

# The Static Hedging of CDO Tranche Correlation Risk<sup>1</sup>

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## Abstract

This article gives examples illustrating the static hedging of CDO correlation risk. Changes in correlation result in changes in the relative portioning out of the total expected loss of a reference portfolio to the different tranches. Thus, portfolios with low correlation risk contain a number of CDO tranches whose weights are adjusted so that the daily changes in the mark-to-market values of the different tranches tends to cancel out. The examples are backtested using iTraxx CDO market spreads for the challenging period following the May 5, 2005 downgrade of Ford and General Motors by Standard and Poor's. The implementation is carried out by using the static loss-surface model of Walker (2005, 2006) and Torresetti, Brigo and Pallavicini (2006).

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# 1 Introduction

A collateralized debt obligation (CDO) is a credit derivative that references a portfolio of credit-risky instruments (for example credit default swaps). The portfolio losses are tranching so that, for example, one has CDO contracts that insure a protection buyer against a given tranche of portfolio default losses. For example, in the case of the standardized iTraxx tranches studied below, the riskiest tranche (called the equity tranche) consists of portfolio losses between 0 and 3% of the total portfolio notional, the next riskiest is the junior mezzanine 3-6% tranche, while the least risky tranche is the super-senior tranche 22-100% tranche. There is also a contract (said to be for protection on the index) that buys protection against all losses to the reference portfolio.

The risk-management of CDO's in practice has been carried out largely in terms of the Gaussian copula model (e.g. see Houdain and Guegan (2006); Jobst (2007); Petrelli et al. (2007)). In the Gaussian copula approach one can compute a hedge ratio describing the delta-hedging of, say, the equity tranche by the index. The mark-to-market value of the equity tranche by itself can undergo large day-to-day fluctuations, depending on market conditions, and the idea of delta-hedging the equity tranche with the index is to form a portfolio of the index and the equity tranche that, one hopes, will exhibit smaller fluctuations in mark-to-market value. In computing the hedge ratio, the Gaussian copula correlation parameter is held constant. A difficulty with this procedure is that one does not hedge the correlation risk, i.e. changes in the correlation parameter. Indeed, during periods of credit market turbulence, such as following the downgrade of General Motors and Ford in May 2005, and during the 'credit crunch' that began in July/August 2007, many hedging procedures used in practice (such as the one described in this paragraph) performed badly due to unhedged correlation changes, and left investors with significant mark-to-market losses (e.g. see Houdain and Guegan (2006); Petrelli et al. (2007); Patel (2007)). Clearly, further study of CDO hedging strategies is desirable.

The main goal of this note will be to describe and backtest simple examples of portfolios constructed in such a way as to lower the correlation risk, while at the same time offering protection against index spread risk and default risk. Although the hedging of the index spread risk of a CDO tranche has been much discussed in the literature (as described above), there has been little discussion of the hedging of correlation risk. In this article the word correlation will be used as a general term describing how the total expected loss of the reference portfolio is portioned out to the different tranches. To effectively hedge correlation risk one should include combinations of tranches in a hedging portfolio, such that the relative changes in mark-to-market values of the tranches tend to cancel out. Since it is expected that the mark-to-market values of the equity and super-senior tranches will change in opposite directions with a change in correlations (e.g. see Willemann (2005)), one of the portfolios tested will consist of the index, and the equity and super-senior tranche. Also, a portfolio formed by adding the junior mezzanine tranche to the previous three will be tested. Finally, a rather obvious choice, which will be called the complete hedging portfolio for the index, and which consists of all tranches, will be investigated.

The examples of the previous paragraph consist of multi-tranche portfolios. The detailed marking-to-market of such portfolios in a consistent manner thus requires the use of a model that can be simultaneously calibrated to all market prices, i.e. for all tranches and maturities, of contracts that reference a given reference portfolio. For this purpose, the static loss-surface

model, introduced in Walker (2005, 2006), and extended to take into account bid-ask spreads in Torresetti et al. (2006), will be used.

Section 2 gives an elementary introduction to the basic hedging procedure, and, in addition, describes one of the main approximations. Section 3 describes the calibration process, gives results measuring its accuracy, and demonstrates its insensitivity to the assumed value of the recovery rate. Section 4 develops the basic profit and loss equations used in the back-testing of the hedging portfolios. Section 5 gives the results of back-testing the hedging procedures using iTraxx market prices for the challenging period following the 5 May 2005 Standard & Poor's downgrade of General Motors and Ford to junk status.

There have been several other recent discussions of the hedging of CDO's. Petrelli et al. (2007) give a review of the copula approach, together with a useful backtesting study of its hedging performance for the period of 22 March to 17 December 2005 (which includes the upheaval associated with the downgrade of Ford and General Motors). Petrelli et al. (2006) have studied the optimal static hedging of defaults in CDO's. Laurent et al. (2007) introduce a dynamic hedging model that focuses on contagion effects and default risk, while neglecting credit spread risk. Frey and Backhaus (2007) also develop a dynamic hedging model, but one that takes account of spread and contagion risk. None of these discuss the hedging of correlation risk.

Copula approaches have been used, to estimate the sensitivities of CDO spreads to changes in the copula correlation parameter (e.g. see Petrelli et al. (2007)). However, copula approaches value one tranche at a time, with each tranche-maturity contract being described by a different risk-neutral measure characterized by the value of its particular correlation parameter. Existing copula approaches are thus not well-suited to the development a consistent picture of multi-tranches portfolios, such as is necessary for the hedging of tranche correlation risk.

Finally, it should be emphasized that the focus of this note is on the description and backtesting of certain portfolios that reduce the risk of holding tranche positions. The complete hedging portfolio described below is an example, and might be used by an institution that wishes to make its money in transaction costs, and aims at holding portfolios that are as risk-free as possible. Also, when turbulent credit markets are anticipated, such as during the period following the downgrade of Ford and General Motors, some institutions might seek to form portfolios that are as insulated against market fluctuations as possible. Thus, the objectives here are complementary to those of an investor who wishes to take on the risk of significant mark-to-market fluctuations in tranche value, in the hope of making a significant profit. For such an investor, a position in, say, an equity tranche delta-hedged with the index using a copula model, might be desirable. Such positions can be used, for example, to make bets on the direction of movement of the tranche correlation.

## 2 Overview

To define the  $nTr = 6$  standardized iTraxx tranches considered in this article introduce a set of discrete losses  $d_k$ ,  $k = 1, \dots, nTr + 1$ , increasing with  $k$ , and given by [0 0.03 0.06 0.09 0.12 0.22 1]. The  $k$ th tranche has a standardized notional  $\mathcal{N}(k) = d_{k+1} - d_k$  and the sum of the standardized notionals of the  $nTr$  tranches equals unity, the the standardized notional of the index. A set of non-overlapping, contiguous standardized tranches such as this, with

total standardized notional equal to the notional of the index, will be called a complete set of tranches. The equity tranche is tranche  $k = 1$  (with  $\mathcal{N}(1) = 0.03$ ) and the most senior tranche is the super-senior tranche  $k = 6$  (with  $\mathcal{N}(6) = 0.78$ ).

Consider a CDO of notional unity (not the standardized notional) on tranche  $k$  which matures in  $T$  years from the present time  $t = 0$ . A fair annualized premium,  $w(k, T)$ , for tranches  $k = 2, \dots, nTr$  is established by equating the present value of the expected losses per unit tranche notional hitting this tranche between time  $t = 0$  and time  $T$ ,  $V_{loss}(k, T)$ , to the present of the expected premium payments. For the equity tranche only, the annualized premium is taken to be  $w(k = 1, T) = 500$  bps, and the tranche is valued in terms of an upfront payment  $u(T)$ . This gives

$$V_{loss}(k, T) = w(k, T)T_{eff}(k, T) + u(T)\delta_{k,1}, \quad k = 1, \dots, nTr, I. \quad (1)$$

Here,  $T_{eff}(k, T)$  is the risky duration of the premium payments and  $\delta_{k,1}$  is the Kronecker  $\delta$  symbol, which is unity if  $k = 1$  and zero otherwise. Also, the notation has been extended so that the index  $k$  can include the letter I, referring to the index.

The fact that the losses on unit notional of the index are equal to the sum of the losses of the standardized tranches (with tranche  $k$  having notional  $\mathcal{N}(k)$ ) yields

$$w(I, T)T_{eff}(I, T) = \sum_{k=1}^{nTr} \mathcal{N}(k)w(k, T)T_{eff}(k, T) + \mathcal{N}(1)u(T). \quad (2)$$

Now consider some time  $t = \tau$  in the future, so that the time to maturity of the CDO's under consideration is now  $T' = T - \tau$ . Also define the changes from time  $t = 0$  to be

$$\begin{aligned} \Delta w(k, T') &= w(k, T') - w(k, T), \\ \Delta T_{eff}(k, T') &= T_{eff}(k, T') - T_{eff}(k, T), \\ \Delta u(T') &= u(T') - u(T). \end{aligned} \quad (3)$$

Note that (here and below) all changes, considered for a fixed initial time to maturity  $T$ , have an implicit dependence on  $\tau$  (since  $T' = T - \tau$ ) which is not shown explicitly in the notation. The change in Eq. 2 can now be found to be, to first order in the changes of Eq. 3,

$$\begin{aligned} \Delta w(I, T')T_{eff}(I, T') &= \mathcal{N}(1)\Delta u(T') + \sum_{k=2}^{nTr} \mathcal{N}(k)\Delta w(k, T')T_{eff}(k, T') + \\ &\left[ \sum_{k=1}^{nTr} \mathcal{N}(k)w(k, T)\Delta T_{eff}(k, T') - w(I, T)\Delta T_{eff}(I, T') \right]. \end{aligned} \quad (4)$$

Now suppose that the term in square brackets in the previous equation can be shown to be small in comparison with the other terms. Then the first line of this equation states that, to a good approximation, the change in the mark-to-market value of unit notional of the index between time  $t = 0$  and time  $t = \tau$  is approximately equal to the change in the market-to-market value of a portfolio of tranches of the same nominal maturity, with  $\mathcal{N}(k)$  being the notional of tranche  $k$ . Thus, provided the term in square brackets can be neglected, the index can be

accurately hedged by this particular portfolio of tranches, and vice versa. The detailed results presented below show that the neglected terms are indeed relatively small. (The percentage changes in  $T_{eff}(k, T')$  are due to changes in the timing of defaults implied by the changes in market spreads of tranches with all maturities up to and including the nominal maturity associated with  $T'$ , and might be expected to be small in comparison with the percentage changes in  $w(k, T')$ , which are proportional to the changes in the expected losses implied by the changes in the spreads.)

It would be possible to base further discussions of the potentially low correlation-risk portfolios directly on the first line of Eq. 4. However, in Section 4, a related result will be derived that gives more insight into the neglected terms.

### 3 Calibration

The hedging strategy described above will be implemented here in terms of the static loss-surface model of Walker (2005, 2006) and Torresetti, Brigo and Pallavicini (2006); readers wishing a more detailed discussion of this model than that given below are referred to these articles. Tranches are valued in terms of a function  $f(k, t)$  giving the expected loss, per unit initial tranche notional, for tranche  $k$  at time  $t$ . For each tranche  $k$ ,  $f(k, t)$  is parameterized as a continuous, piecewise-linear function of time  $t$ . The standardized iTraxx tranches used in the example below are issued for maturities of  $T_1, T_2, T_3$ , and  $T_4$ , which have nominal values of 3, 5, 7 and 10 years, respectively. In this article, the parameterization of Torresetti et al. (2006) is followed, in which  $f(k, t)$ , for each  $k$ , starts from zero at  $t = T_0 \equiv 0$ , and rises linearly in each of the regions  $T_i \leq t \leq T_{i+1}$ ,  $i = 0, 1, 2, 3$ . Thus, the risk-neutral measure is determined by the values of the 24 parameters  $f(k, T_i)$ ,  $k = 1, \dots, 6$ , and  $i = 1, \dots, 4$ . The parameters  $f(k, T_i)$  are subject to certain linear constraints imposing the conditions that losses are non-decreasing with time, and that tranche  $k$  is riskier than tranche  $k + 1$ ,  $k = 1, \dots, nTr - 1$ .

A key feature of the static loss-surface approach is that both the present values of the expected tranche losses, and the risky durations, are linear functions of the parameters  $f(k, T_i)$ . Thus  $V_{loss}(k, T)$  and  $T_{eff}(k, T)$  can be written, for  $k = 1, \dots, nTr - 1$ , as

$$\begin{aligned} V_{loss}(k, T_i) &= \sum_{j=1}^i a(i, j) f(k, T_j), \\ T_{eff}(k, T_i) &= b(i) + \sum_{j=1}^i c(i, j) f(k, T_j). \end{aligned} \tag{5}$$

Here, the coefficients  $a(i, j)$ ,  $b(i)$ , and  $c(i, j)$  are known functions that are evaluated in terms of the risk-free discount factors. For the super-senior tranche (i.e.  $k = nTr$ ), the situation is somewhat more complicated because not only must losses that hit this tranche be subtracted from the bottom of the tranche, but recoveries from all defaults must be subtracted from that top of the super-senior tranche (e.g. see Walker (2006)). This means that  $T_{eff}(nTr, T_i)$  depends linearly on  $f(k, T_j)$  for all  $k = 1, \dots, 6$ , rather than just depending on  $f(nTr, T_j)$ . For the index ( $k = I$ ), the quantities  $f(k, T_j)$  in Eq. 5 are replaced by  $\sum_{r=1}^{nTr} \mathcal{N}(r) f(r, T_j)$ , and the coefficients  $c(i, j)$  are determined not only by the risky discount factors, but contain a factor  $(1 - R)^{-1}$ , where  $R$  is the recovery rate.

To calibrate the model, it is assumed that iTraxx market prices for a given trading day are available for tranches  $k = 1, \dots, nTr - 1$  and for the index, for all standard maturities  $T_i, i = 1, \dots, 4$ . Calibration involves an optimization procedure that finds a “best choice” for the values of the 24  $f(k, T_i)$  parameters defining the risk-neutral measure. The values of the parameters  $f(k, T_i)$  thus obtained can then be used to calculate a set of “theoretical” values for the spreads  $w(k, T_i)$ . For a given theoretical spread  $w(k, T_i)$ , Torresetti et al. (2006) defined a standardized mispricing by

$$MisPr(k, T_i) = \left| \frac{w(k, T_i) - w_{mid}(k, T_i)}{BidAsk_{Spread}/2} \right|. \quad (6)$$

where  $w_{mid}(k, T_i)$  is the mid-point of the bid and ask spreads,  $BidAsk_{Spread}$  is the difference of the bid and ask spreads, and the vertical bars indicate an absolute value. The best choice for the parameters  $f(k, T_i)$ , and hence for the theoretical spreads  $w(k, T_i)$ , is obtained by a least squares minimization procedure applied to the standardized mispricings (see Torresetti et al. (2006) for details).

To backtest the hedging procedure developed below, the paper will work with a data base of iTraxx tranche prices for each of the 95 trading days from 6 May 2005 (the day after the Standard and Poor’s downgrade of Ford and General Motors to junk status) until 19 September, 2005. Each set of prices for a given trading day consists of the 24 different spreads, i.e. the spreads for contracts of 3, 5, 7, and 10-year maturities, for tranches  $k = 1, 2, 3, 4, 5$  and the index. Note that, for a given trading day, the number of spreads calibrated to (which is 24) is exactly equal to the number of parameters  $f(k, T_i)$  which must be determined by the calibration. (Super-senior tranches are not often traded, and hence super-senior spreads are not included in the calibration procedure.) The aim of the calibration procedure is to produce theoretical spreads with standardized mispricings that are as small as possible. For the 95 trading days there will be a total of  $95 \times 24 = 2280$  different theoretical spreads. To assess the quality of the calibration, the resulting mispricings (Eq. 6) will be binned into 6 different bins defined by  $0 \leq MisPr \leq 0.05$ ,  $0.05 < MisPr \leq 0.5$ , etc., as indicated in Table 1. The table shows the dependence of the calibration accuracy on the value of the recovery rate. Since a risk-neutral pricing procedure has been adopted for this article, the recovery rate is a risk-neutral parameter which can be adjusted to improve the quality of the calibration, and is not necessarily closely related to the historical recovery rate. It can be seen that, for recovery rates below 40%, the vast majority of the theoretical spreads obtained by the procedure just described lie very close to the mid-point spread. Furthermore, it will be shown that the mark-to-market values of the hedging portfolios obtained below have a negligible sensitivity to the value of the recovery rate (see the discussion of Fig. 2).

As noted above, the super-senior spreads are not used in the calibration process. The risk-neutral parameters obtained in this calibration can, however, be used to determine a mark-to-model value of the super-senior spread. The average value of the calculated super-senior spread for the 5-year maturity for the 95 trading days used for the calibrations described by the Table is 4.05 bps. The magnitude of the spread difference for  $R = 0\%$  and  $R = 30\%$ , averaged over the 95 trading days, is 0.17 bps. The mark-to-model super-senior spreads are thus relatively insensitive to the assumed value of the recovery rate. The super-senior tranches are necessary for the hedging procedures discussed in this article. It will be assumed that they are available on the market at their mark-to-model spread.

bin	$R = 0\%$	$R = 10\%$	$R = 20\%$	$R = 30\%$	$R = 40\%$
$0 \leq MisPr \leq 0.05$	2159	2151	2113	2011	1600
$0.05 < MisPr \leq 0.5$	94	100	135	228	504
$0.5 < MisPr \leq 1.0$	22	21	22	28	117
$1.0 < MisPr \leq 1.5$	2	4	6	9	40
$1.5 < MisPr \leq 2.0$	1	2	2	2	15
$2.0 < MisPr$	2	2	2	2	4
% outside <i>BidAsk</i>	0.2%	0.3%	0.4%	0.6%	2.6%

Table 1: For a given recovery rate  $R$ , the calibration procedure was carried out for data sets of 24 spreads for each of the 95 trading days between 6 May, 2005 and 19 September, 2005. The 2280 theoretical spreads so obtained were binned according to their degree of mispricing as indicated in the Table. Note that a large majority of the theoretical spreads have a mispricing parameter ( $MisPr$  of Eq. 6) less than or equal to 0.05, i.e. the theoretical spreads in general lie very close to the mid-point spreads. The best choice for the risk-neutral recovery rate appears to be  $R = 0\%$ . For  $R = 0\%$ , only 5 of the 2280 theoretical spreads (i.e. 0.2%) lie outside of the *BidAsk* spread. The fraction of theoretical spreads lying outside of the *BidAsk* spread is tabulated for each value of  $R$  in the last row of the table.

## 4 Derivation of the Profit and Loss Equations

Suppose that, for a given trading day called  $t = 0$ , a set of risk-neutral parameters  $f(k, T_i)$  ( $k = 1, \dots, nTr$ , and  $i = 1, \dots, 4$ ) and a set of theoretical spreads  $w(k, T_i)$  ( $k = 1, \dots, nTr, I$ , and  $i = 1, \dots, 4$ ) have been established by the procedure of the preceding section. Then Eqs. 1 with  $V_{loss}(k, T_i)$  and  $T_{eff}(k, T_i)$  given by Eq. 5 (together with related results for  $k = nTr$  and  $k = I$ ) are satisfied. Now consider the time  $t = \tau$  in the future, so the the set of times to maturity is now  $T'_i = T_i - \tau$ . Define the changes in the  $f$ 's from time  $t = 0$  by

$$\Delta f(k, T'_i) = f(k, T'_i) - f(k, T_i). \quad (7)$$

From Eqs. 1 together with Eqs. 5 one can derive the result

$$\Delta f(k, T'_i) = \sum_{j=1}^i A(k, i, j) \Delta w(k, T'_j) + \Delta B(k, T'_i), \quad k = 2, \dots, nTr - 1. \quad (8)$$

The expressions  $A(k, i, j)$  and  $\Delta B(k, T'_i)$  can be evaluated in terms of  $f(k, T_i)$  and the risk-free discount factors, i.e. these expressions are known at time  $t = 0$ . The quantity  $\Delta B(k, T'_i)$  gives the evolution of  $\Delta f(k, T'_i)$  with time  $\tau$  when all tranche spreads are kept constant. The quantities  $\Delta w(k, T'_j)$  and  $\Delta f(k, T'_j)$  are not known at time  $t = 0$ , and should be regarded as stochastic variables whose values only become known at time  $t = \tau$ . The values of these stochastic variables must, however, be related by Eq. 8. Related expressions for tranche  $k = 1$  (where  $\Delta w(k, T'_j)$  is replaced by  $\Delta u(T'_j)$ ) and for tranche  $nTr$  (where  $\Delta w(k, T'_j)$  is replaced by a linear combination of the  $\Delta w(r, T'_j)$ ,  $r = 1, \dots, nTr$ ) can be obtained.

The relation between the change in the index spread and the change in the tranche spreads

can now be found to have the form

$$\begin{aligned}
\Delta w(I, T'_i) &= \Delta w_0(I, T'_i) + [\Delta\Theta(T'_i)] \\
&+ \left[ \sum_{j=1}^{i-1} \left( U(k=1, i, j) \Delta u(T'_j) + \sum_{k=2}^{nTr} U(k, i, j) \Delta w(k, T'_j) \right) \right], \\
\Delta w_0(I, T'_i) &= U(k=1, i, i) \Delta u(T'_i) + \sum_{k=2}^{nTr} U(k, i, i) \Delta w(k, T'_i). \tag{9}
\end{aligned}$$

This expression is obtained by first using Eqs. 8 and related results for tranches  $k = 1$  and  $k = nTr$  to find  $f(k, T'_i) = f(k, T_i) + \Delta f(k, T'_i)$ ,  $k = 1, \dots, nTr$ . These results are then used to find  $\Delta w(I, T'_i)$  correctly to terms linear in the changes, thus giving Eq. 9. As in Eqs. 8, the coefficients  $U(k, i, j)$ , as well as the contribution  $\Delta\Theta(T'_i)$ , are determined in terms of quantities known at time  $t = 0$ . The quantity  $\Delta w(I, T'_i)$  is expressed in Eq. 9 as a sum of a contribution  $\Delta w_0(I, T'_i)$ , and a the contribution in square brackets which will be found in the example studied below to be much smaller than  $\Delta w_0(I, T'_i)$ . The terms in square brackets in Eq. 9 do not correspond exactly to the terms in square brackets in Eq. 4; the quantities  $\Delta T_{eff}(k, T)$  in Eq. 4 for  $T = T_i$  implicitly contain spread changes  $\Delta w(k, T'_i)$  for the same maturity  $T_i$ , whereas the square brackets in Eq. 9 contain only spread changes  $\Delta w(k, T'_j)$  for which  $j < i$ . Because numerical evaluation shows that the quantities in square brackets in the first and second lines of Eq. 9 are small in comparison with the quantity  $\delta w_0$ , these quantities will be neglected in further work. The stochastic variable  $\Delta w(I, T'_i)$  giving the time evolution of the index spread for nominal maturity corresponding to  $T_i$  is thus, to a good approximation, given in terms of the stochastic variables  $\Delta u(T'_i)$  and  $dw(k, T'_i)$ ,  $k = 2, \dots, nTr$  corresponding to tranches of the same nominal maturity. A practical guide to the goodness of this approximation is the accuracy with which its use allows the index to be hedged with a portfolio of tranches of the same maturity (as investigated in Figs. 2 and 3).

Suppose that one buys protection today (time  $t = 0$ ) on tranche  $k$ ,  $k = 1, \dots, nTr$ ,  $I$ , for the period from today to maturity  $T_i$ ,  $i = 1, \dots, 4$ . The fair spread for this contract is set in such a way that the present value of the contract is zero. The mark-to-market value of this protection at a later time  $t = \tau > 0$  is

$$\begin{aligned}
MtM(k=1, T'_i) &= u(T_i) + \Delta u(T'_i), \\
MtM(k, T'_i) &= \Delta w(k, T'_i) T_{eff}(k, T'_i), \quad k = 2, \dots, nTr, I. \tag{10}
\end{aligned}$$

The second line of Eq. 9 can be transformed into an equation relating these mark-to-market values, thus giving

$$MtM(I, T'_i) = MtM\_Pt(T'_i) \tag{11}$$

where  $MtM\_Pt(T'_i)$  is the mark-to-market value of a portfolio of tranches, given by

$$\begin{aligned}
MtM\_Pt(T'_i) &= \sum_{k=1}^{nTr} \tilde{N}(k, i) MtM(k, T'_i), \\
\tilde{N}(k, i) &= U(k, i, i) T_{eff}(I, T_i) / T_{eff}(k, T_i). \tag{12}
\end{aligned}$$

k \ i	1	2	3	4
1	0.0281	0.0288	0.0288	0.0283
2	0.0297	0.0297	0.0293	0.0284
3	0.0301	0.0301	0.0300	0.0296
4	0.0301	0.0301	0.0301	0.0300
5	0.1006	0.1004	0.1004	0.1005
6	0.7849	0.7841	0.7848	0.7875

Table 2: The table shows  $\tilde{\mathcal{N}}(k, i)$ , i.e. the weight of tranche  $k$  in the complete hedging portfolio for maturity  $i$ , calculated using market spreads for 5 May 2005, and assuming a recovery rate of  $R = 30\%$ . Calculations carried out for  $R = 0\%$  typically agree with these to within 0.3%.

The portfolio defined by the right hand side of the first of Eqs. 12 will be called the complete hedging portfolio (of the index) for maturity  $T'_i$ . Numerical evaluations show that the weights of the different tranches in the complete hedging portfolio for maturity  $i$ ,  $\tilde{\mathcal{N}}(k, i)$ , are approximately equal to the weights  $\mathcal{N}(k)$  defined at the beginning of Section 2 (see Table 2). Eqs. 11 and 12 give a linear relation between the stochastic variables  $MtM(k, T'_i)$ ,  $k = 1, \dots, nTr, I$ , where the coefficients in the linear relation,  $\tilde{\mathcal{N}}(k, i)$ , are known at time  $t = 0$ .

Note that, because  $\tilde{\mathcal{N}}(k, i) \approx \mathcal{N}(k)$ , the default risk will be accurately hedged, since the losses to the index will, to a good approximation, be equal to the losses to the complete hedging portfolio.

Note also that, to go from Eq. 9 to Eqs. 11 and 12 without approximation, one would have to replace  $T_{eff}(I, T_i)/T_{eff}(k, T_i)$  with  $T_{eff}(I, T'_i)/T_{eff}(k, T'_i)$ . However, the aim of this article is to develop a hedge that is correct to first order in the changes, and Eq. 11 with Eq. 12 is correct to first order in the changes. Furthermore, the second line of Eq. 12, written as is, allows the weights  $\tilde{\mathcal{N}}(k, i)$  of the hedging portfolios to be evaluated at the inception of the hedge (i.e. at  $\tau = 0$ ) which is essential.

The portfolios defined by

$$\begin{aligned}
 MtM(1, 6, T'_i) &\equiv \sum_{k=1,6} \tilde{\mathcal{N}}(k, i) MtM(k, T'_i), \\
 MtM(1, 2, 6, T'_i) &\equiv \sum_{k=1,2,6} \tilde{\mathcal{N}}(k, i) MtM(k, T'_i),
 \end{aligned} \tag{13}$$

will also play a role in the discussion of the results in the following section. These are examples of an incomplete hedging portfolios for the index: they hedge the correlation only partly.

The hedging portfolio weights  $\tilde{\mathcal{N}}(k, i)$  which enter into the first line of Eq. 12 obey the formula

$$\tilde{\mathcal{N}}(k, i) = \frac{\Delta MtM_{Pt}(T'_i)}{\Delta MtM(k, T'_i)} \Big|_{(k,i)} = \frac{\Delta MtM(I, T'_i)}{\Delta MtM(k, T'_i)} \Big|_{(k,i)}, \tag{14}$$

where the symbol  $\Big|_{(k,i)}$  indicates that the ratio of the changes is to be taken subject to  $\Delta MtM(r, T'_i) = 0$  for all  $r = 1, \dots, nTr$  except  $r = k$ . Eqs. 11 and 12 thus describe the ‘‘delta-hedging’’ of the index with respect to tranches  $k = 1, \dots, nTr$ .

The risk management of CDO’s will be discussed below in terms of the profit and loss (P&L) of the position of interest. The discussion of P&L in this paragraph is similar to that of Petrelli

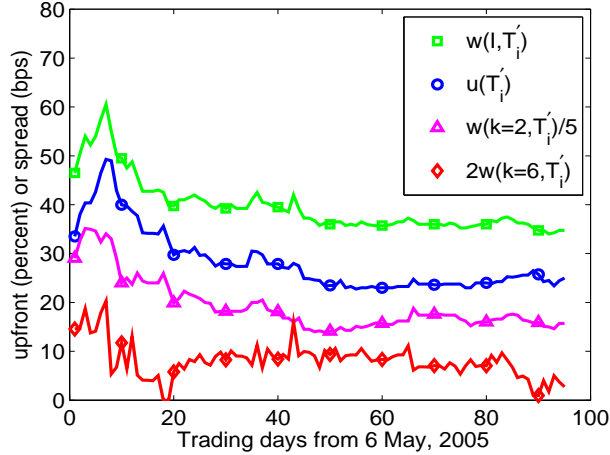


Figure 1: Spread behavior for selected tranches of 5-year maturity from 6 May 2005 (the day after the S&P downgrade of General Motors to junk status) to 19 September 2005. During the initial 20 trading days, spreads for the index, and the junior mezzanine tranche, and the upfront percentage for the equity tranche, first increased strongly, and then decreased strongly. The theoretical spreads, which are essentially identical to the mid-point spreads for the three just-mentioned tranches (e.g. see Table 1), are shown here.

et al. (2007). The P&L at a given time  $t$  has two components, a cash component,  $C(t)$ , and a mark-to-market component,  $MtM(t)$ . Thus, the P&L for a notional amount  $\tilde{N}(k, i)$  of tranche  $k$  (representing the contribution of this tranche to the complete hedging portfolio) at time to maturity  $T'_i$  will be written

$$P\&L(k, T'_i) = \tilde{N}(k, i) [C(k, T'_i) + MtM(k, T'_i)]. \quad (15)$$

For a portfolio that receives cash flows  $c_j$  at times  $t_j$ , the cash position at a time  $t$  is

$$C(t) = \sum_{j, t_j \leq t} c_j \exp \left[ \int_{t_j}^t r(t') dt' \right], \quad (16)$$

assuming that the cash accrues at the short risk-free rate  $r(t)$ . Also, for a premium payment made in arrears at a time  $t_{j+} > t$ , but covering a period from  $t_{j-} < t$  to  $t_{j+} > t$ , an accrued contribution proportional to  $t - t_{j-}$  must be added to this expression for  $C(t)$ . The cash positions  $C(k, T'_i)$  are approximately linear in the time  $\tau$  from contract inception. Thus, the principal effect of adding the cash component to the MtM component in arriving at the figures discussed in the following section, is to change the average slope of the curves.

Finally,  $P\&L_{Pt}(T'_i)$ ,  $P\&L(1, 6, T'_i)$  and  $P\&L(1, 2, 6, T'_i)$  are obtained by adding the appropriate cash component to  $MtM_{Pt}(T'_i)$ ,  $MtM(1, 6, T'_i)$  and  $MtM(1, 2, 6, T'_i)$ , respectively.

## 5 Results

Fig. 1 shows the behavior of selected theoretical spreads for the period 6 May 2005 to 19 September 2005. The theoretical spreads determine the risk-neutral measure for each trading

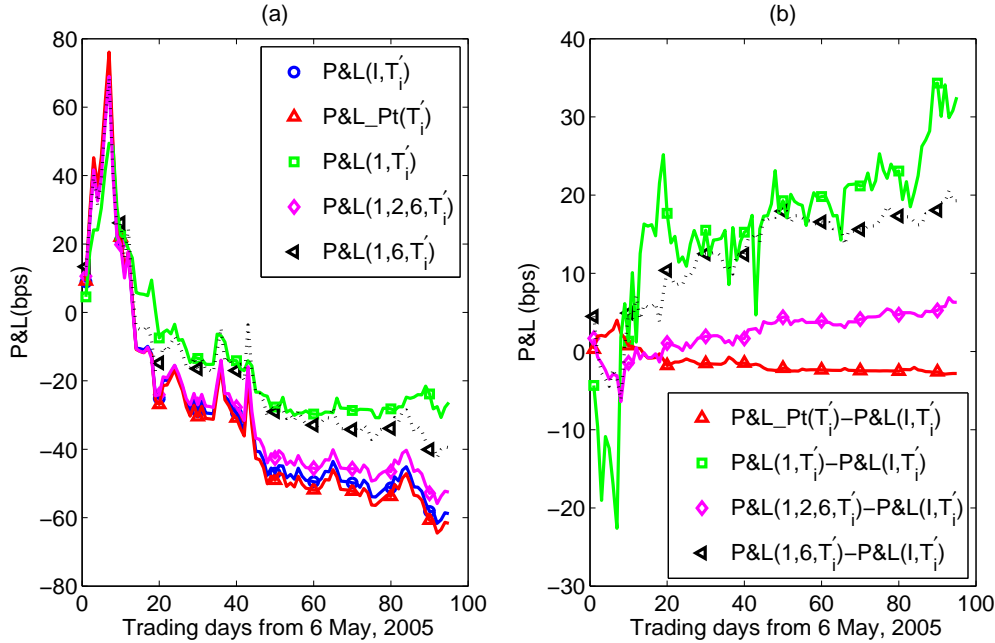


Figure 2: Part (a) of this figure shows the P&L of protection buyers as a function of trading days from the portfolio purchase date of 6 May 2005, for the index, the complete hedging portfolio, and the contribution to the complete hedging portfolio of the equity tranche, of tranches  $k = 1, 2$  and  $6$ , and of tranches  $k = 1$  and  $6$ . The nominal maturities are 5 years. Part (b) shows the results of using the index to statically hedge the four other portfolios shown in part (a).

day, which in turn is used to evaluate the daily P&L shown in subsequent figures. The theoretical spreads are equal to the mid-point spreads to a high degree of accuracy (see Table 1).

Fig. 2(a) shows the P&L, as a function of the number of trading days from 6 May 2005, for protection buyers of the index, as well as the P&L of a number of portfolios exhibiting varying degrees of correlation risk. The P&L's plotted here show strong variations which have their origin in the strong variations in the market spreads, as shown in Fig.1.

Fig. 2(b) shows the result of various efforts made to construct static portfolios from tranches and the index on 6 May 2005, such that these portfolios would have low risk, i.e. small daily fluctuations in mark-to-market values. In all of these portfolios the notional of the index is taken to be unity. Consider first the portfolio that is that is long protection on equity ( $k = 1$ ) tranche and short protection on the index and has a net P&L of  $P\&L(1, T'_i) - P\&L(I, T'_i)$ . This portfolio exhibits relatively strong mark-to-market fluctuations, indeed the strongest of all portfolios shown in the figure. Because the mark-to-market changes are negative initially and positive later, it is clear that no choice for a static hedge ratio of the equity tranche to the index will give good results over the entire time interval shown. Furthermore, it is clear that correlation risk can not be hedged by combining a single tranche with the index, even dynamically. Recall from the discussion in Section 1 that the word correlation in this article is taken to mean the way in which the total expected loss of the reference portfolio is parcelled out to the different tranches. Changes in correlation can thus occur that do not change the mark-to-market value of the index, while nevertheless strongly changing the mark-to-market value of the equity tranche.

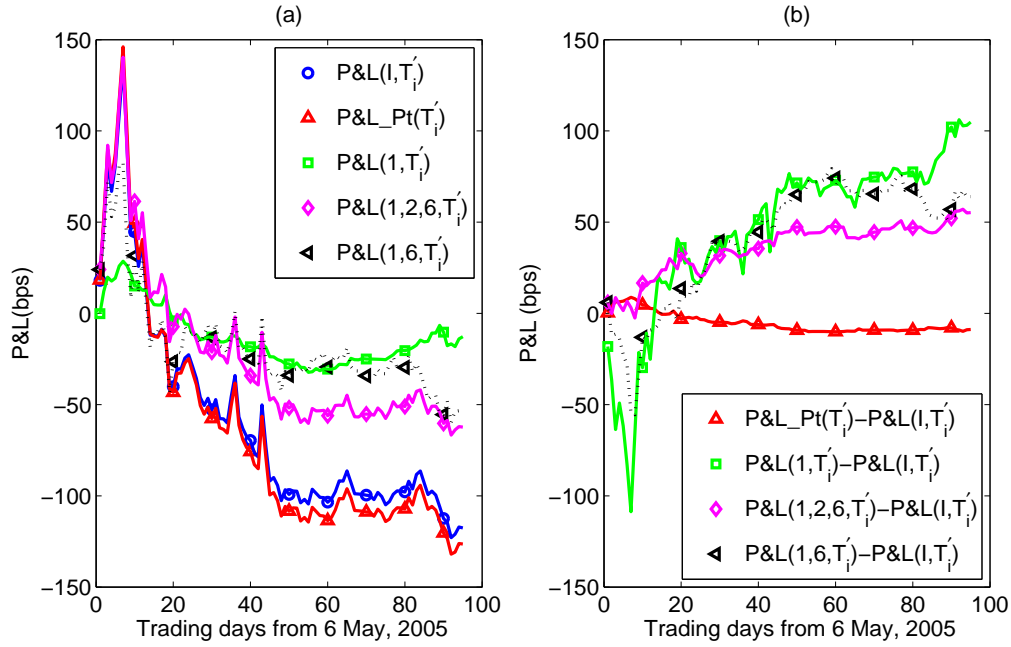


Figure 3: This figure is similar to Fig. 2, except that the nominal maturity of the contracts is 10 years.

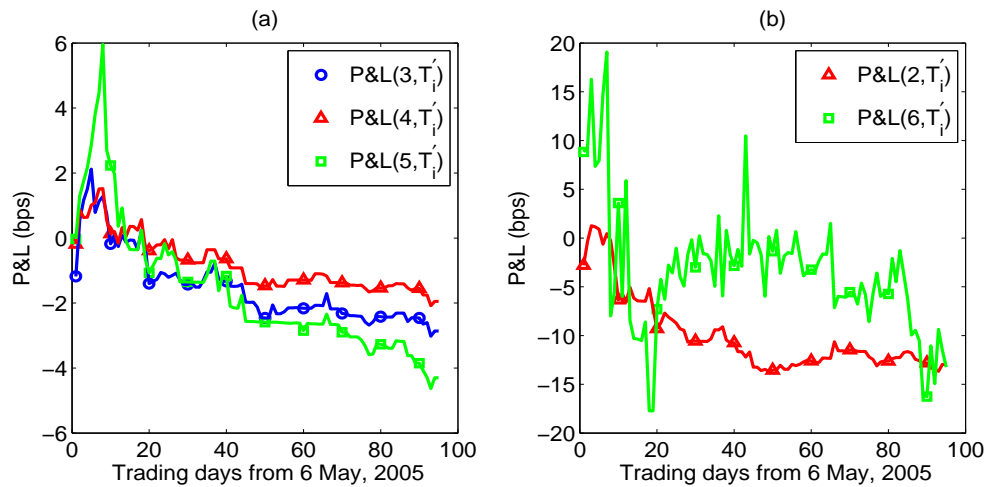


Figure 4: Part (a) shows the P&L contributions to the complete hedging portfolio for tranches  $k = 3, 4$  and  $5$ , while part (b) shows those for tranches  $k = 2$  and  $6$ . In all cases the nominal maturity is 5 years.

In order to even partially hedge the correlation risk associated with the equity tranche, it is necessary to combine it with at least one other tranche in such a way that the relative changes in the expected losses of the two tranches cancel. As a first try, consider combining the equity tranche with the super-senior tranche, since intuition from Gaussian copula models suggests that, for these two tranches, the mark-to-market changes due to a change of the copula correlation parameter will have opposite signs. The resulting plot of P&L for this case (with the index also included) is shown as  $P\&L(1, 6, T'_i) - P\&L(I, T'_i)$ . It can be seen that the inclusion of the super-senior tranche reduces the fluctuations in portfolio value somewhat, particularly during the first 10 trading days following 6 May 2005.

The inclusion of the junior mezzanine tranche in the latter portfolio, to form the new portfolio for which the P&L is given by  $P\&L(1, 2, 6, T'_i) - P\&L(I, T'_i)$ , shows a more dramatic reduction in daily variation (in Fig. 2(b)). Finally, as might be expected, the portfolio in which the index is hedged with what is called here the complete hedging portfolio (for the index), and which has a P&L of  $P\&L_{Pt}(T'_i) - P\&L(I, T'_i)$ , has the smallest daily variations in P&L.

The comments for Fig. 2 also apply to Fig. 3, except that for the latter the nominal maturity is 10 years.

The above few paragraphs have described the fact that, when one starts with a portfolio consisting of the equity tranche and the index and then successively adds other tranches, the magnitude of the daily fluctuations in portfolio mark-to-market values is decreased. This can only occur if the changes in the mark-to-market values of the the different tranches are strongly correlated.

For completeness, the contributions of the P&L's of tranches  $k = 3, 4$  and  $5$  to the complete hedging portfolio are shown in Fig. 4(a). It is clear that these three tranches give relatively small contributions to the complete hedging portfolio (note the difference in scale of a factor of more than 10 between Fig. 2(a) and Fig. 4(a)). Part (b) shows the contributions of tranches  $k = 2$  and  $6$ .

The results described in this section are not sensitive to the assumed value of the recovery rate. For example, the variation in the plot of  $P\&L_{Pt}(T'_i) - P\&L(I, T'_i)$  in Fig. 2(b) in going from  $R = 0\%$  to  $R = 30\%$  is about 0.1 bps.

Although no losses due to default occur in the reference portfolio over the time-period shown in the figures, the portfolios discussed do provide at least partial hedging against default losses. For example the portfolio described by  $P\&L(1, 2, 6, T'_i) - P\&L(I, T'_i)$ , is immunized against losses due to default to tranches  $k = 1$  and  $2$ , while the portfolio described by  $P\&L_{Pt}(T'_i) - P\&L(I, T'_i)$  is immunized to all losses due to default.

## 6 Conclusions

A contract with a given maturity for protection on the index (here taken to be the iTraxx index) can be statically hedged, to a good first approximation, by what was called above the complete hedging portfolio. The complete hedging portfolio consists of a complete set of CDO tranches of the same maturity as the index. This is a form of delta-hedging in which the delta for a given tranche, relative to the index (both having the same maturity) is the change in the mark-to-market value of the index divided by the change in the mark-to-market value of the given tranche, assuming all of the other tranches have unchanged mark-to-market values. A tranche

delta thus defined is approximately equal to the standardized tranche notional for that tranche. Backtesting using iTraxx market spreads during the days following the 5 May 2005 downgrade of General Motors and Ford showed that the index hedged with the complete hedging portfolio was largely immunized against the turbulent market fluctuations of that period. One could thus say that both index spread risk and correlation risk were accurately hedged. Also, the portfolio in question is immunized against default risk. (No defaults occurred in the data so this latter claim was not tested.)

As noted in the introduction, a principal objective of this note has been to study the hedging of correlation risk. Thus, the backtesting study described above treated four different example portfolios in which the index was combined with (i) the equity tranche, (ii) the equity and the super-senior tranches, (iii) the equity, junior mezzanine and super-senior tranches and (iv) all tranches. Choice (i), in which there is no hedging of correlation, was found to be the riskiest (i.e. the daily variations in P&L were the largest). In going from choice (i) to choice (ii) to choice (iii) to choice (iv) the number of tranches in the portfolio is increased and the magnitude of the daily fluctuations in total P&L is decreased, thus indicating a strong correlation between the changes of the mark-to-market values of the different tranches. Individual tranches have strong correlation risk, which results from day-to-day changes in the relative portioning out of the expected losses of the reference portfolio to the different tranches. This correlation risk can be reduced by incorporating a number of different tranches in the same portfolio in such a way that the highly correlated changes in tranche mark-to-market values tend to cancel out.

Since hedging the correlation risk of tranches requires the construction and the consistent marking-to-market of multi-tranche portfolios, it is necessary to implement the procedure with a model that (unlike typical copula models, for example) can be simultaneously calibrated to a number of different tranches and maturities (and preferably to all available market prices). The static loss-surface model (Walker, 2005, 2006; Torresetti et al., 2006) was used for this purpose.

Results were obtained for a range of different assumed values for the recovery rate, and were found to be insensitive to the choice of value for the recovery rate.

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