CDO Models – Towards the Next Generation: Incomplete Markets and Term Structure

Michael B. Walker[1,2,3]

First version: July 27, 2005
This expanded version: May 29, 2006

Abstract

This article describes a new approach to the risk-neutral valuation of CDO tranches, based on a general specification of the tranche loss distributions and the index default distribution. The new model is a term-structure model, and the generality with which the basic distributions are specified allows it to be perfectly calibrated to any set of market prices (for any number of tranches and maturities) that is arbitrage-free. The use of the new model is illustrated by testing market prices for the standardized iTraxx index tranches (for all marketed tranches and maturities) to see if they are arbitrage-free. Other examples include the determination of the arbitrage-free range of prices allowed for an unmarketed standardized tranche and the determination of the cost of exiting a tranche position. For the latter example, both arbitrage-free price ranges, and a preferred price, are obtained. Prices for unmarketed maturities and unmarketed non-standard tranches are also obtained by an interpolation and extrapolation procedure. Because the model is an incomplete-market model characterized by many more parameters than market prices, it was essential to develop an efficient optimization approach to valuation. The article also makes use of a new approach to the problem of unequal recovery rates and notionals.

[1]Department of Physics, University of Toronto, Toronto, ON M5S 1A7, CANADA; email: walker@physics.utoronto.ca; telephone: (416) 978-3821.

[2]I thank David Beaglehole, Mark Davis, Julien Houdain, John Hull, Alex Kreinin, Dan Rosen and Alan White for discussions and encouragement; Efim Katz and the Theory Group of the Institute Laue-Langevin, where some of this work was carried out, for their hospitality; and Julien Houdain and Fortis Investments for providing the market prices used in the examples of this article.

[3]The support of the Natural Sciences and Engineering Research Council of Canada is acknowledged.

1 Introduction

This paper presents a new approach to the risk-neutral modelling of collateralized debt obligations (CDO's). The approach is characterized by the adoption of risk-neutral loss and default distributions for the reference portfolio that are defined in terms of general, model-independent constraints (e.g. probabilities must lie between zero and unity, and losses must increase with time). A general model such as this is essential 1) if one requires a term-structure model that can distinguish between sets of tranche prices that are arbitrage-free and those that are not, and 2) if one requires a term-structure model that can be calibrated to any set of arbitrage-free market prices. Because there can be a relatively large number of relevant market prices available on a given reference portfolio on any given day (e.g. see Table 1 below), the problems of checking to ensure that these market prices are arbitrage-free, and of calibrating a model to them, is a non-trivial one. A key element in the realization of this program is the implementation of the model via an efficient linear-programming optimization approach. The search for a model that can be calibrated to a given set of market prices has been a central topic of recent CDO research (e.g. see Burtschell, Gregory and Laurent (2005a)), and with good reason: the verification of the arbitrage-free character of the set of all relevant market prices, and the precise calibration of the model to these prices, is the starting point for any application of risk-neutral pricing.

The model is first applied to the problem of checking iTraxx market quotes for arbitrage opportunities, and a significant number of cases are found where, according to the model, the market was not arbitrage-free. The problem of pricing an unmarketed standardized tranche is also discussed. Because of the incomplete-market nature of the CDO problem, an unmarketed tranche has a range of potential arbitrage-free prices, and not a single definite price. This article represents a significant evolution from previous presentations [Walker (2004/2005), Walker (2005)] of the CDO problem as an incomplete-market problem. Another example treated below is the use of the model to calculate the windup cost for a tranche position. An arbitrage-free ranges of values is found for the windup cost, and, in addition, a procedure for establishing a "preferred" risk neutral measure giving rise to a preferred price for this cost is established. Finally, interpolation and extrapolation procedures are devised to produce acceptable arbitrage-free prices for standardized tranches at unmarketed maturities, and for non-standard tranches.

A collateralized debt obligation is a financial instrument that transfers the credit risk of a reference portfolio (or basket) of assets. The credit risk on the CDO is tranched, so that a party that buys insurance against the defaults of a given tranche receives a payoff consisting of all losses that are greater than a certain percentage, and less than another certain percentage, of the notional of the reference portfolio. In return for this insurance, the protection buyer pays a premium, typically quarterly in arrears, proportional to the remaining tranche notional at the time of payment; there is also an accrued amount in the event that default occurs between two payment dates. A relatively liquid market for standardized index tranches, based on the CDX and iTraxx credit default swap (CDS) indices has developed recently, and the detailed examples of this article will refer to these index tranches. (For an introduction to standardized CDS indices and index tranches see Amato and Gyntelberg (2005).)

In the modelling of a single defaultable bond, or credit default swap, the risk-neutral fractional loss at default process, and the risk-neutral hazard rate process giving rise to defaults, are regarded as separate processes. Model calibration involves attempting to extract information from market prices that can be used to separately calibrate these processes (Duffie and Singleton, 1999). The counterpart of these ideas in this article is that the tranche-averaged loss distributions on the one hand, and the notional-averaged default distribution on the other, are regarded as separate distributions. The problem of calibrating to different risk-neutral expected losses of the individual names in the reference portfolio, all of which are, in general, functions of time, is not an easy one within the framework of a copula model. This problem is transformed in the present article to the problem of imposing the requirement that the value of the loss distribution for the index can not exceed the notional averaged default distribution for the index (see the discussion surrounding Eq. 6).

At present, the most commonly used models for the risk-neutral pricing of collateralized debt obligations (CDO's) are copula models. It has been found to be possible to find copulas that reproduce very well, or even perfectly, selected market prices for a number of different tranches of the same reference portfolio (or basket) and which have the same maturity (e.g. see Andersen and Sidenius (2004/2005), Guegan and Houdain (2005), Hull and White (2005), Joshi (2005), and for a recent review see Burtschell et al. (2005a)). Whereas a given copula model describes only a subset of the the possible deterministic risk-neutral measures, and thus can not probe for arbitrage opportunities within this class, it is the goal of this article to establish a framework that can probe all possible deterministic risk-neutral measures.

Although the present article incorporates two features that should be incorporated into the next generation of CDO models: incomplete markets and term structure, it does so within the framework of a deterministic risk-neutral measure. While deterministic risk-neutral measures have demonstrated their usefulness in practice in the pricing of credit defaults swaps, and in the copula approach to CDO's it would nevertheless be a further significant step forward to extend CDO modelling to models accounting for the dynamical evolution of the term structure. Recently, a number of models presenting frameworks allowing for dynamical evolution have been introduced (Albanese et al. (2005), Bennani (2005), Hull et al. (2005), Schönbucher (2005), and Sidenius et al. (2005)).

2 CDO Tranche Valuation

A CDO tranche refers to a particular loss segment of a basket of N reference entities. The total notional of the basket will be taken to be unity and will be divided into contiguous segments called tranches labelled $k = 1, 2, ..., nTr$, where nTr is the total number of tranches. Losses $\ell(k)$ are introduced such that tranche k is associated with losses from $\ell(k-1)$ to $\ell(k)$; also $\ell(0) \equiv 0$ and $\ell(nTr) \equiv 1$.

Tranches can be valued if the loss distribution is known (e.g. see Laurent and Gregory (2003), Hull and White (2004)). The loss distribution $F(\ell, t)$ is the probability that the total losses associated with defaults of any of the names in the reference basket exceed ℓ at time t. To value tranche k, it is convenient to use the tranche-averaged loss distribution $f(k, t)$, defined by

$$f(k,t) = \frac{1}{L(k)} \int_{\ell(k-1)}^{\ell(k)} F(\ell, t) d\ell, \tag{1}$$

where $L(k) = \ell(k) - \ell(k-1)$. It can be shown that $L(k)f(k, t)$ is the expected value of the losses associated with tranche k at time t.

Consider a CDO contract of maturity T which provides protection against losses associated with tranche k in return for premium payments made at times $t_j, \quad j = 0, 1, \ldots, N_p(T)$. The j-th payment is for protection for the interval $\delta_j = t_j - t_{j-1}$ $(t_{-1} \equiv 0)$ and has a magnitude equal to $w(k, T)\delta_j$ times the remaining tranche notional at time t_j, where $w(k, T)$ is the annualized premium for tranche k. Typically the premium payments are made quarterly, so that $\delta_j = 0.25$ for $j \geq 1$, but one can have $\delta_0 < 0.25$. Furthermore, if a default occurs at some time t in the interval (t_{j-1}, t_j), an accrued payment of $w(k,T)\delta_t\phi_t$ times the default loss associated with the tranche notional must be made. Here, $\delta_t = \delta_j$ and $\phi_t = (t - t_{j-1})/\delta_j$ for t in (t_{j-1}, t_j). The fair value of the annualized premium $w(k,T)$ for the tranche is calculated by balancing the present value of the expected losses against the expected present value of the premium payments, which gives

$$\int_0^T e^{-rt} df(k,t) = uf(k,T) + w(k,T)\left\{\sum_{j=0}^{N_p(T)} \delta_j[1 - f(k,t_j)]e^{-rt_j} + \int_0^T \delta_t\phi_t e^{-rt} df(k,t)\right\}. \quad (2)$$

An upfront premium payment $uf(k,T)$ has been included.

The calculation of the present value of the expected premium payments on the index is quite different from that for the tranches, since the premium payment at a given time t is proportional to the existing notional, and independent of the losses. The fair value of the premium $s(T)$ for the index is also found by balancing the present values of the expected losses and premium payments, giving

$$\sum_{k=1}^{nTr} L(k) \int_0^T e^{-rt} df(k,t) = s(T)\left\{\sum_{j=0}^{N_p(T)} \delta_j[1 - q(t_j)]e^{-rt_j} + \int_0^T \delta_t\phi_t e^{-rt} dq(t)\right\} \quad (3)$$

where

$$q(t) = \sum_{i=1}^{N} \mathcal{N}_i q_i(t). \quad (4)$$

is called the notional-averaged default distribution. The quantity $\mathcal{N}_i$ is the notional of name i, $\sum_{i=1}^{N} \mathcal{N}_i = 1$, and $q_i(t)$ is the probability that the i-th name defaults before time t.

It would perhaps be possible to evaluate the premium for the index in terms of the premiums for the CDS's on the individual names in the reference portfolio. However, it is customary to regard the contract for protection on the index as a separate contract that trades on the market at its own price, and not at a price determined theoretically in terms of the market prices of individual CDS's (Chaplin, 2005).

3 The Risk-Neutral Distributions

The approach of this article is to consider all tranche-averaged loss distributions and notional-averaged default distributions, subject only to certain necessary general constraints. The relations constraining the distributions are

$$\begin{aligned} 0 < f(k,t) &< 1, \quad \text{all k}; \\ f(k,t) &> f(k+1,t), \quad k = 1,\ldots,nTr-1; \\ \frac{\partial f(k,t)}{\partial t} &> 0, \quad \text{all k}; \\ 0 < q(t) &< 1; \\ dq(t)/dt &> 0; \\ \sum_{k=1}^{nTr} L(k) f(k,t) &< q(t). \end{aligned} \tag{5}$$

These constraints should be self-evident, except for the last one. For the final constraint of Eqs. 5, note that the expected value of the losses occurring for the index in time interval $(t, t+dt)$ can be written

$$d\mathcal{L}^I(t) = \sum_{i=1}^{N} \mathcal{N}_i \ell_i(t) \frac{dq_i(t)}{dt} dt. \tag{6}$$

Here, $\ell_i(t)$ is the risk-neutral expected fractional loss given default for name i at time t. Since the maximum allowed value for $\ell_i(t)$ is unity, the maximum allowed value of the expected losses at time t is obtained by putting $\ell_i(t) = 1$ (for all i and t) in Eq. 6 and integrating between times 0 and t. This gives the final constraint of Eqs. 5.

For the standardized index tranches, there are a total of nTr market premiums available (for tranches $k = 1, 2, \ldots, (nTr-1)$ and for the index).[4] However there are $nTr+1$ functions of t to be determined (the distributions $f(k,t),\ k = 1,\ldots,nTr$, and $q(t)$). It is clear that, even with the assumption that market premiums are available for all maturities between zero and some maximum maturity, there is not enough information to completely determine the distributions. The tranche problem is thus an incomplete-market problem.

[4] Burtschell et al. (2005b) state (in the caption to their Table 1) that the iTraxx 22-100% tranche is not traded.

iTraxx Tranche Quotes - 21 June 2005				
Tranche	3 year	5 year	7 year	10 year
1: 0-3%	7.3	27.38	43.8	53.25
2: 3-6%	26	91	245	455
3: 6-9%	7.4	31.5	60	123
4: 9-12%	2.4	19	30	56
5: 12-22%	2.3	12.5	19.5	35.5
0-100%	23.25	39.25	49.75	60.5

Table 1: The table shows the quotes (mid-point of bid and ask) for the standardized iTraxx tranches for 21 June, 2005. The 0 to 3% (equity) tranche is quoted as a percentage upfront payment, assuming that subsequent payments are made quarterly at a rate of 500 basis points per year. The other tranches are quoted as basis points per year, again assuming quarterly payments. The 0-100% tranche is the iTraxx index. The quotes are for CDO contracts having maturities of 3, 5, 7, and 10 years. Source: Julien Houdain and Fortis Investments.

For the purposes of numerical implementation, the tranche-averaged loss distributions and the notional-averaged default distribution will be taken to have a continuous piecewise-linear form. Thus, consider a sequence of times $T_m, m = 0, 1, \ldots, m_{Max}$ with $T_0 = 0$ and $T_{m_{Max}}$ being the maximum maturity for which CDO contracts on the tranches of the particular basket of interest are considered. The parameterized form of the distributions is defined piecewise by

$$\begin{aligned} f(k,t) &= f(k,T_{m-1}) + g(k,m)(t - T_{m-1}), \\ q(t) &= q(T_{m-1}) + \lambda(m)(t - T_{m-1}), \\ T_{m-1} < t &\leq T_m, \quad m = 1, 2, \ldots, m_{Max}, \end{aligned} \tag{7}$$

with $f(k, T_0) = 0$. The relevant distributions are thus completely determined by a knowledge of the sets $\{g\}$ of all $g(k,m)$, and $\{\lambda\}$ of all $\lambda(m)$.

4 Calibration and Arbitrage Opportunities

An example of market quotes for the iTraxx index and for the standardized iTraxx tranches for 21 June, 2005 is shown in Table 1. This section addresses the question of whether or not a given set of quotes, such as those in the example, can be reproduced by an appropriate choice of the sets of parameters $\{g\}$ and $\{\lambda\}$ defining the risk-neutral loss distribution. If this is the case, then, according to the model, the set of quotes in question will have been shown to be arbitrage free. On the other hand, if no risk-neutral loss distribution can be found to reproduce the quotes, then, according to the model, the quotes should admit arbitrage possibilities.

To determine whether or not a risk-neutral measure can be found that reproduces a given set of market prices, consider the linear-programming problem of maximizing the quantity zero over all model parameters in the sets $\{g\}$ and $\{\lambda\}$, subject to the relevant constraints. More specifically,

$$\begin{aligned}
&\text{Maximize}_{[\{g\},\{\lambda\}]}\ 0, \quad \text{subject to:} \\
&\qquad \text{Eq. 2 for all tranches } k \text{ and maturities } M \\
&\qquad\qquad \text{for which there are market prices,} \\
&\qquad \text{Eq. 3 for all maturities } M \text{ for which} \\
&\qquad\qquad \text{there are market prices,} \\
&\qquad \text{and the constraints of Eq. 5.}
\end{aligned} \tag{8}$$

The maximum maturity is 10 years and the number of time steps in this interval is taken to be $m_{Max} = 40$; this determines the number of parameters in the sets $\{g\}$ and $\{\lambda\}$. The time-step intervals are taken to be quarterly, with the possible exception of the first. Also, in this and in all other examples, a constant risk-free rate of $r = 5\%$ is assumed. This linear programming problem has a solution for the iTraxx market prices listed in Table 1, indicating that this set of market prices is arbitrage-free.

In a similar manner, an attempt was made to calibrate the model to the iTraxx quotes for each of the 97 trading days between 6 May 2005 and 19 September 2005. Calibration was successful for 76 of these days and unsuccessful for 21 of these days, indicating 21 instances in which the market was not arbitrage-free.

Both Eq. 3 and the final constraint of Eqs. 5 contain all tranches as well as the index. From these equations it it clear that it is necessary to consider the full set of market prices that are available for all tranches and for all maturities (as has been done here) when checking to ensure the absence of arbitrage opportunities, and when calibrating the model. One can not, for example, just consider all tranches at a single maturity, or all maturities for a single tranche.

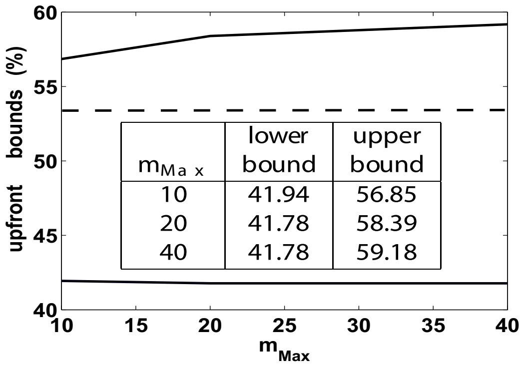

Figure 1: The solid lines in the Figure show the upper and lower bounds obtained on the arbitrage-free range of values of the upfront payment for a certain CDO contract (see text for details). The bounds are plotted for piecewise-linear interpolation schemes containing $m_{Max} = 10$, 20 and 40 steps in a period of 10 years. The Table shows the same values that are plotted, but with greater accuracy. The dashed line shows the market price (not used as input data) which is clearly inside the bounds, and thus arbitrage-free.

5 An Unmarketed Standardized Tranche

As an example of the pricing an unmarketed standardized tranche, suppose that the iTraxx equity tranche with the 10-year maturity is not available on the market, but that all of the 23 other tranche-maturity combinations shown in Table 1 are on the market, and have the market prices indicated in the Table. This section determines the upper and lower bounds of the arbitrage-free price range for the unmarketed tranche. If a prospective buyer of this tranche makes known to a dealer a desire to buy this unmarketed tranche, its price within the bounds set by risk-neutral pricing will be determined by negotiation between the buyer and the dealer, and can not be reliably predicted by the model (or, indeed, by any model).

The price of the unmarketed tranche in question, $uf(k_0, M_0)$ for $k_0 = 1$, M_0 corresponding to the 10-year maturity, is determined as a linear function of the model parameters by Eq. 2. The bounds on the arbitrage-free range of values available to $uf(k_0, M_0)$ can thus be determined by a linear-programming optimization procedure similar to that of the preceding Section.

To assess the accuracy of the piecewise-linear method of representing the tranche-averaged loss distributions, the bounds are obtained by using the three different values, 10, 20, and 40, for m_{Max}. Recall that m_{Max} is number of linear time steps between time zero and a time equal to the maximum maturity (10 years). The results are shown in Fig. 1. For any given value of m_{Max}, the bounds obtained give a price range within which all prices are arbitrage-free. As the value of m_{Max} is increased, the bounds expand somewhat, as expected. However, the changes in the bounds are not large, and the value of $m_{Max} = 40$ would appear to give a reasonably accurate determination of the bounds, at least for the purposes of the discussions of this article.

As a second example, consider the identical problem to that just discussed, but for market prices for 12 consecutive trading days beginning with 6 May 2005. Fig. 2 shows the results. Note that the market price of the 10-year equity tranche contract (not used in the calculation) is close to the calculated upper bound. For days 4, 5, and 6, the actual market price is above the upper bound, indicating that there arbitrage opportunities. For days 7, 8, and 9, there is no solution, again indicating the existence of arbitrage opportunities. Had this knowledge been freely available in May 2005, it might have had an impact on the pricing of the tranche in question.

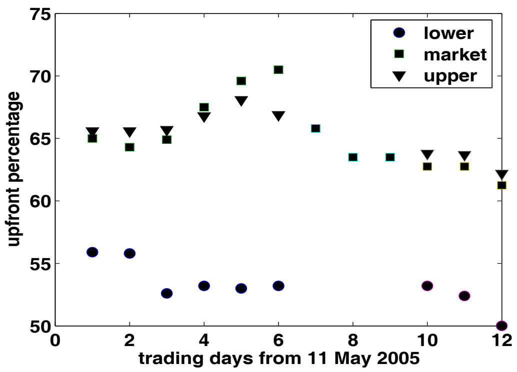

Figure 2: The upper and lower bounds on the arbitrage-free price range, and the actual market price, for the upfront premium for the 10-year equity tranche of the iTraxx index, for 12 consecutive trading days beginning with 11 May, 2005.

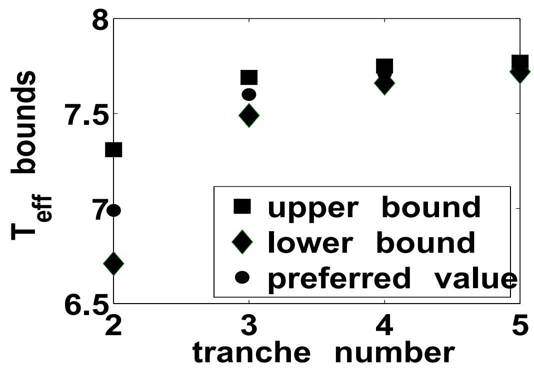

Figure 3: The upper and lower bounds on the arbitrage-free range of values for the quantity T_{eff} used in the calculation of the payment necessary to exit, on 21 June 2005, a position in one of the standardized iTraxx 10-year maturity tranches $k = 2, \dots, 5$. The preferred value is also shown. The quantity T_{eff} is smaller for lower tranche numbers because the possibility of more defaults makes the expected number of premium payments smaller.

6 Exiting a Tranche Position

Consider an investor that has sold protection on a CDO tranche to a dealer, and suppose that the investor wishes to exit this position. The investor then buys protection on the same tranche from the dealer. The loss payments in the event of default cancel each other for these two positions. In the case of an equity tranche, the investor will have to pay the upfront payment on buying protection, and after this, the premium payments (which are 500 basis points in each case) will cancel each other, so the original position has been cancelled out. In the case of a tranche k, with $k = 2, \dots, nTr - 1$, the premium for the new contract $w(k, M)$ may be different from the premium $w_{old}(k, M)$ agreed to for the original contract. The present value, per unit tranche notional, of the difference of the remaining premium payments can be calculated as

$$PV = [w(k, M) - w_{old}(k, M)]T_{eff}(k, M), \tag{9}$$

where $T_{eff}(k, M)$, an effective time to maturity, is given by the term in curly brackets in Eq. 2.

As an example, suppose that the investor had sold protection on a standardized iTraxx 10-year maturity contract on 5 April 2005, and that on 21

June 2005 the investor wanted to buy protection from a dealer so as to cancel the original contract. The iTraxx quotes for 21 June 2005 are shown in Table 1, and have been shown above to be arbitrage-free, so that a procedure very similar to that of the previous section can be used to evaluated the upper and lower bounds of the arbitrage-free range of values for the payment given by Eq. 9 (or, equivalently, T_{eff}) necessary to exit the original position. The results for the bounds on T_{eff} are shown in Fig. 3. The calculation of the preferred value is described below.

7 Calibration to a Preferred Measure

The previous section has described how to calculate the arbitrage-free ranges of values allowed for the tranche windup cost. The choice of a definite price within this range is a matter of the personal preferences of individual buyers and sellers, and of negotiation in the market. It is often useful to have a method of imposing personal preferences on the model, and this section describes one way of doing this. It must be emphasized that the individual market participants will have there own preferences, and that the example given here is for illustrative purposes only. Readers dissatisfied with the particular interpolation scheme described below can impose their own preferences in a similar way.

Table 1 tabulates market prices for tranches for maturities of 3, 5, 7, and 10 years. This section uses an interpolation and extrapolation procedure to obtain an extended set of prices for the 10 equally spaced maturities from 1 to 10 years for the tranches listed in Table 1, including the index. The preference expressed in this section is that the plot of tranche prices versus maturity should be relatively smooth (see below for more details). Also, a choice of annual interpolation steps from 1 to 10 years is made, so that the interpolation steps coincide with the intervals between maturities for the extended price set. The model is then calibrated, as in Section 4, by choosing the risk-neutral measure in such a way that the prices in the extended price set are perfectly reproduced. The risk-neutral measure produced by this procedure is called the preferred measure, and can be used to obtain definite prices (called the preferred prices) for tranche windup costs. These preferred prices are shown in Fig. 3. Note that the preferred prices fall within the arbitrage-free bounds, as expected.

In more detail, cubic spline interpolation was used to obtain market prices

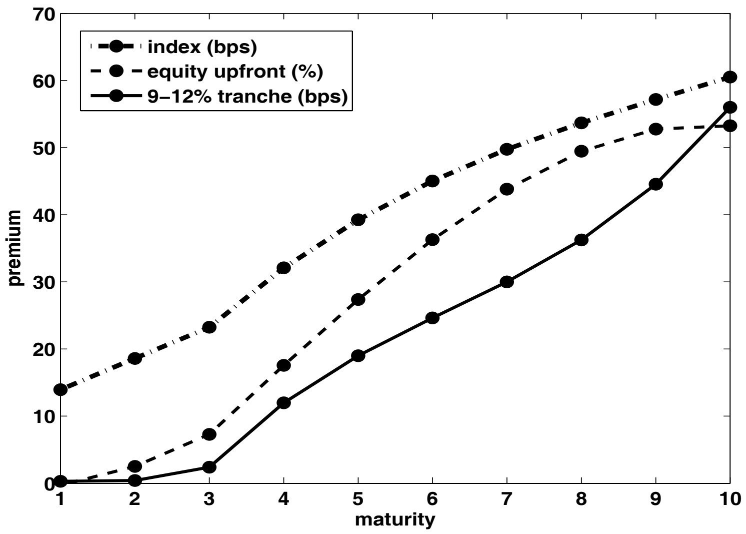

Figure 4: The premiums obtained by interpolation and extrapolation from the iTraxx quotes of Table 1 for the index, and for 0-3% and 9-12% tranches are indicated by the solid circles. These three tranches were chosen for display because their numerical values allows them to be shown on the same graph. Similar curves are found for the 3-6%, 6-9% and 12-22% tranches. The preferred risk-neutral measure is determined by calibrating to the prices represented by the 10 solid circles, for each of the 6 tranches.

for each of the tranches $1, \ldots, nTr - 1$, and the index, for maturities of 4, 6, 8, and 9 years, from the market prices of Table 1 for maturities of 3, 5, 7 and 10 years. The calibration procedure of Section 4 was then applied, and it was found that the resulting set of 48 market prices is arbitrage-free. The next task is to extrapolate to determine prices for the maturities 1 and 2 years, and it is clear that, without market prices for at least the one-year maturity as a guide, there will be a high degree of arbitrariness in this procedure. A choice of prices was made based on the criteria that the chosen prices should be arbitrage-free, and that the premiums should not decrease with increasing maturity. The method used in Section 5, of finding the arbitrage-free range of premiums for an unmarketed tranche, was an essential tool in the procedure. This procedure could be improved by estimating the 1 and 2-year maturity premiums for the index from CDS premiums for the names making up the index. The sets of prices finally arrived at for the index, the upfront equity payment, and for the premium for 9-12% tranche are shown in Fig. 4. These prices represent an acceptable (i.e. smooth and arbitrage-free) way of extending the original price set to non-standard maturities.

8 Pricing Non-Standard Tranches

The previous section described one way of arriving at tranche prices for non-marketed maturities, given a set of tranche prices for the standard marketed maturities of Table 1. It is also possible to devise interpolation schemes to estimate acceptable prices for unmarketed non-standard tranches (i.e. having non-standard attachment and detachment points) in terms of the set of prices obtained in the previous section.

As an example, consider the set of $(nTr - 1)$ "shifted" tranches $[1.5 - 4.5\%, 4.5 - 7.5\%, 7.5 - 10.5\%, 10.5 - 17\%, 17 - 61\%]$. Note, for example, that the price of the shifted $1.5 - 4.5\%$ tranche would be expected to lie in between the prices of the $0 - 3\%$ and $3 - 6\%$ standard tranches. The procedures developed earlier in the article have been used to determine a smooth set of arbitrage-free shifted tranche prices, by interpolation from the tranche prices given in the previous section, for all shifted tranches and for all integral maturities from one to ten years. A sample of the results, for maturities of 4, 5 and 6 years, is shown in Fig. 5. Note that there are now a total of 10 tranches, one index, and 10 maturities, for a total of 110 distinct prices. The model has been calibrated so as to perfectly reproduce all of the 110 prices.

Finally, it should be noted that the calibration procedure of this section produces a parameterized form for the full loss distribution, $F(\ell, t)$, introduced in Section 2. The model can therefore price tranches with non-standard attachment points other than those considered here.

9 Conclusions

This article describes a term-structure model for the risk-neutral valuation of CDO tranches. Examples show how to use the model to determine whether or not a given set of market prices is arbitrage-free, and how to calibrate the model to any set of arbitrage-free market prices. These two operations

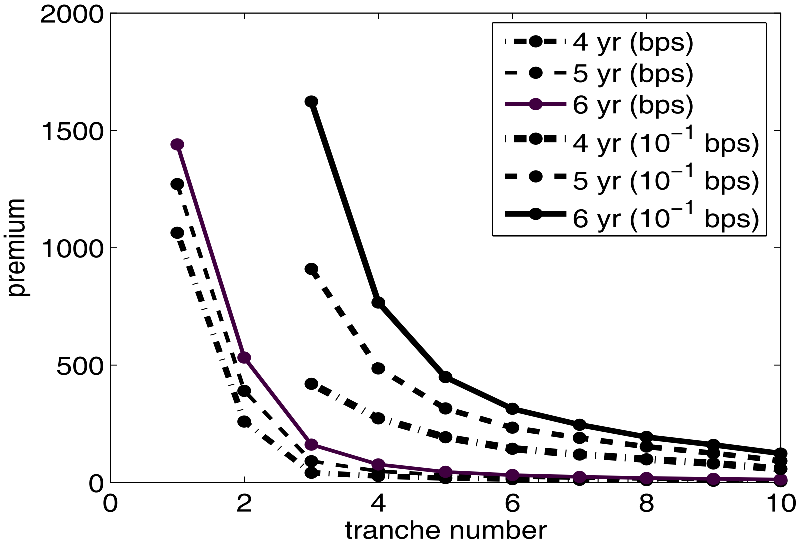

Figure 5: The premiums shown are for the tranches [0-3%, 1.5-4.5%, 3-6%, 4.5-7.5%, 6-9%, 7.5-10.5%, 9-12%, 10.5-17%, 12-22%, 17-61%], labelled 1 to 10, respectively, in the figure. Only the results for maturities 4, 5, and 6 years are shown, and the three curves for tranches 3 to 10 towards the top right have been expanded vertically by a factor of 10 relative to the corresponding curves towards the bottom left. For comparison purposes, the 0-3% tranche is valued here in terms of the appropriate quarterly premium given zero upfront payment. The prices to which the model has been calibrated are indicated by the solid circles. Note that the results for the shifted tranches fit smoothly in between those for the standard tranches, as intended.

represent the starting point for all risk-neutral pricing applications. Other examples discuss the use of the model to find the arbitrage-free range of values for the premium of an unmarketed standard tranche and the windup cost of a tranche position. The construction of a preferred risk-neutral measure giving precise values for windup costs is also discussed, as is the problem of pricing, by interpolation, of both standard and non-standard tranches at both standard and non-standard maturities.

An essential feature of the new approach is the development of a linear-programming optimization procedure that deals efficiently with the incomplete-market aspects (i.e. the fact the general specification of the risk-neutral measure requires many more parameters than there are market prices) of the model. Also, the fact the the risk-neutral expected loss may vary with obligor, and is in general time-dependent, is accounted for in a simple manner in terms of a constraint.

References

Albanese, Claudio, Oliver Chen and Antonio Dalessandro, 2005, "Dynamic Credit Correlation Modeling," www.defaultrisk.com/pp_corr_80.htm

Amato, Jeffery D. and Jacob Gyntelberg, 2005, "CDS Index Tranches and the Pricing of Credit Risk Correlations," BIS Quarterly Review, March.

Bennani, Norddine, 2005, "The Forward Loss Model: A Dynamic Term Structure Approach for the Pricing of Portfolio Credit Derivatives," www.defaultrisk.com/pp_crdrv_95.htm

Andersen, Leif and Jakob Sidenius, 2004/2005, "Extensions to the Gaussian Copula: Random Recovery and Random Factor Loadings," *Journal of Credit Risk* **1**, 1 (Winter), 29-70.

Burtschell, X., J. Gregory and J.-P. Laurent, 2005a, "A Comparative Analysis of CDO Pricing Models," www.defaultrisk.com/pp_crdrv_71.htm

Burtschell, X., J. Gregory and J.-P. Laurent, 2005b, "Beyond the Gaussian Copula: Stochastic and Local Correlation," www.defaultrisk.com/pp_corr_76.htm

Chaplin, Geoff, 2005, *Credit Derivatives: Risk Management, Trading and Investing*, p. 205 (Wiley, Chichester).

Duffie, Darrell and Kenneth J. Singleton, 1999, "Modeling Term Structures of Defaultable Bonds," *The Review of Financial Studies*, Vol. 12, No. 4, 687-720.

Guegan, Dominique and Houdain, Julien, 2005, "Collateralized Debt Obligations Pricing and Factor Models: A New Methodology Using Normal Inverse Gaussian Distributions," www.defaultrisk.com/pp_crdrv_93.htm

Hull, John and Alan White, 2003, "The Valuation of Credit Default Swap Options," *Journal of Derivatives* **10** (3), spring, 40-50.

Hull, John and Alan White, 2004, "Valuation of a CDO and an nth to Default CDS without Monte Carlo simulation," *Journal of Derivatives* **12**(2), winter, 8-23.

Hull, John, Mirela Predescu, and Alan White, 2005, "The Valuation of Correlation-Dependent Credit Derivatives Using a Structural Model," www.defaultrisk.com/pp_crdrv_68.htm

Hull, John and Alan White, 2005, "The Perfect Copula," www.defaultrisk.com/pp_crdrv_83.

Joshi, Mark S. and Alan M. Stacey, "Intensity Gamma: A New Approach to Pricing Portfolio Credit Derivatives, 2005, www.defaultrisk.com/pp_crdrv_84.

Laurent J.-P., and Gregory, J., 2003, "Basket Default Swaps, CDOs and Factor Copulas," Working paper, ISFA Acturarial School, University of Lyon & BNP Paribas, www.defaultrisk.com

Schönbucher, Philipp J., 2004, "A Measure of Survival," Risk, August, 79.

Schönbucher, Philipp J., 2005, "Portfolio Losses and the Term Structure of Loss Transition Rates: A New Methodology for the Pricing of Portfolio Credit Derivatives," www.defaultrisk.com/pp_model_74.htm

Sidenius, Jakob, Vladimir Piterbarg, and Leif Andersen, 2005, "A New Framework for Dynamic Credit Portfolio Loss Modelling," default-risk.com/pp_model_83.htm

Walker, Michael B., 2004/2005, "Risk-Neutral Correlations in the Pricing and Hedging of Basket Credit Derivatives," *Journal of Credit Risk* **1**(1), Winter.

Walker, Michael B., 2005, "An Incomplete-Market Model for Collateralized Debt Obligations," http://www.physics.utoronto.ca/~qocmp/incompleteCDO.2005.10.05.pdf