1999-2000 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 6: AC Circuits and Electronics

1) Ohm my!

You might be able to do this one by inspection: if r → ∞ then *no current* flows in the circuit, so no power is dissipated through r. On the other hand, as r → ∞ power dissipation increases for a constant current. So a few equations are in order.

In the series circuit

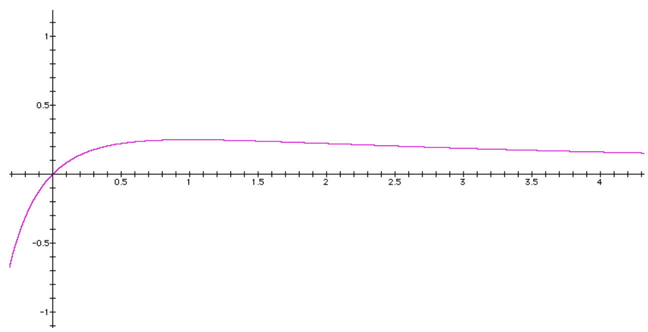
$V = I\,(R + r)$ *(voltage V across resistors in series, current I through resistors)*

$P = I \bullet V_r = I^2\, r$ *(power P dissipated in r, current I through r same as R)*

$$= \left(\frac{V}{R+r}\right)^2 r$$

For an extremum:

$$0 = \frac{dP}{dr} = \left(\frac{V}{R+r}\right)^2 \left(1 - \frac{2r}{R+r}\right)$$



which happens at $r = R$. However, this is a *maximum*, not a minimum.

This is easy to see by graphing it, by finding the second derivative, or by simply noting:

$$\lim_{r \to 0} \left\{ \left(\frac{V}{R+r}\right)^2 r \right\} = 0$$

$$\lim_{r \to \infty} \left\{ \left(\frac{V}{R+r}\right)^2 r \right\} = \lim_{r \to \infty} \left\{ V^2 \frac{r}{(R+r)^2} \right\} = V^2 \lim_{r \to \infty} \left\{ \frac{1}{2(R+r)} \right\} = 0$$

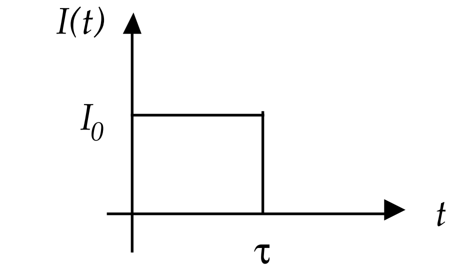
…so if it is zero at the origin and at infinity, has one extremum, and cannot be negative-valued, it must be a maximum. The minima you're asked for correspond to r = 0 and r = ∞.

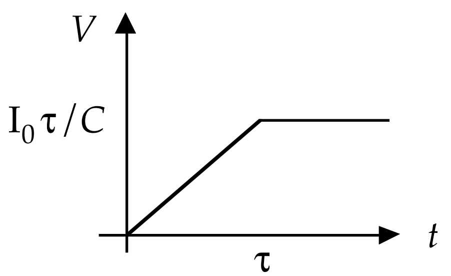
How about in parallel? It's even easier to see: $P_r = V_r\, I_r = (V_r)^2/r$. The voltage $V_r = V_o$ the battery voltage. Clearly this power decreases monotonically as r increases — the miminum is at r = ∞. *[Robin]*

2) A current affair

Usually questions like this one are given with a specified *voltage* applied — which would be easier! Here it is a *current* that is specified (without telling you how they made sure such a current was produced, at whatever voltage needed). This makes some parts tricky, unfortunately.

For the first case, all the current must go through the resistor, and onward to charge the capacitor. Therefore, the capacitor charge goes up linearly respect to time, and the voltage across it would be $V = \frac{q}{C} = \frac{I_0 t}{C}$. After time τ no current flows, so the charge remains on the the capacitor (except perhaps for *leakage* due to some large resistance through the capacitor which ideally should be infinite. Leakage can happen through moisture in the air, too).





For the case where the capacitor and resistor are connected in parallel, the current I_0 is divided through both of them. Initially most of the current will pass into the capacitor and increase the capacitor voltage as it charges up (since we've assumed a constant current flow, this the source voltage will have to increase to match the capacitor voltage, to keep current constant). As this voltage increases, the current through the resistor in parallel also will increase, according to $V = IR$, and this will make the current through the resistor increase — a bigger share of the current flows through the resistor.

Even if the current were constant for all time, the capacitor cannot charge up forever: at some voltage the current drain through the resistor would be equal to the whole current supply, and the capacitor will not take any of the current. Then the circuit will reach 'steady state'. This helps us figure out how to solve the following equation which describes the circuit (a famous physicist once said: "Never write down an equation until you've figured out what the answer should be.")

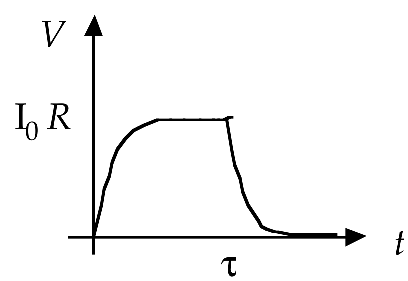
$$I_o = I_{capacitor} + I_{resistor}$$
$$= \frac{dQ}{dt} + \frac{V}{R}$$
$$= C\frac{dV}{dt} + \frac{1}{R}V$$

Where V is the voltage across both the capacitor and the resistor (in parallel), and Q is the charge on the capacitor. The current Io is constant, so we can solve this equation either systematically or by guessing from what we figured out about voltages, above.

We try a solution which has exponentials in it (because the derivative of the exponential is again an exponential, which maybe would cancel with the original). In steady-state, the dV/dt term will be zero (i.e., nothing is changing), so finally $V(t \to \infty) = V_o = I_o R$. If the voltage approaches this value asymptotically, then maybe it should have the form:

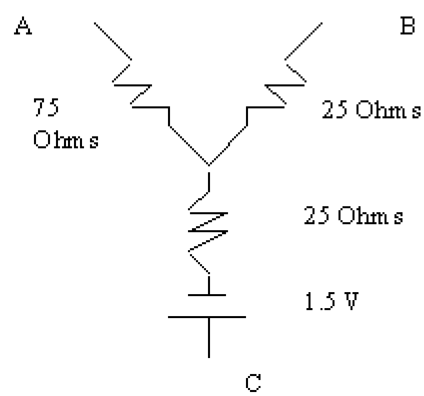
$$V(t) = V_o\left(1 - e^{\alpha t}\right)$$

Try it in the differential equation: it works, as long as $\alpha = -1/RC$. Sounds OK, because if R → 0 (short circuit) or C → 0 (no capacitor), then it takes no time at all to reach the final state.



In fact what happens is that the voltage follows this curve *V(t)* until the current is turned off. Then the charge on the capacitor leaks away through the resistor until it goes back to zero, exponentially with the same time-constant $\alpha = 1/RC$. *[Yaser & Robin]*

3) Pandora's box o' electronics



There is no unique solution to this problem (for example, a 100 Ω resistor could also be drawn as two 50 Ω resistors, etc.).

Trial and error always works, and here's a possible solution. The battery has a value of 1.5V, resistance at A is 75 Ω, resistance at B is 25 Ω and the resistance at C is also 25 Ω.

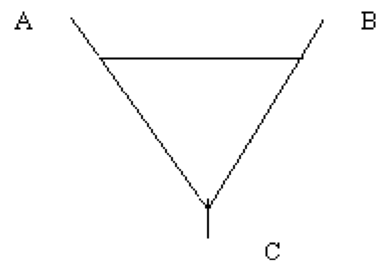
Using these values we first of all see that the resistance from B to C is 100 Ω (within error of the 99 Ω given); measuring resistance from A to C or B to C yields an error, as we're going across the battery [an aside: the way a multimeter measures resistance is to send a small current across the circuit, measure the potential difference created and find the resistance from Ohm's Law. However, when a battery is in the way it will change the current, and perhaps even reverse it, which will yield in very strange values shown by the multimeter].

The current from A to B is 0, and the voltages from A to C and B to C will be 1.5 V (within error of 1.6V), as given [the internal resistance of the multimeter is extremely high when measuring voltage]. The current from A to C (remember that the resistance of the multimeter is very small when measuring current) is 1.5 / 100 = 0.015 A and the current from B to C will be 1.5 / 50 = 0.030 A, again within error of the given values.

If you like a more systematic approach, try this:

- First, try to keep things as simple as possible (i.e., start with the simplest components, like resistors and batteries, adding more complicated ones later).
- Next, the most complex circuit possible is when all terminals are interconnected.

Now, it may be shown that such an 'outer' triangle is equivalent to an 'inner' triangle, just as I've drawn before. So, we re-draw the diagram to the one I had first (this is not necessary; I'm only doing it so that this solution is the same as the one above). Now, it is pretty clear that there has to be at least 1 battery between A-C and B-C. The easiest way to satisfy this is to put one at C.



- As to the remaining resistors, you can either guess the values or solve 3 equations and 3 unknowns, as below:

$$R_A + R_B = 100$$
$$R_A + R_C = 1.5/0.015 = 100$$
$$R_B + R_C = 1.5/0.030 = 50$$

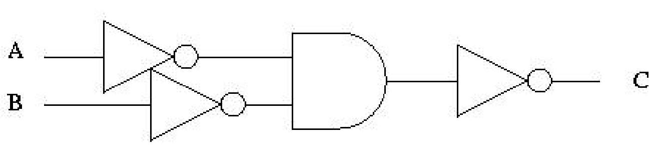
(RA, RB and RC are the resistances in the arms A, B and C, respectively, in the 1st diagram). And this indeed does it… *[Peter]*

4) Only logical

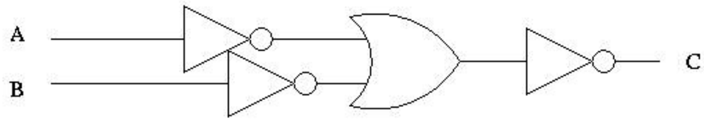
The truth table for the OR gate is shown at right: If either A *or* B is true (1) then the output is true (1).

A	B	C
1	1	1
1	0	1
0	1	1
0	0	0

b) Using NOT and AND gates the following OR gate can be constructed:

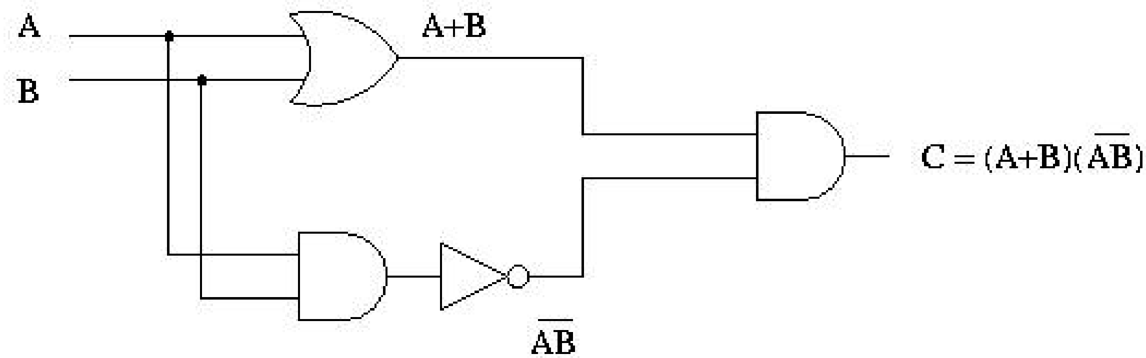


c) Using NOT and OR gates the following AND gate can be constructed:



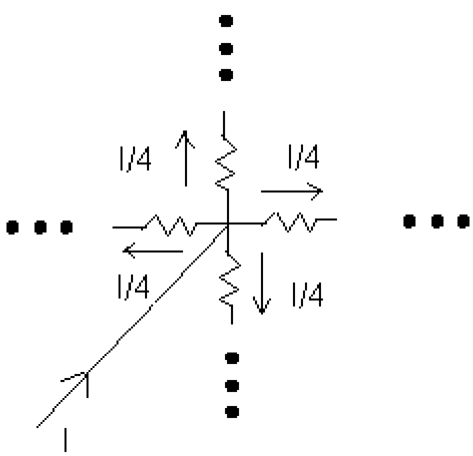
BONUS

d) One possible EXCLUSIVE OR gate built out of AND, OR and NOT gates:



[Carrie]

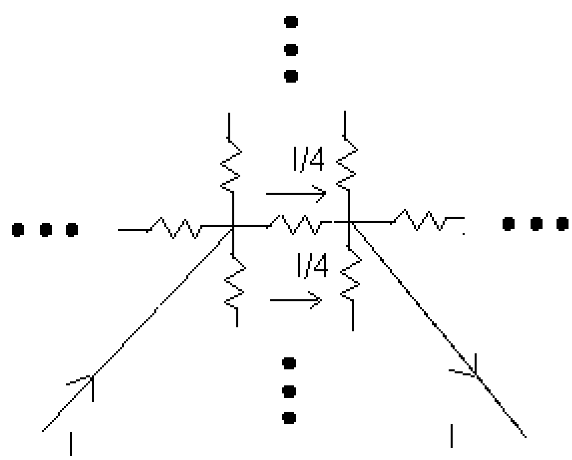
5) Net worth



iv) Let me do the general problem first. Consider an infinite, homogenous grid where at any junction N resistors are connected together. Note that a voltage source is really equivalent to a current source, *i.e.*, something which injects & draws current.

Consider injecting a current I into the junction. Because the whole system is N-symmetric, the current will split evenly and I/N will flow into every resistor. See the example, above left.

Consider now drawing a current I *from* a junction. Again, a current I/N will be drawn from every resistor. Now *superimpose* the 2 current flows, offset by 1 arm (see right). We



will have a current I coming in, and a current I coming out one arm away — equivalent to a battery. The current in the given arm will be I/N + I/N = 2I/N. The resistance is r. Thus, the voltage drop is 2Ir/N which must also equal RI, where R is the equivalent resistance. From this we see that R = 2r/N. This also solves the remainder of the questions…

This question is inspired by a Polish Olympiad question, from way back when…

Answers:

i) r/2

ii) r/3

iii) 2r/3

iv) 2r/N

BONUS: it is *not possible* to draw such a figure in 2D; in 2 dimensions, the only possible figures are those for which N = 3, 4 or 6. To convince yourself of this you can note that at each junction the sum of all the internal angles has to be 360 degrees (of course, you don't have to be able to draw this figure for part (iv) to hold…). *[Peter]*

6) My current analysis...

Here's what you have to know:

i) when the driving frequency ω is small, the current is almost steady (DC). The inductor behaves like a piece of wire, with zero resistance. The capacitor, on the other hand, doesn't let any current through and behaves like a break in the circuit.

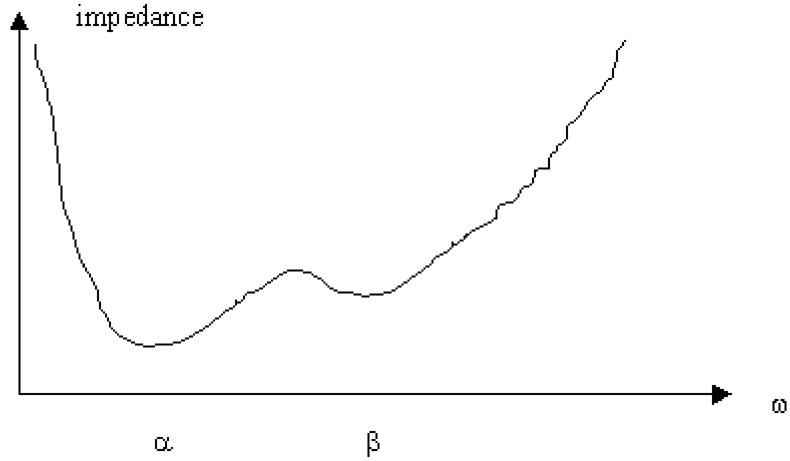
ii) when ω is very large, the opposite is true: the capacitor lets current through but the inductor doesn't (the e.m.f. that builds up inside is very large)

iii) an LC or RLC circuit will resonate at a frequency of $\omega^2 = \frac{1}{LC}$. At this point the power dissipated is maxi-mum, and the impedance is minimum.

Now, the circuit in question contains 2 circuits (the R, L_2, C_2 circuit and the C_1, L_1 circuit), each with their own resonant frequencies. We can thus expect that the overall impedance will have minima near the resonant frequencies. In fact, the impedance of the R, L_2, C_2 circuit will be higher than that of the other circuit, due to the resistor R. Also, the impedance will never quite reach zero, as the two resonant frequencies are not equal ($\alpha \neq \beta$).

From all this, we 'guesstimate' the curve on the following page (note that I have arbitrarily assumed $a < b$).

For an analytical treatment I will employ the complex number technique. What happens here is that a sinusoidal driving source (e.g., cos(ωt)) can be conveniently represented as a complex number ($\cos(\omega t) = \mathrm{Re}(\exp(i\omega t))$, where Re denotes the real part). This is useful as derivatives of the exponential are really easy to calculate. This, in the end, is useful to solve the differential equations present when dealing with capacitors and/or inductors.



The gist of all this is the fact that the resistor, capacitor and inductor, in a circuit driven by a sinusoidal source, behave as resistors whose generalized 'resistance' (called *impedance*) is a complex number (the number has to be complex because capacitors and inductors change the phase of the current; the phase is represented by the angle the impedance makes with the real axis in the complex plane).

The values turn out to be:

Resistor R: $Z = R$

Capacitor C: $Z = \dfrac{1}{i\omega C}$

Inductor L: $Z = i\omega L$

(Z is the impedance; $i^2 = -1$)

Finally, the standard Kirchoff rules apply to impedances (*i.e.*, they're added when in series and the reciprocals of the sum of their reciprocals is taken in parallel).

After this crash course, let's apply this to our circuit. The total impedance will be

$$Z = (i\omega L_1) + \left(\frac{1}{i\omega C_1}\right) + \frac{1}{\left(\frac{1}{R}\right) + \left(\frac{1}{i\omega L_2 + \frac{1}{i\omega C_2}}\right)}$$

$$= i\omega L_1\left(1 - \frac{\alpha^2}{\omega^2}\right) + \frac{1}{\frac{1}{R} + \frac{1}{i\omega L_2\left(1 - \frac{\beta^2}{\omega^2}\right)}}$$

Real nasty… now, what we really want to plot is the magnitude of this thing, $|Z|$. This yields some pretty ugly expressions, and not much can be done so simplify them.

You can take my word for the fact that this yields very similar results to the graph above; the second minimum isn't really a minimum, more of an inflection point, but it's close enough for this rough analysis. *[Peter]*