1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

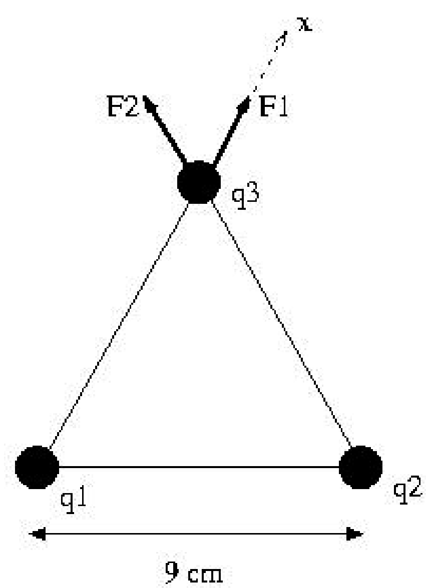
Solution Set 5: Electricity and Magnetism

1) Vile away the hours

i) The 2.0 g sphere contains 1.99×10^{22} atoms. The charge on the nucleus of each atom is 29 *e*. Thus the sphere contains $(1.99 \times 10^{22}) \bullet 29 = 5.77 \times 10^{23}$ electrons. One electron has a charge of 1.6×10^{-19} C. The electrons removed = $(3 \times 10^{-6}$ C$)/(1.6 \times 10^{-19}$ C per electron$) = 1.875 \times 10^{13}$ electrons. So the fraction of electrons removed is

$$(1.875 \times 10^{13}) \; / \; (5.77 \times 10^{23}) = 3.25 \times 10^{-11}$$

b) The figure at right illustrates the configuration of the charged spheres with q1 = 7 μC, q2 = 5 μC and q3 = 3 μC. Let the positive x direction be as shown, and the y direction perpendicular to it, towards the upper left. The net force on q3 is the vector sum of the repulsive forces F_1 and F_2. The magnitude of F_1 is given by:



$$\begin{aligned} F_1 &= \frac{k q_1 q_3}{r^2} \\ &= \frac{\left(9 \times 10^9 N\, m^2 C^{-2}\right) \bullet 7 \times 10^{-6} C \bullet 3 \times 10^{-6} C}{(0.09\, m)^2} \\ &= 23.3\, N \end{aligned}$$

Similarly F_2 is:

$$\begin{aligned} F_2 &= \frac{k q_2 q_3}{r^2} \\ &= \frac{\left(9 \times 10^9 N\, m^2 C^{-2}\right) \bullet 5 \times 10^{-6} C \bullet 3 \times 10^{-6} C}{(0.09\, m)^2} \\ &= 16.7\, N \end{aligned}$$

The vector sum $\vec{F} = \vec{F}_1 + \vec{F}_2$ has components along x and along y. In the x direction:

$$\begin{aligned} F_x &= F_{1x} + F_{2x} \\ &= 23.3\, N + 16.7\, N \bullet \cos(60°) \\ &= 31.7\, N \end{aligned}$$

and in the y direction,

$$
\begin{aligned}
F_y &= F_{1y} + F_{2y} \\
&= 0\ N\ +\ 16.7\ N \bullet \sin(60°) \\
&= 14.5\ N
\end{aligned}
$$

Thus the magnitude of the resultant force is:

$$
\begin{aligned}
F &= \sqrt{(31.7\,N)^2 + (14.5\,N)^2} \\
&= \ 34.9\,N
\end{aligned}
$$

The direction is at an angle

$\arctan(14.5/31.7) = 25°$ from the x axis. *[Carrie]*

2) Capacity for thought

The charge of each capacitor is $Q=CV$. When the capacitors are connected to each other and the separation between the plates change, the total charge is still conserved so $2Q=q_1+q_2$. Since the capacitors are parallel to each other, the voltage across them is equal to $V = \frac{q_1}{C_1} = \frac{q_2}{C_2}$. We know that the capacitance is inversely proportional to the separation of the plates. Therefore $C_1 d_1 = C_2 d_2$. Since the plates are moving together or apart by speed v for the two capacitors, we have

$$
d_1 = d + vt \qquad\qquad d_2 = d - vt
$$

Therefore $\frac{C_1}{C_2} = \frac{d_2}{d_1} = \frac{d - vt}{d + vt}$

On the other hand, $q_1 = \frac{C_1}{C_2} q_2 = \frac{d - vt}{d + vt} q_2$.

Substitute the value of $q_2 = 2Q - q_1$ in the above formula we have

$$
q_1 = Q\frac{d - vt}{d} \quad \text{and} \quad q_1 = Q\frac{d + vt}{d}
$$

The current in the circuit is the rate of change of charge, which is equal to

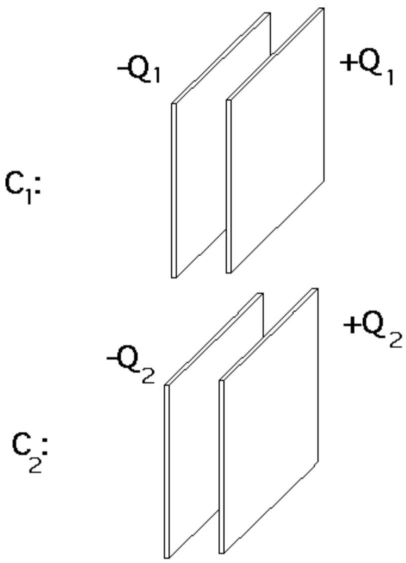
$$
I = \frac{dq_2}{dt} = -\frac{dq_1}{dt} = \frac{Qv}{d}.
$$

[Yaser]

3) Dielectric City

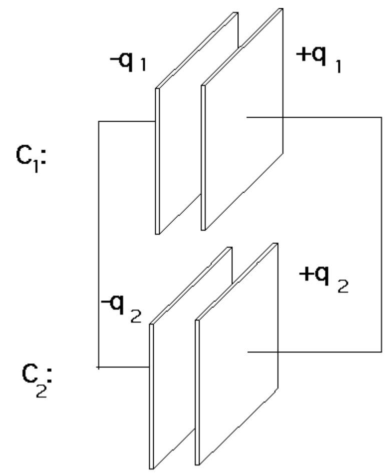
a) The energy stored in a capacitor with capacitance C and charge Q is $\frac{Q^2}{2C}$. So the TOTAL energy of a system consist of two capacitors is:

$$
E_1 = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2}
$$



b) At first, I'll find the equivalent capacitor and the total charge on it.

Those two capacitors are connected as parallel capacitors and the total charge on the right-hand-side plates is conserved (there is no external current) we have $q_1 + q_2 = Q_1 + Q_2$ (or similarly for the left plates: $(-q_1) + (-q_2) = (-Q_1) + (-Q_2)$)



$$C_{equivalent} = C_1 + C_2$$
$$Q_{total} = Q_1 + Q_2$$

Now I can find the new energy of the system of one equivalent capacitor.

$$E_2 = \frac{Q_{total}{}^2}{2C_{equivalent}} = \frac{(Q_1 + Q_2)^2}{2(C_1 + C_2)}$$

c) One may find the difference between E_2 and E_1 as:

$$E_1 - E_2 = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} - \frac{(Q_1 + Q_2)^2}{2(C_1 + C_2)}$$
$$= \frac{C_2(C_1 + C_2)Q_1^2 + C_1(C_1 + C_2)Q_2^2 - C_1C_2(Q_1^2 + Q_2^2 + 2Q_1Q_2)}{2C_1C_2(C_1 + C_2)}$$
$$= \frac{C_2^2Q_1^2 + C_1^2Q_2^2 - 2(C_1Q_2)(C_2Q_1)}{2C_1C_2(C_1 + C_2)}$$
$$= \frac{(C_2Q_1 - C_1Q_2)^2}{2C_1C_2(C_1 + C_2)} > 0$$

which is always equal or larger than zero.

The energy loss is because of the emission of accelerated particles due to connection of two wires with different voltages. When you connect the first two wires (say negative plates) nothing will happen and you will just have one ground to measure voltage on each part of the two circuits. You can then calculate the voltage difference of two positive plates.

Connecting the other two wires will create a large electric field between two wires (when they are separated by a small distance) and then the electrons will be accelerated. If we could put a resistor between two capacitors, this energy could have been changed into heat. Often, the energy loss can be found in the form of electromagnetic waves in visual range (you'll see it as a spark!).

d) Capacitance of a capacitor full of water is ε times capacitance of an empty capacitor. (Recall the formula for capacitance $C = \varepsilon\varepsilon_o \frac{A}{c}$, where ε is susceptibility of water.)

Energy of full and empty capacitor can be found by using that formula:

$$E_{empty} = \frac{Q^2}{2C_{empty}}$$

$$E_{full} = \frac{Q^2}{2C_{full}} = \frac{Q^2}{2\varepsilon C_{empty}}$$

and the energy difference is:

$$\Delta E = E_{empty} - E_{full} = \frac{Q^2}{C_e}(1 - \frac{1}{\varepsilon})$$

$$C_f = \varepsilon\varepsilon_o \frac{A_1}{d}; \quad C_e = \varepsilon\varepsilon_o \frac{A_2}{d}$$

$$E_{capacitor} = \frac{(\text{total charge})^2}{2(\text{equivalent capacitance})}$$

in which I've assumed that an equivalent capacitor have the total charge (initial charge).

$$\Rightarrow E = \frac{Q^2}{2\frac{\varepsilon_0}{d}(\varepsilon A_1 + A_2)}$$

Since $A_1 = wh$ and $A_2 = w(H\text{-}h)$ we can write:

$$E_c = \frac{Q^2 d}{2\varepsilon_0 w(\varepsilon h + H - h)}$$

On the other hand, gravitational energy of water is:

$$E_g = mg \times \frac{h}{2} = \rho g w h d \times \frac{h}{2} = \frac{\rho w d g}{2} h^2$$

And total energy of the system is:

$$E_{\text{total}} = E_{\text{c}} + E_{\text{g}} = \frac{Q^2 d}{2\varepsilon_o w(H + (\varepsilon - 1)h)} + \frac{\rho w d g}{2} h^2$$

Since the energy tends to be minimized in physics world (and defines the equilibrium state):

$$\frac{dE_{total}}{dh} = 0 \Rightarrow \frac{-Q^2 d(\varepsilon-1)}{2\varepsilon_o w(H+(\varepsilon-1)h)^2} + \rho w d g h = 0$$

$$\Rightarrow Q^2(\varepsilon-1) = 2\varepsilon_o \rho w^2 g h (H+(\varepsilon-1)h)^2$$

$$\Rightarrow \left(\frac{h}{H}\right)^3 + \frac{2}{\varepsilon-1}\left(\frac{h}{H}\right)^2 + \frac{1}{(\varepsilon-1)^2}\left(\frac{h}{H}\right) - \frac{Q^2}{2\varepsilon_o(\varepsilon-1)\rho w^2 g H^3} = 0$$

*Imaginary-valued solutions have no physical meaning, here.

*Answers with $\frac{h}{H} > 1$ mean that the capacitor will be completely filled.

*Answers with $\frac{h}{H} < 0$ mean that the water won't rise at all. (in our equation this will not happen, but if one could find a medium with ε <1, $\frac{h}{H}$ could be negative).

The condition that capacitor is completely filled with water is:

$$1 + \frac{2}{\varepsilon-1} + \frac{1}{(\varepsilon-1)^2} - \frac{Q^2}{2\varepsilon_0(\varepsilon-1)\rho w^2 g H^3} \geq 0$$ *[Amir]*

4) Super loopy

The main point of this question is that the loops are superconductors. Superconductors have *no* resistance. Therefore, when the loops are brought together, the magnetic flux, which passes through each of them, must not change. Otherwise there would be a voltage across each loop, which would cause an infinite current! To avoid infinite current, the *currents* in the loops change so as to keep the *flux fixed*.

When the loops are far apart the flux passing through each loop is equal to

$$\Phi_i = LI_i$$

which is because of the self-inductance of each loop. When the two loops are brought close together, the mutual inductance is equal to the self-inductance. The flux passing through each loop is combined in equal parts. One is from the self-inductance and the other is from the mutual inductance. Therefore

$$\Phi_f = LI_f + LI_f = 2LI_f$$

To avoid infinite current,

$$\Phi_i = \Phi_f \Rightarrow I_f = \frac{I_i}{2}$$

The energy of an inductance is equal to

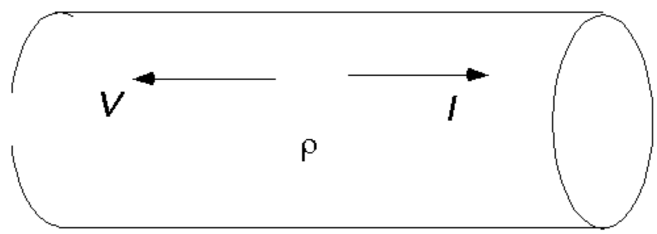
$$U = \frac{1}{2} L I^2$$

Therefore the difference in the energies of the system is equal to

$$\Delta U = U_f - U_i = \frac{1}{2} L I_f^2 + \frac{1}{2} L I_f^2 - \frac{1}{2} L I_i^2 - \frac{1}{2} L I_i^2 = -\frac{3}{4} L I_i^2$$

It can be shown that this is equal to the work needed to bring the loops together. Here the work is negative because the loops attract each other.. *[Yaser]*

5) Keeping it together...

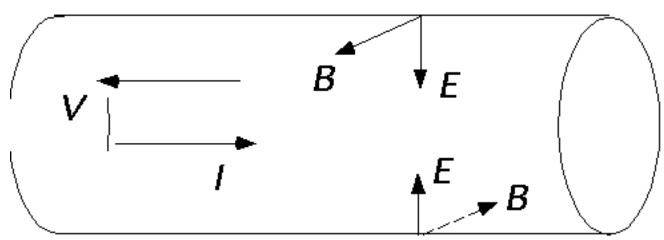


i) I'll find the density of electrons inside the beam.

$$I = \rho v(\pi r^2) \ \Rightarrow \ \rho = \frac{-I}{\pi r^2 v}$$

The electric field at the edge of beam can be found by using Gauss's law.

$$\vec{E} = \frac{\lambda}{2\pi r \varepsilon_o} = \frac{\rho \cdot \pi r^2}{2\pi \varepsilon_o r} = \frac{\rho r}{2\varepsilon_o} = \frac{-I}{2\pi \varepsilon_o r v} \hat{r}$$



Using Ampere's law, I can find B:

$$B = \frac{\mu_o I}{2\pi r}$$

The radial force on electrons on the edge is

$$\vec{F} = -e(\vec{E} + \vec{v} \times \vec{B})$$

$$\Rightarrow \vec{F} = e\left(\frac{I}{2\pi \varepsilon_o r v} - \frac{\mu_o I v}{2\pi r}\right) \hat{r}$$

$$\Rightarrow \vec{F} = \frac{I e}{2\pi \varepsilon_o r v} (1 - \mu_o \varepsilon_o v^2)$$

as you may know, the quantity $\mu_o \varepsilon_o = \frac{1}{c^2}$ where *c* is the speed of light.

$$\Rightarrow \vec{F} = \frac{I e}{2\pi r \varepsilon_o v} (1 - \frac{v^2}{c^2}) \approx \frac{I e}{2\pi r \varepsilon_o v}$$

The change in radial momentum can be found by using

$$\Delta P_r = F_r \, \Delta t = \frac{I\,e}{2\pi\varepsilon_0 \, r\, v} \bullet \frac{100\,r}{v}$$

$$\Delta P_r = \frac{100\,I\,e}{2\pi\varepsilon_0 \, v^2}$$

$$\Delta v_r = \frac{\Delta P_r}{m_e} \quad \frac{100\,I\,e}{2\pi\varepsilon_0 \, m_e v^2}$$

in which I've assumed that F_r, r and v won't change too much during this process. One may find the ratio:

$$\frac{\Delta v_r}{v} = \frac{100\; I\; e}{2\pi\varepsilon_0 \, m_e v^3} = \frac{100 \bullet 10^{-3} \bullet 1.6 \times 10^{-19}}{2\pi \bullet 8.85 \times 10^{-12} \bullet 9.1 \times 10^{-31} \bullet (10^7)^3}$$

$$tan\vartheta = \frac{\Delta v_r}{v} \approx 0.3$$

$$\Rightarrow \vartheta = 0.3 \; radians \approx 17^\circ$$

(The speed in the problem set was wrongly written as 1000 m/s. It is 10,000 km/s. Sorry!).

BONUS:

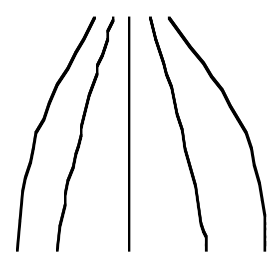
In part (c) we found that the force acting on electrons at the border is

$$\vec{F} = \frac{I\,e}{2\pi\varepsilon_0 \, v\, r}\left(1 - \frac{v^2}{c^2}\right)$$

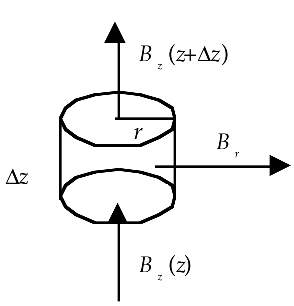
putting this force to be zero will result:

$$1 - \frac{v^2}{c^2} = 0 \;\; \Rightarrow \;\; v = c \qquad [Amir]$$

6) A feel for fields



a) Try to draw field lines, which have to be denser as the z changes. The field lines must bend, which shows the presence of a radial component. Use Gauss's Law for this part. The magnetic field lines that enter a cylinder must go out so as to make a closed loop.



The flux from the top and bottom surfaces is

$$\Phi_1 = \pi r^2 (B(z+\Delta z) - B(z) = \pi r^2 \frac{dB_z}{dz} \Delta z$$

The flux from the sides is

$$\Phi_2 = 2\pi r \Delta z B_r$$

For the total flux to be zero we should have

$$\Phi_1 = \Phi_2 \quad \Rightarrow \quad B_r = \frac{r}{2}\frac{dB_z}{dz}$$

b) There are two forces acting on the loop, gravitational force and the magnetic force. The rate of change of the magnetic flux respect to time is

$$\frac{d\Phi}{dt} = \pi r^2 \frac{dB_z}{dt} = \frac{\pi r^2}{v}\frac{dB_z}{dz}$$

where v is the velocity of the loop falling down. For the current in the loop we have

$$i = \frac{V}{R} = \frac{d\Phi / dt}{R} = \frac{\pi r^2 v}{R}\frac{dB_z}{dt}$$

The force exerted by the magnetic field, which is due to radial component of the magnetic field and upward, is equal to

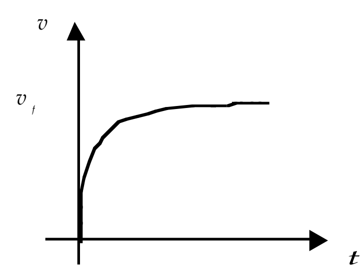
$$F = i l B_r = i(2\pi r) B_r$$

Substitute the value of current and magnetic field we already derived we have

$$F = \frac{\pi^2 r^4}{R}\left(\frac{dB_z}{dz}\right)^2 v$$

Therefore the equation of the motion of the loop is

$$m\frac{d^2 v}{dt^2} = mg - \frac{\pi^2 r^4}{R}\left(\frac{dB_z}{dz}\right)^2 v$$



The magnetic force is negative because of Lenz's law. The magnetic force is proportional to v and negative, causes damping. Initially, the velocity increases but reaches a terminal velocity which is when the acceleration is zero.

$$\frac{d^2 v}{dt^2} = 0 \Rightarrow v_f = \frac{mgR}{\pi^2 r^4 (dB_z / dz)^2}$$

The velocity of the loop versus time is plotted. *[Yaser]*