1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 4: Optics and Waves

1) Bob-bob-bobbin' along

This question was a sneaky way to get you solve a second order differential equation quite painlessly.

a) The forces involved included the restoring force, –kx and the inertial force, ma, where $a = \frac{d^2x}{dt^2}$. These two balance each other (the net force is zero). Thus the force equation for our system is,

$$m\left(\frac{d^2x}{dt^2}\right) = -kx$$

Which is a second-order ordinary differential equation.

b) We are given x = sin(ωt) as a solution to our force equation. Taking the first time derivative we obtain,

$$\frac{dx}{dt} = \omega\cos(\omega t)$$

Differentiating again with respect to time yields,

$$\frac{d^2x}{dt^2} = -\omega^2\sin(\omega t)$$

Plugging this expression for the acceleration into our force equation, along with x=sin(ωt) yields

$$m\left(-\omega^2\sin(\omega t)\right) = -kx = -k\sin(\omega t)$$

Therefore, solving for ω we get $\omega^2 = \frac{k}{m}$.

So this oscillation x = sin(ωt) isn't always a solution — only for a certain ω will there be a solution. So we find that the frequency of oscillation depends this way on both the spring constant of the spring (its strength) and the mass of the bob attached.

c) Putting the values k = 80 N/m and m = 100 g into our equation for ω, gives us a value

$$\omega = \sqrt{\frac{80}{0.1}} = 28/\text{sec},\ \text{or } \mathbf{4.5\ Hz}$$ *[Carrie]*

2) Keeping up an image

Since the object distance u and the image distance v can be exchanged in the lens law and their ratio shows the magnification to be $(v/u)^2 = 9$ (*or* 1/9), then $v/u = 3$ (or 1/3). Thus, the object distance is 30 cm (or 90 cm) and the new image distance is 90 cm (or 30 cm). The focal length can be calculated from the lens law: $f = 22.5$ cm.

If the same amount of light passed through the lens in both cases then the 9 times smaller image would be 81 times brighter, as the smaller image occupies a surface 81 times smaller on the screen than the larger one. However, the lens placed at a greater distance receives only *one ninth* of the light reaching the nearby lens, therefore the small image is only nine times brighter that the large one.

It can be shown in general that in such cases the small image is as many times *brighter* than the large one as the large one is *larger*. *[Gnädig/Honyek]*

3) Shifty radar

a) The radar gun uses the *Doppler shift* to detect the speed of cars. For light the correct doppler effect involves relativistic considerations. For cars, which have relatively small source velocities, we skip those. Thus we have for the officers viewing at rest,

$$f_L \approx (1 + v/c)f_S$$

Where f_L is the frequency perceived by the officers (or listeners at rest), v is the velocity of the source, c the speed of light, and f_S the source frequency (1000 MHz in this case). If waves are reflected from a moving car the reflected frequency, f_R, is that of the a source moving with twice the car velocity. So,

$$f_R \approx (1 + 2v/c)f_S.$$

The waves which are reflected from the moving cars are beat against the transmitted waves, yielding a frequency difference given by,

$$F = f_R - f_S \approx 2vf_S/c$$

Solving for the velocity of the car we get, $v \approx Fc/2f_S$.

b) Plugging in the numbers to our velocity equation we get,

$$v = [(330\ \text{Hz})(3 \times 10^8\ \text{m/s}) \ / \ (2\ (10^9\ \text{Hz}))](1\ \text{km}/1000\text{m})(3600\ \text{s}\ /1\ \text{hr})$$

$$= 178\ \text{km}\ /\text{hr}\ !!!!$$

Book him, Dano! *[Carrie]*

4) To air is human

a) We know that in an adiabatic process PV^γ is constant.

$$P_0 = \frac{nRT_0}{V_0} = \frac{nRT_0}{Ah_0}$$

$$P_1 = P_0 + \frac{Mg}{A}$$

$$P_1 V_1^\gamma = P_0 V_0^\gamma \quad \Rightarrow \quad P_1 h_1^\gamma = P_0 h_0^\gamma$$

$$\Rightarrow \; h_1 = h_0 \left(\frac{P_0}{P_1}\right)^{1/\gamma} = h_0 \left(\frac{P_0 A}{P_0 A + Mg}\right)^{1/\gamma}$$

$$h_1 = \frac{nRT_0}{P_0 A} \left(\frac{P_0 A}{P_0 A + Mg}\right)^{1/\gamma}$$

b) Exactly the same as part (a) you can see that:

$$h_2 = \frac{nRT_0}{P_0 A} \left(\frac{P_0 A}{P_0 A + (M+m)g}\right)^{1/\gamma}$$

$$P_2 = P_0 + \frac{(M+m)g}{A}$$

c) The force acting in a mass-spring system is $-k\Delta x$. Consider that we change the equilibrium state by moving the piston x away from equilibrium. The force acting on the piston to push it back will be:

$$F = P_0 A - PA + (M+m)\,g$$

Since $P(h_2 + x)^\gamma = P_2 h_2^\gamma = P_1 h_1^\gamma = P_0 h_0^\gamma$

$$P = P_2 \frac{h_2^\gamma}{(h_2 + x)^\gamma}$$

and since x is quite small compared to h_2,

$$P = P_2 \left(1 + \frac{x}{h_2}\right)^{-\gamma} \approx P_2 \left(1 - \frac{\gamma x}{h_2}\right)$$

$$\Rightarrow F = P_0 A + (M+m)g - P_2 A \left(1 - \frac{\gamma x}{h_2}\right)$$

$$\Rightarrow F = \gamma\, P_2 \frac{A}{h_2}\, x = \; k\, x$$

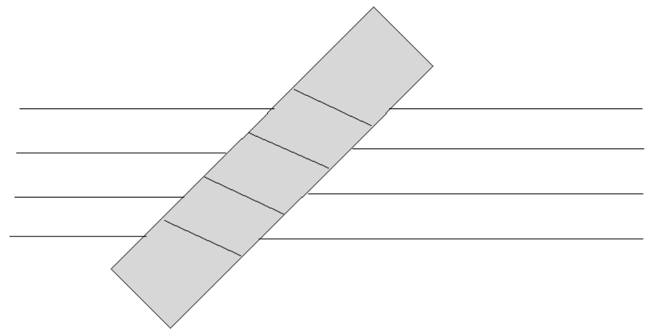
the spring coefficient of the system will be $\frac{\gamma P_2 A}{h_2}$ and the corresponding mass will be (M + m).

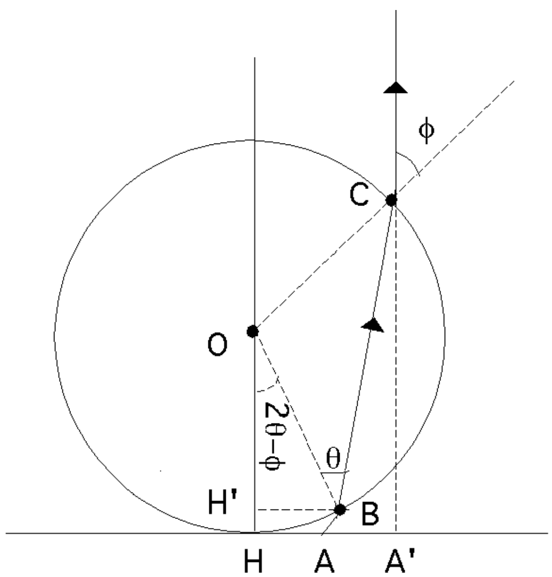
d) Since the harmonic oscillator goes up and down with the same distance, and the upper level of this oscillator is $(h_2 - h_1)$ above the equilibrium, the lower level will be $h_2 - h_1$ under the equilibrium.

$$h_{min} = h_1 - (h_2 - h_1) = 2h_1 - h_2$$

5) Bending the truth

a) A rod is a cylindrical lens which basically magnifies in one direction only, and leaves the other direction unchanged, so the line will be magnified in the direction perpendicular to the rod, and won't change in the direction of parallel to the rod.





b) Since we are looking far from the rod, the outgoing beams (going to your eyes) are parallel. The point A is emitting light in all directions, but we are concerned about the beam that will go out vertically. The beams are sketched at right.

The point A will be seen as A′ and we can find the ratio of $\frac{AH'}{AH}$ by trigonometry. First we assume that AH is approximately equal to the BH' line:

$$HA = BH' = R\sin(\theta - (\varphi - \theta)) = R\sin(2\theta - \varphi)$$
$$HA' = R\sin\varphi$$

$$\text{Magnification} = \frac{HA'}{HA} = \frac{\sin\varphi}{\sin(2\theta - \varphi)}$$

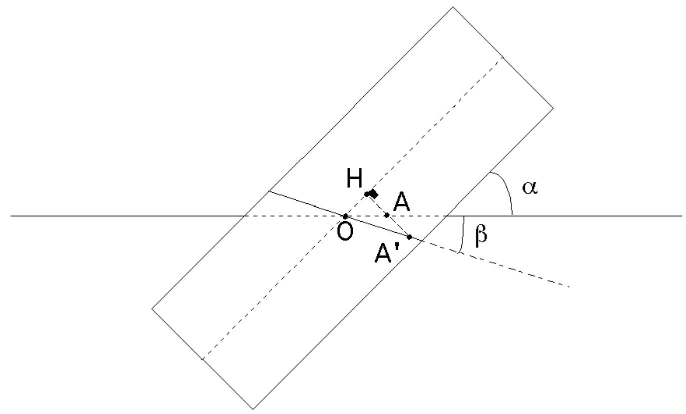
We neglect rays which pass either far from the centre, or at large angles from the axis (the *paraxial approximation*). In this case, θ and φ are small and $\begin{cases}\sin\theta \cong \theta \\ \sin\varphi \cong \varphi\end{cases}$.

$$\sin\varphi = n\sin\theta \quad \Rightarrow \quad \varphi = n\theta$$

$$\Rightarrow \text{Magnification} = \frac{n\theta}{2\theta - n\theta} = \frac{n}{2 - n}$$

In practice:

The rod is magnifying in the direction of the line perpendicular to the axis of rod. The point A will be seen as A' .



$$\frac{HA'}{HA} = m \text{ (Magnification)}$$

$$HA' = OH \tan(\alpha + \beta)$$

$$HA = OH \tan(\alpha)$$

$$\Rightarrow \frac{HA'}{HA} = \frac{\tan(\alpha+\beta)}{\tan(\alpha)} = \frac{n}{2-n} \quad \Rightarrow \quad n = \frac{2\tan(\alpha+\beta)}{\tan(\alpha) + \tan(\alpha+\beta)}$$

Where α is the angle between rod and parallel lines and β is the angle between bent lines and parallel lines. One can find n by measuring α and β. *[Amir]*

6) Cool abrrrrations!

a) $$\frac{1}{f} = (n-1) \bullet \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\Delta\left(\frac{1}{f}\right) = \Delta(n) \bullet \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$= \left(\frac{1}{f}\right) \bullet \frac{\Delta(n)}{(n-1)}$$

b) When two different lenses are placed together, their inverse focal lengths add up:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

We want the aberration to be minimum. So

$$\Delta\left(\frac{1}{f}\right) = 0$$

=> $$\frac{1}{f_1} \bullet \frac{\Delta(n_1)}{n_1 - 1} + \frac{1}{f_2} \bullet \frac{\Delta(n_2)}{n_2 - 1} = 0$$

Consider two wavelength 488 nm and 890 nm (*nanometers*).

$$\Delta(n_1) = 1.4877 - 1.4758 = 0.0119$$

$$\Delta(n_2) = 1.5793 - 1.5634 = 0.0159$$

Now we have two equations, two unknowns

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f} = \frac{1}{10}$$

$$\frac{1}{f_1} \bullet \frac{0.0119}{(1.48 - 1)} + \frac{1}{f_2} \bullet \frac{0.0159}{(1.57 - 1)} = 0$$

Therefore

$f_1 = 2.5$ cm

$f_2 = -3.3$ cm *[Yaser]*