1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 2: Mechanics

1) Going for a spin

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{15 \cdot 2 \cdot \pi / 60}{3} = 0.5 rad / s^2$$

Angular momentum is conserved (but energy is not!!)

$$I_o \omega_o = (I_o + mr^2)\omega$$

$$\frac{\omega}{\omega_o} = \frac{2000}{2000 + 20(2.5)^2} = 0.9\%$$

Ignoring the fact that Matt is sitting off the whirly-ride, what happens is that the force of friction is providing the centripetal acceleration necessary to keep him on the ride. The force is $\leq \mu mg = 137.2 N$.

Thus, at maximum,

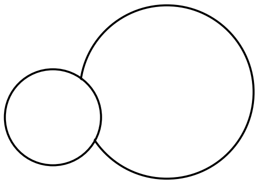
$$\frac{v^2}{4.5} = 0.7 \cdot 9.8,\ v = 6\ \ m s^{\pm1} \text{ and } \omega = \frac{v}{4.5} = 1\ \ rad s^{\pm1}$$ *[Carrie]*

2) Bubble bonanza

a) Consider 2 hemispheres of a bubble of radius r. They are repelled by a force of $F = \Delta PA = \Delta P \pi r^2$, where A is the area of the projection of half a sphere and $\Delta P = P_o - P$ (difference between outer (atmospheric) and inner pressure). The hemispheres are pulled together by surface tension (which is given as force per unit length); moreover, there are 2 surfaces to a bubble, so the total force here is $F = 2(2\pi r\sigma)$. Solving yields

$$P = P_o + \frac{4\sigma}{r}$$

b) Note that pressure is inversely proportional to the radius. Thus, the smaller bubble will actually have a higher pressure inside! This means that the film between the 2 bubbles will be pushed into the larger bubble.



If the film breaks, the bubbles will either break (but that's so boring), or merge. Using the fact that the numbers of moles inside the bubble stays constant, assuming that temperature stays constant and using the ideal gas law, one gets

$$PV = P_1V_1 + P_2V_2$$

$$\left(\frac{4\sigma}{r} + P_o\right)r^3 = \left(\frac{4\sigma}{r_1} + P_o\right)r_1^{\,3} + \left(\frac{4\sigma}{r_2} + P_o\right)r_2^{\,3}$$

(P, V, r refer to the final bubble, the remaining variables refer to the 2 initial bubbles)

The equation could in theory be solved; note that for $P_o = 0$, the final bubble will be bigger than either of the 2 initial bubbles…

The system is obviously not in equilibrium. Since the smaller bubble is at a higher pressure, it will in fact contract and force air into the bigger bubble. Thus, we will end up with a membrane at one end and a big bubble at the other. *[Peter, Amir]*

3) Siphoning cellar

Solutions for this problem will be provided with Set #3 *[Robin]*

4) Gimme a break...

First I have to comment that this question was so hard that nobody got it right, including some of us here at POPTOR…

a) Ok, that was easy. Conserving energy yields

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m(0.5)^2 + mg(6-4)$$

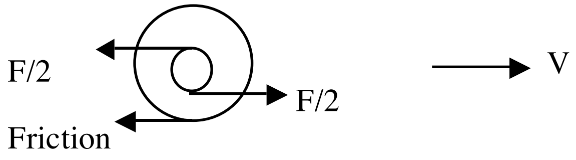
From this, v = 6 m/s.

b) This is where things get difficult. Clearly we want to stop the cart using static friction, as these coefficients are higher than those for kinetic friction, and thus the force will also be larger. Something has to slip, however (you can't just stop the cart dead) – since the coefficient of kinetic friction between the wheels and the rail is lower than that between the brakes and the wheel, we want to avoid letting the wheels slip (it will yield the lowest force).

Now, if we have an object of mass m lying on a level plane, where the coefficient of static friction is μ it certainly is true that the maximum force we can apply so that the object doesn't start moving (and kinetic friction doesn't kick in) is $F_{MAX} = F_{friction} = \mu mg$.

But a rolling object is different!! For one, it is not stationary, it is already moving. So what force can we apply? Well, we have to ensure that the cart keeps on rolling, i.e. V(t) = ω(t)R, and not slipping…

I am also going to assume that there are 2 forces acting on the wheel – each of magnitude F/2 - pointing in different directions.



We have to analyze the linear and rolling motions (about the axle) while the brakes are applied, knowing we start with the cart rolling (m is the mass of the cart, M the mass of the wheel, I the moment of inertia of, F the applied force, R the radius of the wheel and μ the coefficient of static friction between the wheel and the track).

$$ma = -\mu mg$$
$$I\alpha = \mu mgR - FR$$

From this we find

$$v(t) = v_o + at = v_o - \mu gt$$
$$\omega(t) = \omega_o + \alpha t = \omega_o - \frac{F/2 \cdot Rt}{I} + \frac{\mu mgRt}{I}$$

We have that $v_o = \omega_o R$ at t=0, but to ensure that this is true at later times, we clearly need that

$$\mu g = \left(\frac{F/2 \cdot R}{I} - \frac{\mu mgR}{I}\right)R$$

The moment of inertia is that of the wheel, *i.e.,* of 2 cylinders, one of radius R and the other of radius R/2. Thus, $I = \tfrac{1}{2}MR^2 + \tfrac{1}{2}M\left(\frac{R}{2}\right)^2 = \frac{5}{8}MR^2$ (I've assumed for simplicity that both sections of the wheel weigh the same amount; it would probably make more sense to assume that their densities are the same, but oh well…)

$$F/2 = \frac{5}{8}\mu Mg + \frac{8}{5}\mu mg \approx \frac{8}{5}(0.78)mg$$

(assuming the wheels are not very heavy)

This is the force felt by the wheel; the actual applied force is

$F = F_a(0.65)$ (the brakes have to be slipping)

Solving for the applied force gives

$$F_a = 2 \cdot 18.8m = 38m$$

NOTE: most people put that $F = 2\mu mg$, which is less than the above. Plugging this into the equations above we can see that $V(t) \neq \omega(t)R$, and so it seems that the cart slips… but you can do some experiments at home to convince yourself otherwise. What's going on here? Well, the force of friction is actually $F_{friction} \leq \mu mg$, and what will in fact happen is that it will adjust itself to a lower value so that the cart doesn't slip… The above force is the maximum that can be applied so that the cart doesn't slip; a bigger force and it's all over…

c) The above force does work ($W = F_a d$) to slow the cart down. It starts with an energy $\frac{1}{2}m(0.5)^2 + mg6$. After a distance d it will stop; here, d turns out to be 1.6 m. Thus, the cart never even makes it to the hill…

d) If the cart was on the hill, we need it to be in static equilibrium. From the above diagram we see that this amounts to $mg\sin\theta \leq 0.78mg\cos\theta$, or $\theta \leq 38^o$. Eyeballing the diagram, this seems to be false, and thus the cart will probably fall… *[Peter]*

5) It's a ball? What a gas!

a) Doing the question in the reference frame of the wall really simplifies things. In the rest frame of the wall, the ball velocity is the same before and after the collision but opposite in the direction. In this system, the ball's initial velocity is v + u. So the ball bounces back with v + u in reference to the wall. This velocity is v + 2u in reference to the laboratory.

b) The change in kinetic energy = $\frac{1}{2}m(v+2u)^2 - \frac{1}{2}mv^2 = 2mu(u+v)$

The work done by the wall = Fx

But $F = \frac{\Delta p}{\Delta t} = \frac{m(v+2u-(-v))}{t} = \frac{2m(u+v)}{t}$; $x = ut$

Therefore work = $Fx = 2mu(u+v)$, same as the change in kinetic energy above…

<u>BONUS</u>: (solution based on Feynman's); "< >" denotes an averaging

Let us work in 1D (results in 3D are very similar); let ρ be the density (#) of atoms per unit volume.

Assume the compression is slow (so the gas stays in equilibrium). Then, the wall is practically stationary. The force exerted by the wall = change in momentum / Δt is

$$F = 2\rho mv^2 A$$

We want to average this over all the speeds, v. Notice that only _ of all atoms (balls) are moving toward the wall – we do not want to average over them, as they do not contribute to the force on the wall. This yields

$$P = F/A = \rho < mv^2 >$$

Multiplying by the volume, V gives,

$$PV = N < mv^2 >$$

In 1D we have that $\frac{1}{2}kT = \frac{1}{2} < mv^2 >$ and thus

$$PV = NkT$$

If you want to know how temperature changes with volume, you can note that this is an adiabatic process and look up the formula in any book (or derive it using the above). *[Peter, Amir]*

6) Dimensional thinking

a) First of all, figuring out how much water to add is equivalent to figuring how much water is actually leaving. There a couple of ways to do this, but dimensional analysis is perhaps the most instructive. First, since the hole is small, the amount of water flowing out in a short time will change the volume in the vessel by very little; thus, the area A is irrelevant.

Now, we need to combine the remaining variables into a flow rate R that has units [kg/s]. We now assume that the formula is of the form

$$R = C\rho^x H^y r^z g^w$$

The units must combine to [kg/s]. It's pretty easy to see that x=1 and w=1/2. Now, it would seem logical that the flow is proportional to the area of the hole, *i.e.,* r^2. Since the flow also grows as H to some power (not as inverse of H), the only logical choice is to set z=2 and H=1/2.

Thus, $R = C\rho r^2\sqrt{gH}$, where C is some constant.

b) Using Bernoulli's Law to find the flow rate of the water leaving the vessel leads one to

$$\tfrac{1}{2}\rho v^2 = \rho g H + \tfrac{1}{2}\rho U^2$$

Here, v is the speed of flow in the pipe and U is the speed of water flow in the tank. For a large area A (compared to the area of the hole), the U term is negligible. However, we don't have to make this assumption. Since the flow is steady, the continuity equation applies and we have $UA = v\pi r^2$. The rate R is $R = \rho v \pi r^2$. Solving gives

$$R = \rho\pi r^2 \frac{\sqrt{2gH}}{\sqrt{1-\left(\frac{\pi r^2}{A}\right)^2}}.$$

(for $A >> \pi r^2$, this is indeed equivalent to a), if $C = \pi\sqrt{2}$) *[Peter, Yaser]*