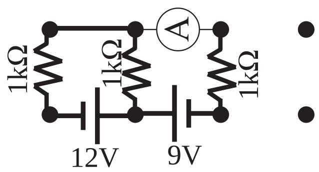
So this is a NOT gate.

b) The second gate (from the left) is a NOT, from above. The first gate is a NAND, by a similar argument as above. Thus, the gate is a NOT (NAND) = AND.

Note that using transistors the output and current voltage are constant. There is still, however another problem, but I'll let you figure it out… *[Peter]*

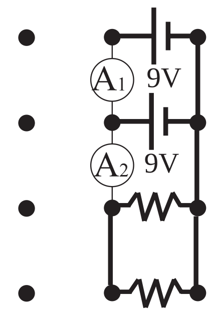
3) Small, but wirey…

a) Set I=2mA.



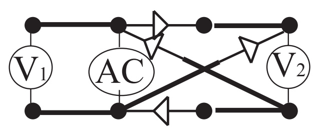
Be careful about the polarity of the 12V source!

b) Set I_2=18mA. What is I_1?



So what is I_1? The normal voltage/current laws do not help you. The only answer that I can argue would be using symmetry, and thus I_1=9mA.

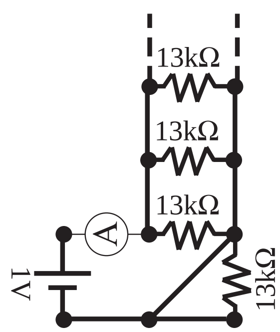
c)



This is known as a full-wave rectifier and can be used to chance AC voltage into DC voltage (with an extra smoothing filter)

d) Set I=47.5 mA.

total: 618 resistors



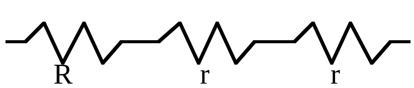
I had a more complicated answer but this one, handed in by a few students, is more straightforward.

[James]

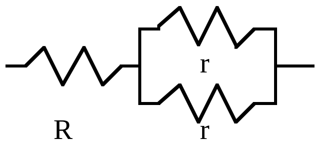
4) Is resistance futile?

Let's try all 6 possibilities:

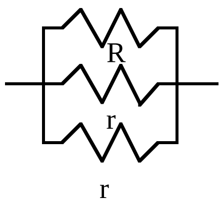
a)



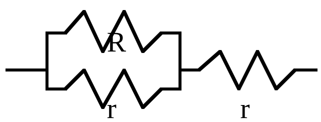
b)



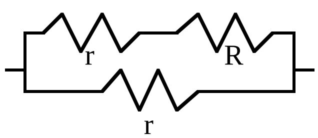
c)



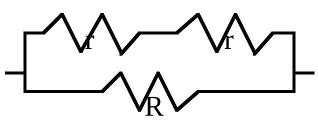
d)



e)



f)



From a) we get: $R = R + 2r$, which is impossible (r > 0)

From b) we get: $R = r/2 + R$, which is again impossible

From c) we get: $r = r + 2R$ – impossible.

From d) we get: $R = r\left(\frac{1+\sqrt{5}}{2}\right)$

From e) we get: $R = r\left(\frac{-1+\sqrt{5}}{2}\right)$

From f) we get: $2r = 2r + R$ – impossible.

So it is possible, using (d) or (e).

A sophisticated aside for question 4:

Note that we have a way of making resistances *r* that are rational. But note that both the solutions (d) and (e) in question 4 are irrational — if our arbitrary *r* is rational then it follows that *R* is irrational. But to make *R* we only used *r*'s in some combinations – we only used rational numbers.

Let's recall what kinds of circuits we can make using resistors:

We could have resistors in parallel and series. But the equivalence formulas involve additions (series), or divisions (parallel), so there is no way to obtain a rational number.

In the general case, we need to solve Kirchoff's Laws, which are linear and create linear systems of equations. Solving such a system involves adding / subtracting rows, and multiplying and/or dividing numbers – once again there is no way to obtain an irrational number.

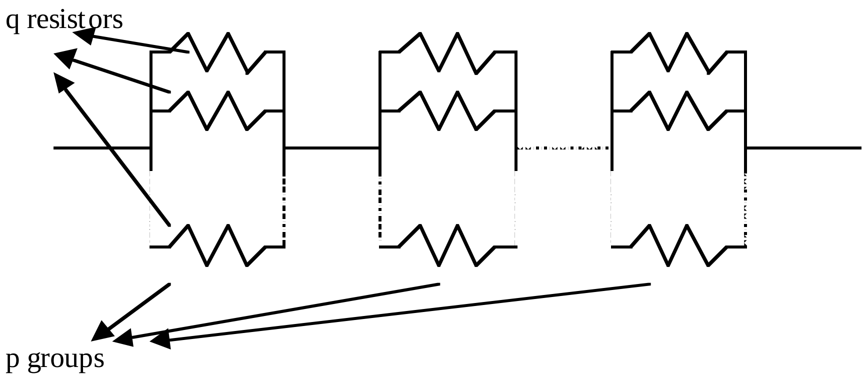
And thus we must conclude that theoretically it is <u>not</u> possible to make the circuit from question 4, unless perhaps we could make *R* from an *infinite* number of resistors *r*.

Experimentally, on the other hand it would be possible. For in experiments we can only measure things to a finite precision. Thus, there are no irrational values – since we only measure a certain amount (say 10) digits, every number is finite and hence rational. And hence we could achieve the circuits from either (d) or (e). *[Peter]*

5) ***'E pluribus unum'***

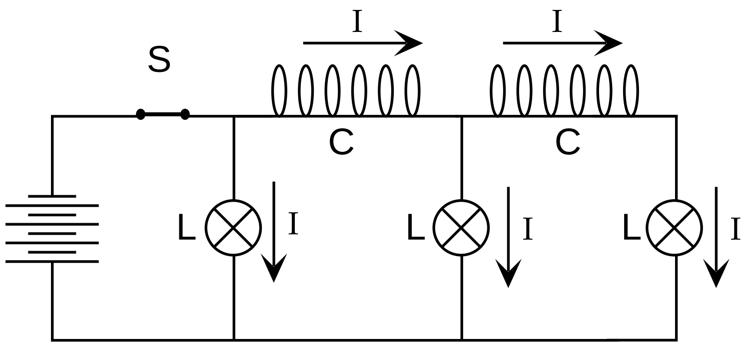
Within experimental error, the other resistor (call it *R*) can only be measured to a finite number of digits. Given any such number, we can express it as a rational number – i.e., in the form p/q where both p and q are integers.

To create a circuit to express p/q we build the following circuit – we connect in series p groups of q groups containing 1 Ω resistors in parallel.

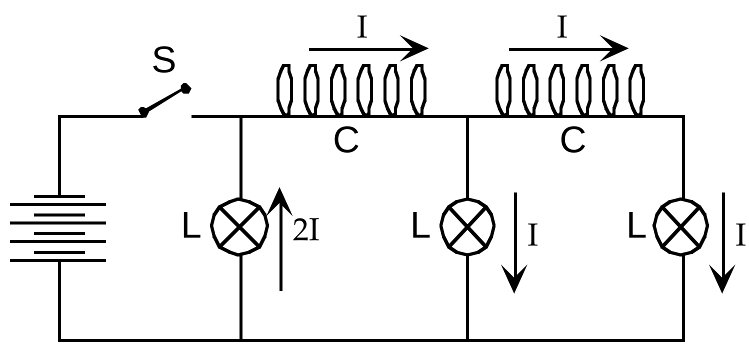


[solution from a Polish Olympiad book "Olimpiady Fizyczne XXI I XXII"] *[Peter]*

6) Inductive reasoning



When the switch is closed, currents as shown at left flow round the circuit. (The value of the current is determined by the voltage of the battery and the resistances of the lamps.) A very short time after turning the switch off the current flowing in the coils is practically unchanged. (If this were not the case, there would be a rapid change in the magnetic flux, which would induce a very high voltage in the coil.) Currents of 2*I* and *I* continue to flow therefore in the coils, and these determine the currents flowing through the lamps (figure at right).



This means that the lamp closest to the switch suddenly flashes and the brightness of the two other lamps does not change. (This is only true for a short time, later all three lamps fade and go out.) *[Gnädig/Honyek]*