

**CDO Valuation:
Term Structure, Tranche Structure, and Loss Distributions¹**

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Abstract

This article describes a new approach to the risk-neutral valuation of CDO tranches, based on a general specification of the loss distribution, and the expected loss at zero default, for the reference portfolio. The new approach can describe tranche term-structures, and the generality with which the basic distributions are specified allows it to be perfectly calibrated to any set of market prices (for any number of tranches and maturities) that is arbitrage-free. Also, given a set of arbitrage-free market prices, arbitrage-free interpolated term structures (plots of tranche price versus maturity for a given tranche) and tranche structures (plots of tranche price versus tranche for a given maturity), as well as implied loss distributions, can be obtained, allowing bespoke tranches to be priced. The marking to market of tranche prices, and the establishment of forward-start premiums for the index are discussed. An efficient linear programming approach to valuation, essential to the implementation, is also described. The article also makes use of a new, simple yet general, approach to the problem of unequal notionals, and of random, time-dependent, risk-neutral, recovery rates.

keywords: collateralized debt obligations, risk-neutral valuation, term structure

1 Introduction

This paper presents a new approach to the risk-neutral valuation of collateralized debt obligations (CDO's). The approach permits precise calibration to any set of market prices that is arbitrage-free. (In the example of Table 1, which gives quotes for tranches on the iTraxx Europe index, there are 24 different market quotes that are simultaneously and precisely calibrated to.) Precise calibration to all available market prices is essential for accurate marking to market of tranche prices, and for the establishment of accurate prices for forward-start contracts for protection on the index. Precise calibration is also essential for the accurate establishment of interpolated term structures (curves of market price versus maturity for a given tranche) and tranche structures (curves of market price versus tranche at a given maturity); these interpolated structures are used to value bespoke tranches on the basket. The search for a model that can be calibrated to a given set of market prices has been a central topic of recent CDO research (e.g. see Burtschell, Gregory and Laurent (2005)).

The new approach is characterized by the adoption of risk-neutral loss and default distributions for the reference portfolio (or basket) as its fundamental distributions. (The default distribution is referred to more precisely below as either the notional-averaged default distribution, or the loss distribution at zero recovery.) This results in a very simple computational structure that bypasses all of the problems that occur when one tries to construct the basket loss distribution from the loss distributions of the individual names (such as the appropriate choice of an approximate dependence structure and how to model and calibrate to what is in reality a very complex correlation structure for the individual names of the basket).

In the domain of physics, one often studies the properties of large objects (e.g. a glass of water) made up of many smaller individual components (e.g. individual H_2O molecules). An approach that tries to understand the properties of the larger object in terms of the properties of its smaller individual components is called a microscopic approach. On the other hand, an approach that uses general arguments to discover relations between the properties of the larger object, without appeal to the fact that the large object may be made up out of smaller components, is called a macroscopic approach. By analogy, the approach of this article to CDO's on baskets is a macroscopic approach, whereas copula approaches are microscopic approaches.

It should be noted that the relative valuation of tranches and the index via a risk-neutral measure, for which this article develops a macroscopic approach, is intrinsically different from the valuation based on the use of a historical measure. In the latter case, where the properties of the individual basket names, such as their historically estimated default rates and correlations, would be expected to play a central role, a microscopic approach might be favored.

The paper addresses a number of specific problems. It begins with a review of risk-neutral CDO tranche pricing. Risk-neutral tranche pricing is relative pricing, and the contracts that are priced relative to one another refer to the tranches and the index of different maturities for a given reference basket. It will

be seen that the basic formulae for these prices contain only quantities describing the basket as a whole (e.g. the total basket loss distribution and the basket default distribution). Thus, this article considers these basket distributions as the fundamental quantities, and does not try to build them up from the multivariate distribution describing the losses at an individual name level, as is most often done. There is no loss of generality in this description, and it is a significant simplification relative to a description in terms of a general multivariate default distribution for the individual names, which is difficult (if not impossible) to deal with in an accurate fashion for baskets that typically have 100 or more names. The first application discussed below is that of checking sets of iTraxx prices to see if they are arbitrage-free. Next the problem of determining the arbitrage-free bounds for a given tranche price is discussed. Tranche prices are market prices that often lie well within their arbitrage-free bounds, and thus judgement and supply and demand should have the central role in their determination. It should nevertheless be useful to market participants to know to what extent these prices are constrained by arbitrage considerations. The article then goes on to describe a method for establishing interpolated term structures (interpolation across maturities for a given tranche), tranche structures (interpolation across losses for a given maturity), as well as implied loss distributions. The technique for obtaining the arbitrage-free bounds on the price of a given tranche plays an important role in the establishment of arbitrage-free interpolation schemes. Also, because of the popularity of base correlations (McGinty et al., 2004), an analogous quantity more appropriate to our approach, the reciprocal base tranche premium, is introduced and discussed. Once an accurate risk-neutral measure has been established by calibration to all available market prices, and by interpolation, tranche prices can be accurately marked to market, as described in Section 8.

There is a developing interest in instruments that require dynamical models (see below) for their valuation, such as forward-start CDO's on tranches. The second last section of this article gives some useful results concerning forward-start contracts on the index. Forward-start contracts on the index, in contrast to those on tranches, can be valued in terms of a static model, but require that the model be accurately calibrated to all available market prices, and particularly to market prices for different maturities. The model of this article is thus ideally suited for valuing forward-start contracts on the index.

The macroscopic approach of this article has been developed in such a way that it lends itself easily to an efficient linear-programming implementation. This is extremely important as it is the key development behind the ability to calibrate to any set of market prices that is arbitrage-free. Already the popular copula approaches can not be calibrated to all currently available tranche-maturity prices, and this problem will only become more severe as the number of maturities and tranches of the standardized indices that are publicly marketed increases.

A collateralized debt obligation is a financial instrument that transfers the credit risk of a reference portfolio (or basket) of assets. The credit risk on the CDO is tranching, so that a party that buys insurance against the defaults of

a given tranche receives a payoff consisting of all losses that are greater than a certain percentage (the tranche attachment point), and less than another certain percentage (the tranche detachment point), of the notional of the reference portfolio. In return for this insurance, the protection buyer pays a premium, typically quarterly in arrears, proportional to the remaining tranche notional at the time of payment; there is also an accrued amount in the event that default occurs between two payment dates. A relatively liquid market for standardized index tranches, based on the CDX and iTraxx credit default swap (CDS) indices, has developed recently, and the detailed examples of this article will refer to these index tranches. (For an introduction to standardized CDS indices and index tranches see Amato and Gyntelberg (2005).)

In the modelling of a single defaultable bond, or a credit default swap, the risk-neutral fractional loss at default process, and the risk-neutral hazard rate process giving rise to defaults, are regarded as separate processes, each of which must be obtained by calibration to market prices (Duffie and Singleton, 1999). The counterpart of these ideas in this article is that the tranche loss distributions on the one hand, and the notional-averaged default distribution on the other, are regarded as separate distributions. The problem of calibrating to different loss processes of the individual names in the reference portfolio, which are in general random functions of time, is not an easy one within the framework of a copula model. This problem is transformed in the present article to the problem of imposing the requirement that the value of the expected loss rate for the basket can not exceed the expected loss rate given zero recovery (see the discussion following Eqs. 15).

At present, the most commonly used models for the risk-neutral pricing of collateralized debt obligations (CDO's) are copula models. For recent examples see Andersen and Sidenius (2004/2005), Guegan and Houdain (2005), Hull and White (2006), Joshi and Stacey (2006), Albrecher, Ladoucette and Schoutens (2006), and for a recent review see Burtschell et al. (2005). The approach of this article has two significant advantages over copula models. Firstly, individual obligor properties, such as obligor hazard-rate curves and risk-neutral recovery rates, do not appear, leading to increased efficiency. Secondly, in adopting a copula approximation to the multivariate individual-obligor default-time distribution, one adopts a dependence structure that is far from being the most general. As a result, much research has been devoted to trying to find a copula that can be calibrated to a number of different market prices. Effectively, calibration to the type of copula, which is a difficult process not guaranteed to be successful, becomes a part of the calibration problem. There is no such problem in this article because the most general basket loss and default distributions are used.

Two dynamical models of particular relevance to this article are Schönbucher (2005), and Sidenius et al. (2005)). Both make use of a loss distribution as a fundamental quantity, as does this article, although their loss distribution is dynamical. Both handle recovery rates differently from this article. Both articles take as given the loss distribution as seen from time $t = 0$. The present article gives a way of determining this initial loss distribution. Other approaches to

dynamical models have been given in Albanese, Chen and Dalessandro (2005), Bennani (2005), Brigo, Pallavicini and Torresetti (2006), Di Graziano and Rogers (2006) and Hull and White (2006). Dynamical models are useful for valuing certain new products that are beginning to appear on the CDO market, such as forward-start contracts and options on CDO tranches (e.g. see Andersen (2006) and Hull and White (2006)). As noted above, the static approach of this article can value one of these new products, namely forward-start contracts on the index (see Section 9 for details), but a dynamic model is required to value forward-start contracts on the tranches.

Recently, Torresetti, Brigo and Pallavicini (2006) have studied the macroscopic approach as proposed in Eqs. 15 and 16, and in the original working paper by the author (Walker, 2006), (but using constant equal recovery rates). Rather than calibrating so as to precisely reproduce the midpoint tranche prices, however, they suggest minimizing a sum of squared standardized mispricings.

2 CDO Tranche Valuation

CDO tranches can be valued if the loss distribution of the reference basket is known (e.g. see Andersen et al. (2003), Hull and White (2004), and Laurent and Gregory (2003)). The index can be valued if, in addition, the expected loss at zero default (defined below) is known. The valuation formulae developed in this section depend explicitly only on the loss distribution and on the expected loss at zero default, and, while it is in principle possible to build these distributions up from the multivariate distribution describing the detailed statistical behavior of the individual names in the basket, this will be a difficult to do accurately in practice, and will not be the approach of this article. Thus, the fundamental quantities on which all of the valuations of this article are based are the basket loss distribution, and the expected basket loss at zero default.

The loss distribution⁵ $F(\ell, t)$ is the probability that the loss associated with defaults of any of the n names in the reference basket, exceeds ℓ at time t . The total notional of the basket will be taken to be unity and will be divided into contiguous segments called tranches labelled $k = 1, 2, \dots, nTr$, where nTr is the total number of tranches. A discrete set of losses $\ell_k, k = 0, \dots, nTr$, is introduced such that tranche k is associated with losses from ℓ_{k-1} to ℓ_k ; also $\ell_0 \equiv 0$ and $\ell_{nTr} \equiv 1$. Finally, the time $t = 0$ width of tranche k is $\Delta_{k,0} = \ell_k - \ell_{k-1}$. The notation just described is suitable for an analysis of the standardized tranches of the iTraxx and CDX indices. For the iTraxx index, which is pertinent to the examples of this article, the standard sequence of ℓ_k values is 0, 0.03, 0.06, 0.09, 0.12, 0.22 and 1.0.

The random variable giving the total loss of the reference basket at time t can be written

$$L_t = \sum_{i=1}^n I(\tau_i < t) N_i (1 - R_i(t)). \quad (1)$$

⁵What is called the loss distribution in this article, is sometimes called the excess loss distribution elsewhere.

Here, the individual names in the basket are labelled by i , the total number of names is n , the notional associated with the i th name is N_i , and the default time of the i th name is τ_i . The total notional of the basket is taken to be unity so that

$$\sum_{i=1}^n N_i = 1. \quad (2)$$

The quantity $R_i(t)$ is the risk-neutral random recovery rate for name i , which is a function of time (Duffie and Singleton, 1999). Also, the quantity $I(\tau_i < t)$ is the default indicator for name i , and is unity if name i defaults before time t , and zero otherwise. The expected loss at time t , per unit initial notional, for tranche k is

$$f(k, t) = \frac{1}{\Delta_{k,0}} E[(L_t - \ell_{k-1})^+ - (L_t - \ell_k)^+] = \frac{1}{\Delta_{k,0}} \int_{\ell_{k-1}}^{\ell_k} F(\ell, t) d\ell. \quad (3)$$

The $t = 0$ present value of the expected losses occurring between times $t = 0$ and $t = T$, per unit initial tranche notional, for tranche k , is now

$$V_{loss}(k, T) = \int_0^T e^{-rt} df(k, t) \quad (4)$$

where $df(k, t) \equiv [\partial f(k, t)/\partial t]dt$.

Consider a CDO contract of maturity T which provides protection against losses associated with tranche k in return for premium payments made at times t_j , $j = 0, 1, \dots, N_p(T)$. The j -th payment is for protection for the interval $\delta_j = t_j - t_{j-1}$ ($t_{-1} \equiv 0$) and has a magnitude equal to $w(k, T)\delta_j$ times the remaining tranche notional at time t_j , where $w(k, T)$ is the annualized premium for tranche k . Typically the premium payments are made quarterly, so that $\delta_j \approx 0.25$ (depending on the day-count convention) for $j \geq 1$, but one can have $\delta_0 < 0.25$. Furthermore, if a default occurs at some time t in the interval (t_{j-1}, t_j) , an accrued payment of $w(k, T)\delta_t\phi_t$ times the default loss associated with the tranche notional must be made. Here, $\delta_t = \delta_j$ and $\phi_t = (t - t_{j-1})/\delta_j$ for t in (t_{j-1}, t_j) .

As just noted, tranche k , which has initial notional $\Delta_{k,0}$, may incur losses due to defaults as time proceeds. For $k = 1, \dots, nTr - 1$ the expected width of tranche k at time t can be evaluated as

$$\begin{aligned} \Delta_{k,t} &= E[(\ell_k - L_t)^+ - (\ell_{k-1} - L_t)^+] = \Delta_{k,0}(1 - f(k, t)), \\ k &= 1, \dots, nTr - 1. \end{aligned} \quad (5)$$

For the super-senior tranche, i.e. tranche $k = nTr$ with detachment point $\ell_{nTr} = 1$, a small correction applies to Eq. 5. This is because, when defaults occur, there is a recovery, and the maximum loss for the super-senior tranche at time t is then no longer unity, but $1 - A_t$ where

$$A_t = \sum_{i=1}^n I(\tau_i < t)R_i(t). \quad (6)$$

Taking this amortization of tranche nTr into account yields a time t width for tranche nTr of

$$\Delta_{nTr,t} = \Delta_{nTr,0}(1 - h(nTr, t)), \quad (7)$$

where

$$h(nTr, t) = \frac{1}{\Delta_{nTr,0}} [q(t) - \sum_{k=1}^{nTr-1} \Delta_{k,0} f(k, t)]. \quad (8)$$

In this last equation

$$q(t) \equiv E(L_t + A_t) = \sum_{i=1}^n N_i q_i(t) \quad (9)$$

where $q_i(t) \equiv E[I(\tau_i < t)]$ is the probability that name i defaults before time t . Thus, $q(t)$ can be interpreted either as the expected basket loss at zero recovery at time t , or as a notional averaged default distribution.

Given this result, the expected value of the premium payments, per unit initial tranche notional, can be written

$$V_{premium}(k, T) = u_f(k, T) + w(k, T) T_{eff}(k, T) \quad (10)$$

where $T_{eff}(k, T)$ is an effective time to maturity (also called a risky duration, or a DV01) given by

$$T_{eff}(k, T) = \sum_{j=0}^{N_p(T)} \delta_j [1 - h(k, t_j)] e^{-rt_j} + \int_0^T \delta_t \phi_t e^{-rt} df(k, t). \quad (11)$$

Here, $h(k, t) = f(k, t)$ for $k = 1, \dots, nTr - 1$ and $h(nTr, t)$ is given by Eq. 8. In addition to the periodic premium payment, $w(k, T)$, the possibility for a time $t = 0$ upfront payment, $u_f(k, T)$ (which may be zero) is included in Eq. 10.

The fair value of the annualized premium $w(k, T)$ for a tranche is calculated by balancing the present value of the expected losses against the expected present value of the premium payments, which gives the equation

$$V_{loss}(k, T) = V_{premium}(k, T). \quad (12)$$

Here, $V_{loss}(k, T)$ is given by Eq. 4, while $V_{premium}(k, T)$ is given by Eqs. 10 and 11.

The calculation of the present value of the expected premium payments on the index is quite different from that for the tranches, since the premium payment at a given time t is proportional to the existing notional, and independent of the losses. The fair value of the premium $s(T)$ for the index is also found by balancing the present values of the expected losses and premium payments, giving

$$\sum_{k=1}^{nTr} \Delta_{k,0} \int_0^T e^{-rt} df(k, t) = s(T) T_{eff}^I(T), \quad (13)$$

where

$$T_{eff}^I(T) = \sum_{j=0}^{N_p(T)} \delta_j [1 - q(t_j)] e^{-rt_j} + \int_0^T \delta_t \phi_t e^{-rt} dq(t) \quad (14)$$

is the risky duration for the index.

It would perhaps be possible to evaluate the premium for the index in terms of the premiums for the CDS's on the individual names in the reference portfolio. However, it is customary to regard the contract for protection on the index as a separate contract that trades on the market at its own price, and not at a price determined theoretically in terms of the market prices of individual CDS's (Chaplin, 2005). In addition to the CDO spread curves for the individual names, one should take into account some estimated contribution to the index premium resulting from the cost of forming the index out of individual names, as well as a (generally negative) contribution resulting from the index in general being more liquid than an equivalent collection of names.⁶

3 Parameterization

It is clear from the previous section that the standardized tranches and the index of a given reference basket can be priced if the expected tranche losses, $f(k, t)$, for the different tranches, and the notional-averaged default distribution, $q(t)$, are known as a functions of time. These are risk-neutral quantities and their determination must therefore be by calibration to all available market prices for derivatives on the basket. The approach of this article is to consider the most general possible form for these functions, subject only to certain necessary general constraints. For example, note that the relation $f(k, t) > f(k + 1, t)$ must hold. This follows from the fact that $f(k, t)$ is an average of the excess loss distribution $F(\ell, t)$ over the losses ℓ satisfying $\ell_{k-1} < \ell < \ell_k$ (see Eq. 3) and the fact that $F(\ell, t)$ is a decreasing function of ℓ at fixed t . Thus, the relations constraining the distributions are

$$\begin{aligned} 0 &< f(k, t) < 1, \text{ all } k; \\ f(k, t) &> f(k + 1, t), \quad k = 1, \dots, nTr - 1; \\ \frac{\partial f(k, t)}{\partial t} &> 0, \text{ all } k; \\ 0 &< q(t) < 1; \\ dq(t)/dt &> 0; \\ dE(L_t) &< dq(t). \end{aligned} \quad (15)$$

where $E(L_t) = \sum_{k=1}^{nTr} \Delta_k f(k, t)$ is the expected value of the total loss of the basket at time t . The last of the above inequalities deserves further comment. The left hand side of this inequality gives the increase in the expected loss of the basket in a small time interval dt . Since $q(t)$ can be interpreted as the

⁶I am indebted to Dan Rosen for a discussion of this point.

expected loss at time t for zero recovery, it is clear that the stated inequality must be satisfied. Because the recovery rate processes $R_i(t)$ are risk-neutral quantities constrained such that $0 < R_i(t) < 1$ (Duffie and Singleton, 1999), one has the freedom to choose them arbitrarily, subject to the last constraint of Eqs. 15, provided that all relevant market prices can be calibrated to. Often in the literature one finds that quite restrictive constraints are placed on the recovery rates. For example, they are generally taken to be constant in time, and to be non-random variables; they might also be taken to be the same for all names, or they might be taken to be equal to the historical default rates for the names (in so far as one can estimate these). The treatment of recovery rates in this article is thus considerably more general than is usually the case. Furthermore, its implementation in terms of a constraint is just as simple and straightforward as is that for the less general assumption of recovery rates that are constant in time, and are the same for all names, and are known in advance.

For the purposes of numerical implementation, the expected tranche loss functions $f(k, t)$ and the notional-averaged default distribution $q(t)$ will be taken to be continuous piecewise-linear functions of time. This piecewise-linear parameterization is somewhat unusual, but gives a perfectly acceptable risk-neutral measure. Furthermore, it is this piecewise-linear parameterization that allows the calibration problem to be solved by the use of linear programming techniques. And the use of linear programming allows efficient and precise calibration to the relative large number of tranche-maturity price combinations that are on the market for the standardized indices. Thus, consider a sequence of times $T_m, m = 0, 1, \dots, m_{Max}$ with $T_0 = 0$ and $T_{m_{Max}}$ being the maximum maturity for which CDO contracts on the tranches of the particular basket of interest are considered. The parameterized form of the distributions is defined piecewise by

$$\begin{aligned} f(k, t) &= f(k, T_{m-1}) + g(k, m)(t - T_{m-1}), \\ q(t) &= q(T_{m-1}) + \lambda(m)(t - T_{m-1}), \\ T_{m-1} &< t \leq T_m, \quad m = 1, 2, \dots, m_{Max}, \end{aligned} \tag{16}$$

with $f(k, T_0) = 0$ and $q(T_0) = 0$. The relevant distributions are thus completely determined by a knowledge of the sets $\{g\}$ of all $g(k, m)$, and $\{\lambda\}$ of all $\lambda(m)$. The smaller the time steps $T_m - T_{m-1}$, the more finely the distributions $f(k, t)$ and $q(t)$ can be specified. The importance of the discretization error can be assessed by successively reducing the size of the time steps (as in Fig. 1 of the following section).

4 Calibration and Checking for Arbitrage

The data set studied in this article is a set of iTraxx tranche prices for 97 trading days between 6 May 2005 and 19 September 2005. (The data was provided by Julien Houdain and Fortis Investments.) A typical example from this data set is the set of midpoint market quotes for the iTraxx index and for the standardized

iTraxx Tranche Quotes - 21 June 2005				
Tranche	3 year	5 year	7 year	10 year
1: 0-3%	7.3	27.38	43.8	53.25
2: 3-6%	26	91	245	455
3: 6-9%	7.4	31.5	60	123
4: 9-12%	2.4	19	30	56
5: 12-22%	2.3	12.5	19.5	35.5
0-100%	23.25	39.25	49.75	60.5

Table 1: The table shows the quotes (mid-point of bid and ask) for the standardized iTraxx tranches for 21 June, 2005. The 0 to 3% (equity) tranche is quoted as a percentage upfront payment, assuming that subsequent payments are made quarterly at a rate of 500 basis points per year. The other tranches are quoted as basis points per year, again assuming quarterly payments. The 0-100% tranche is the iTraxx index. The quotes are for CDO contracts having maturities of 3, 5, 7, and 10 years. Source: Julien Houdain and Fortis Investments.

iTraxx tranches for 21 June, 2005 shown in Table 1. This section addresses the question of whether or not a given set of prices, such as those in the example, can be reproduced by an appropriate choice of the sets of parameters $\{g\}$ and $\{\lambda\}$ defining the risk-neutral loss distribution. If this is the case, then the set of prices in question will have been shown to be theoretically arbitrage-free. If this is not the case, then these prices are not arbitrage-free, or else, given the fact that numerical work is never perfect, they are very close to being not arbitrage-free. Having established that a given set of market prices is arbitrage-free, one can then go on in following sections to other applications, such as the establishment of interpolated term structures and tranche structures that can be used to interpolate prices for bespoke tranches and maturities.

To determine whether or not a risk-neutral measure can be found that reproduces a given set of prices, consider the linear-programming problem of maximizing the quantity zero over all model parameters in the sets $\{g\}$ and $\{\lambda\}$, subject to the relevant constraints. More specifically,

$$\begin{aligned}
& \text{Maximize}_{\{g\},\{\lambda\}} 0, \text{ subject to:} \\
& \text{Eq. 12 for all tranches } k \text{ and maturities } M \\
& \quad \text{for which there are market prices,} \\
& \text{Eq. 13 for all maturities } M \text{ for which} \\
& \quad \text{there are market prices,} \\
& \text{and the constraints of Eq. 15.} \tag{17}
\end{aligned}$$

By choosing an objective function of zero for this linear-programming problem, one is simply checking to see if the constraint equations can be satisfied. The maximum maturity is 10 years and the time-step intervals $T_m - T_{m-1}$ in Eq. 16 are taken to be quarterly for the calculations of this section. Also, in this and in all other examples, a constant risk-free rate of $r = 3.5\%$ (which is a

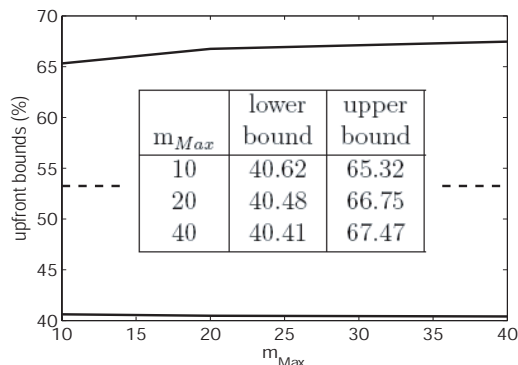


Figure 1: The solid lines in the Figure show the upper and lower bounds obtained on the arbitrage-free range of values of the upfront payment for a certain CDO contract (see text for details). The bounds are plotted for piecewise-linear interpolation schemes containing $m_{Max} = 10, 20$ and 40 steps (annual, semi-annual and quarterly time steps) in a period of 10 years. The Table shows the same values that are plotted, but with greater accuracy. The dashed line shows the market price (not used as input data) which is clearly inside the bounds, and thus arbitrage-free.

rough average of the interest rate swaps curve for euros) is assumed. This linear programming problem has a solution for the iTraxx market prices listed in Table 1, indicating that this set of tranche prices is arbitrage-free.

In a similar manner, an attempt was made to calibrate the model to the iTraxx midpoint quotes for each of the 97 trading days between 6 May 2005 and 19 September 2005. Calibration was successful on all but 8 of these 97 days. On each of these 8 days it was possible to obtain an arbitrage-free set of prices by replacing selected midpoint quotes by other values within the bid-ask range.

Both Eq. 13 and the final constraint of Eqs. 15 contain all tranches as well as the index. It is thus clearly necessary to consider a full set of all available market prices (for all tranches and for all maturities) when checking to ensure that the prices are theoretically consistent, and when calibrating any model. One can not, for example, just consider all tranches at a single maturity, or all maturities for a single tranche.

5 Arbitrage-Free Bounds on Tranche Prices

This section describes how to determine the arbitrage-free bounds on a tranche price. This procedure has at least two uses. Firstly, it could be of use to a buyer of a tranche to know what constraints the no-arbitrage principle puts on the price, particularly if the tranche is not very liquid. In this case, even if a market maker has listed a bid-ask spread, it would be of interest to know to

what extent the listed bid-ask spread is imposed by no-arbitrage considerations, and hence what freedom there is for a negotiation to determine a different price. It turns out that, in the example considered here, the quoted market price is well within the the arbitrage-free bounds, which tells the tranche buyer that the price is a market- (i.e. negotiation-) determined quantity, and that model predictions should not affect this price. Secondly, the procedure for determining the arbitrage-free price bounds on a tranche is useful in the establishment of arbitrage-free interpolated term structures and tranche structures. How this works will be demonstrated in detail in section 6.

As an example of the the determination of the arbitrage-free price bounds on a tranche, consider the iTraxx equity tranche with the 10-year maturity. Suppose that this determination is carried out on 21 June 2005 and that the tranche price quotes are as given in Table 1. The price of the unmarketed tranche in question, $u_f(k_0, T_0)$ for $k_0 = 1$, $T_0 = 10$ years, is determined as a linear function of the model parameters by Eq. 12. The bounds on the arbitrage-free range of values available to $u_f(k_0, T_0)$ can thus be determined by a linear-programming optimization procedure similar to that of the preceding Section (i.e. by using Eq. 17). However, the objective function must be replaced by $u_f(k_0, T_0)$, given as a function of the model parameters. Also, the tranche and index price constraints imposed through Eqs. 12 and 13 include all prices listed in Table 1 except for those corresponding to the 10-year equity tranche (i.e. a total of 23 prices).

To assess the accuracy of the piecewise-linear method of representing the functions $f(k, t)$ and $q(t)$, the bounds are obtained by using the three different values, 10, 20, and 40, for m_{Max} of Eq. 16. This corresponds to using yearly, semiannual, and quarterly time steps, respectively. The results are shown in Fig. 1. For any given value of m_{Max} , the bounds obtained give a price range within which all prices are arbitrage-free. As the value of m_{Max} is increased, the bounds expand somewhat, as expected, since one is sampling more finely determined risk-neutral measures. However, the changes in the bounds are not large, and the value of $m_{Max} = 40$ would appear to give a reasonably accurate determination of the bounds, at least for the purposes of the discussions of this article.

6 Interpolated Term Structures, Tranche Structures and Loss Distributions

The first job of this section is to establish smooth term structure curves for each of the standardized tranches and for the index. In Table 1, prices for only the 3, 5, 7 and 10 year maturities are given, and the objective is to propose reasonable prices for contracts having maturities in between the four standard maturities. The procedure adopted here is to use cubic spline interpolation to establish a smooth interpolation of prices from the market prices for maturities 3, 5, 7 and 10 years, to the maturities 4, 6, 8, and 9 years, and then to extrapolate linearly

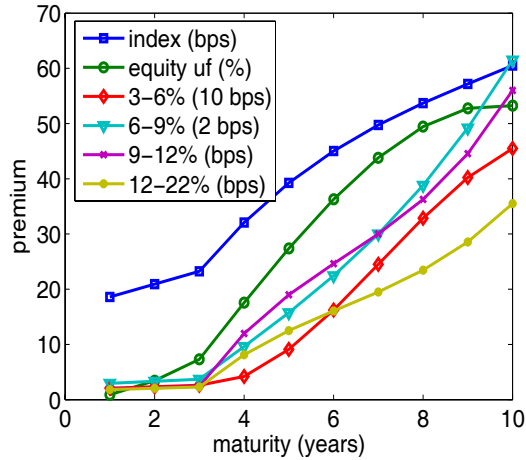


Figure 2: The term structures obtained by interpolation and extrapolation from the iTraxx quotes of Table 1. Note the units given in the legend: for example, the 10-year premium for the 3–6% tranche is approximately $45 \times (10 \text{ bps}) \approx 455 \text{ bps}$, in agreement with the value quoted in Table 1. The set of premiums indicated by the markers has been checked by the method section 4 and has been found to be free of arbitrage.

to maturities 1 and 2 years. While the interpolation should yield reasonable, although not unique, suggested prices, the extrapolation below 3 years is more problematic as there is no market price at very small maturities to serve as a guide. This new interpolated price set that has just been arrived at is not necessarily acceptable because it is not necessarily arbitrage free. Therefore, each of the new target prices is examined in turn to see if it lies within its arbitrage-free bounds (using the technique of Section 5), and if not, its price is adjusted to a new one that is within its arbitrage-free bounds, and (if possible) is close to the target price. A final check of the prices established for all tranches and the index, for all annual maturities, is then made using the method of Section 4 to ensure freedom of the total (market plus interpolated and extrapolated) price set from arbitrage opportunities. The term structures thus obtained for each of the standardized tranches, and for the index, are shown in Fig. 2.

The above procedure has established an interpolated term structure for each tranche, i.e. for each tranche the price is given as a function of maturity. In doing this, values for the expected tranche losses $f(k, T)$ for $k = 1, 2, 3, 4, 5$ and $T = 1, 2, \dots, 10$ were determined. What will be done now, is to establish an interpolated tranche structure, i.e. for each maturity, an interpolated price as a function of tranche number will be established. To do this, consider the set of $(nTr-1)$ “shifted” tranches [1.5–4.5%, 4.5–7.5%, 7.5–10.5%, 10.5–17%, 17–61%]. Note that each of the shifted tranches lies in between two of the standard tranches, and thus would be expected to have a premium which is bracketed

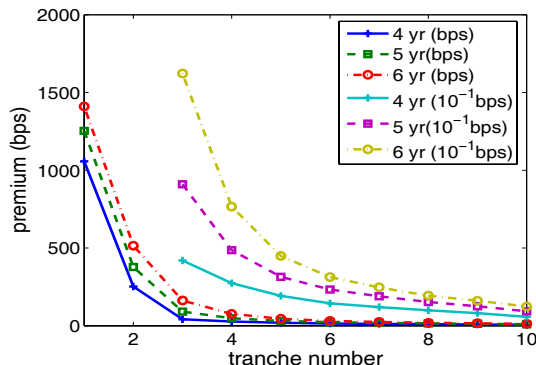


Figure 3: The interpolated tranche structure for maturities 4, 5 and 6 years. The tranches [0-3%, 1.5-4.5%, 3-6%, 4.5-7.5%, 6-9%, 7.5-10.5%, 9-12%, 10.5-17%, 12-22%, 17-61%] are labelled 1 to 10, respectively, in the figure. The markers indicate the prices to which the model has been calibrated.

by the premiums of its two neighboring tranches. For example, the premium of the shifted 1.5 – 4.5% tranche should be chosen to lie in between the premiums for the standard 0 – 3% and 3 – 6% tranches. To carry out the details of this procedure, the full loss distribution $F(\ell, T)$ is parameterized at a discrete set of losses for each annual maturity T . More precisely, $F(\ell, T)$ is assumed to be known on a grid of ℓ and T values, where the maturities T are in the set [1 2 3 4 5 6 7 8 9 10] years, and the loss values of ℓ are in the set [0 0.015 0.03 0.045 0.06 0.075 0.09 0.105 0.12 0.17 0.22 0.61 1.0]. This set of loss values contains all attachment and detachment point of the standardized and shifted tranches. The function $F(\ell, T)$ is now defined on a grid of 130 values. These values of $F(\ell, T)$ are constrained by the fact that they must reproduced the values of $f(k, T)$ identified at the beginning of this paragraph. The linear programming approach used to determine the interpolated term structure above, can also be used here to determine an interpolated tranche structure by fixing the prices of the shifted tranches. Instead of fixing target prices for the shifted tranches beforehand, however, each of the shifted tranche is examined in turn, and its price is fixed to be in the middle of its arbitrage-free range. This gives a smooth variation of the tranche prices versus tranche number for a given maturity, as can be seen in Fig. 3. A plot of premium versus tranche number for a given maturity, such as this one, will be called a tranche structure, by analogy with the phrase term structure.

Note that there are now a total of 10 tranches, one index, and 10 maturities, for a total of 110 distinct prices. The model has been calibrated so as to perfectly reproduce all of the 110 prices, and this set of 110 prices has been shown to be arbitrage-free.

The loss distribution $F(\ell, t)$ obtained from these calculations is plotted in Fig. 4. A knowledge of $F(\ell, t)$ allows the premiums for bespoke tranches

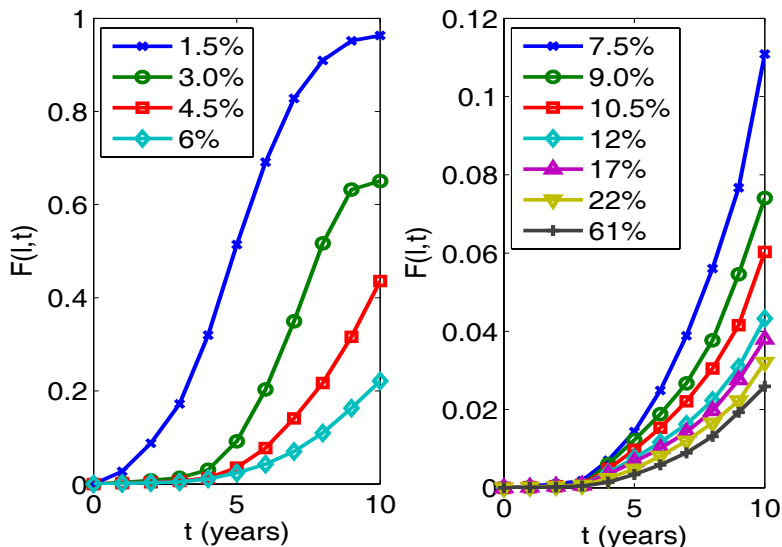


Figure 4: The loss distribution as a function of time t at selected values of the loss ℓ , as indicated in the legends.

(tranches with nonstandard attachment and detachment points, and non-standard maturities) on the same basket, to be obtained. Furthermore, the loss distribution as seen from time $t = 0$, i.e. the loss distribution determined here, is an important initial input for the dynamical models being developed in Schönbucher (2005) and Sidenius et al. (2005).

It is also of interest to investigate the last of the constraints of Eq. 15. The expected loss of the index at time t , $E(L_t)$, and the expected loss given zero recovery, $q(t)$, are plotted in Fig. 5. The inequality $E(L_t) \leq q(t)$ is clearly satisfied at all $t > 0$. (A close look at numerical values is necessary to confirm this for small t .) Also, at each time t the slope of the $q(t)$ plot is greater than that of the $E(L_t)$ plot, as is required. It is convenient to define a time-dependent effective risk-neutral recovery rate for the basket, $R_{eff}(t)$, by $dE(L_t) = (1 - R_{eff}(t))dq(t)$. The time-dependent effective recovery rate as calculated from this expression is also shown in Fig. 5. (The effective risk-neutral recovery rate, $R_{eff}(t)$ should not, of course, be confused with any analogous historical quantity.)

Currently, base correlations (McGinty et al., 2004) are popular. Base correlations are associated with base tranches (tranches with attachment points of zero). The base correlation curve for a given maturity can be viewed as depending on the base tranche premium (i.e. the premium at which the base tranche would be sold if it were on the market). An analogue of the base correlation in the present approach is the reciprocal of the base tranche premium. Given the loss distribution $F(\ell, T)$ established in the determination of the interpolated term and tranche structures, the base tranche premiums are easily calculated.

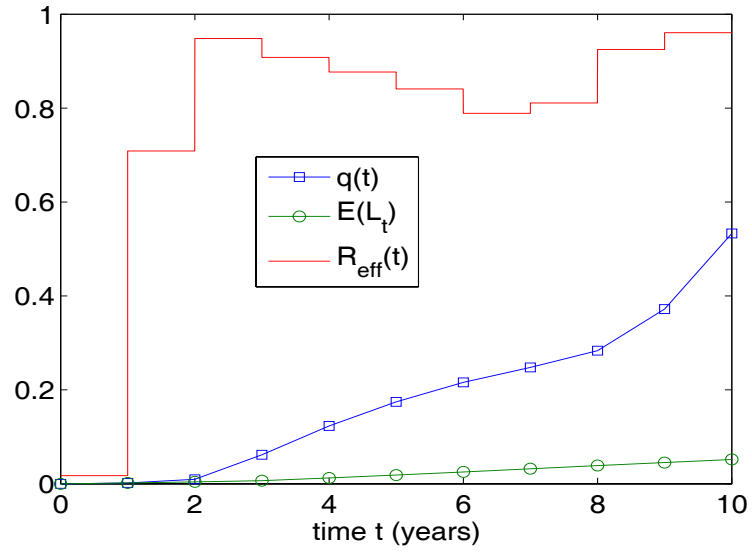


Figure 5: The expected loss of the index $E(L_t)$, the expected loss at zero recovery $q(t)$, and the effective recovery rate, $R_{\text{eff}}(t)$, are plotted as functions of time t .

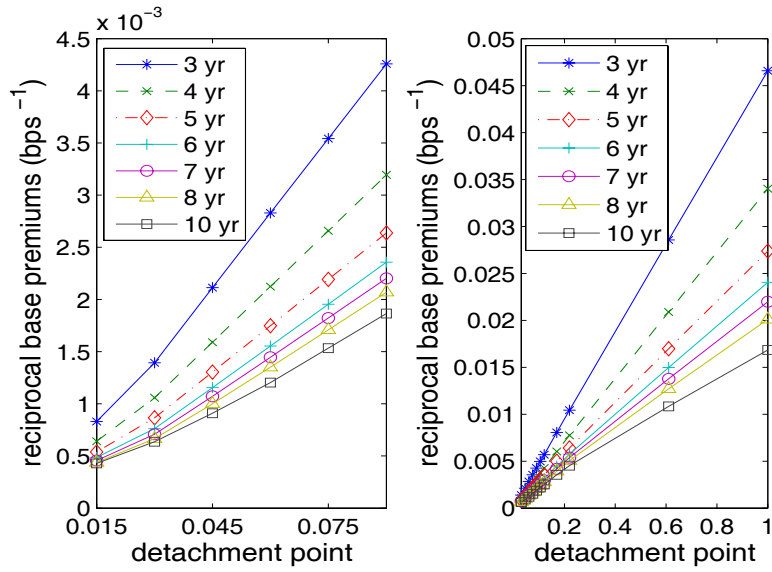


Figure 6: The reciprocal base-tranche premiums versus detachment point.

iTraxx Tranche Quotes - 10 November 2006			
Tranche	5 year	7 year	10 year
1: 0-3%	12.375	26.875	41.125
2: 3-6%	53	129	341.5
3: 6-9%	15.25	37.5	99
4: 9-12%	6	19	40.25
5: 12-22%	2.5	6.625	13.5
6: 22-100%	0.75	1.4	2.45
0-100%	24	32	42

Table 2: The table shows the quotes (mid-point of bid and ask) for the standardized iTraxx tranches for 10 November 2006. The 0 to 3% (equity) tranche is quoted as a percentage upfront payment, assuming that subsequent payments are made quarterly at a rate of 500 basis points per year. The other tranches are quoted as basis points per year, again assuming quarterly payments. The 0-100% tranche is the iTraxx index. In contrast to the data from Table 1, the data source contained quotes for the super-senior (22-100%) tranches, but did not contain 3-year quotes. Source: Julien Houdain and Fortis Investments.

In Fig. 6, the reciprocal base tranche premiums are plotted versus base tranche detachment point. Note the approximate linearity of the curves all the way up to detachment points of 1.0 (corresponding to the index). The base tranche premiums obtained in this way are consistent with each other not only within a given maturity, but also across all maturities. One of the uses of the base correlation approach has been to smoothly interpolate from standard detachment points to non-standard detachment points. In the present case it is of interest that there is a smooth curve joining base tranche premiums corresponding to detachment points of shifted tranches, to those of standardized tranches. This is another demonstration of the smoothness of the interpolation from standard tranches to shifted tranches.

7 Inclusion of Super-Senior Tranches

The example discussed in Section 6 made use of the quotes in Table 1, which did not contain quotes for the super-senior (22-100%) tranches. Here a similar analysis is carried out for the relatively recent data of Table 2, which does contain quotes for the super-senior tranches. There are thus now 7 quotes for each quoted maturity (6 tranche quotes plus the index). It might be thought that, since the expected loss for all 6 tranches is equal to the expected loss of the index, that it is superfluous to consider the quotes for tranche 6 if quotes for the first 5 tranches and the index are already considered. However, this is not the case. There are $nTr = 6$ tranche quotes plus the index for each maturity and, correspondingly, there are $nTr = 6$ expected tranche loss functions of time, $f(k, t)$, $k = 1, \dots, nTr$ plus the expected loss at zero recovery, $q(t)$ to

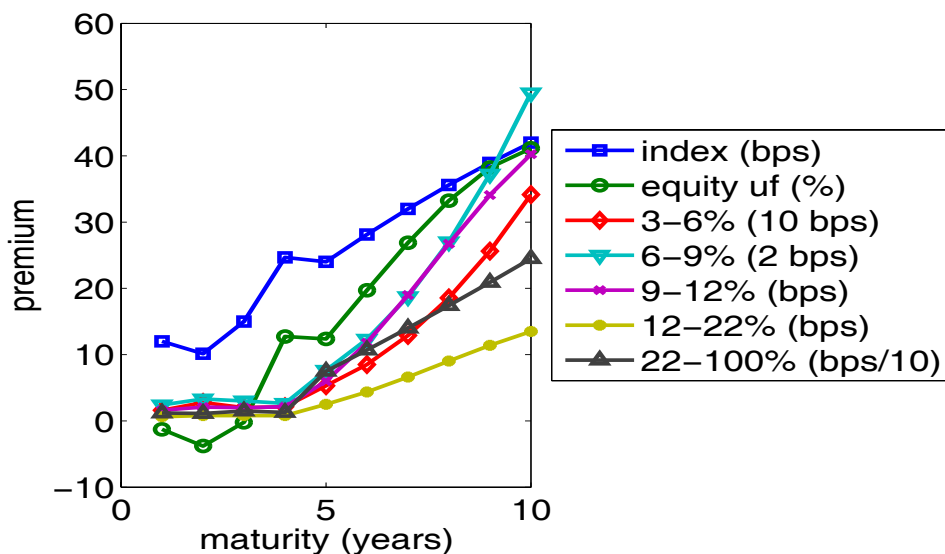


Figure 7: The term structures obtained by interpolation and extrapolation from the data of Table 2.

be fitted to, so there is enough freedom to fit to $nTr = 6$ tranche prices plus an independent market price for the index. This argument depends critically on the approach introduced previously to allow for time-dependent risk-neutral recovery rates. The constraints are now more rigorous, however, and it will be seen that this makes it more difficult to obtain smooth interpolated term structures.

As a first step, the procedure of Section 4 has been applied and has demonstrated that the quotes of Table 2 are arbitrage-free. One can also obtain more complete term structures (interpolated and extrapolated to fixed values of premiums at all annual maturities, for example), following the approach of Section 6. However, it is not easy to obtain interpolated term-structure curves that are smooth, and arbitrage-free, and precisely reproduce the market prices of Table 2. Usually one likes smooth interpolated curves because one believes that if tranches were on the market at the interpolated maturities, supply and demand in a liquid market would in general smooth out the term structure curves. However, supply and demand for the marketed maturities does not necessarily pay attention to what the potential arbitrage-free prices for the unmarketed maturities would be, and so there are no obvious market forces that act to smooth out the interpolated curves. Also, when one has quotes from a complete set of tranches as well as the index (7 quotes for a given maturity), the freedom one has to choose the prices for the unmarketed maturities is significantly reduced. A set of interpolated and extrapolated term structure curves that precisely reproduce the market prices quoted in Table 2, and that has been shown to be

arbitrage-free, is shown in Fig. 7. It can be seen that the term structure curves are largely reasonably smooth, but there are some bumps.

Finally, it should be noted that, in contrast to the term-structure curves, there is no difficulty in obtaining smooth tranche structure curves. The tranche structure curves obtained for the market quotes of Table 2 are just as smooth as the corresponding curves from Section 6, i.e. Figs. 3 and 6.

One might be able to obtain smoother interpolated term structures by relaxing somewhat the requirement that the interpolated term structures exactly reproduce the midpoint quotes, as in Torresetti et al. (2006), for example. It would appear that at present Torresetti et al. (2006) consider a maximum of only 6 quotes for each maturity [6 quotes = (5 tranches + 1 index) or 6 quotes = (6 tranches + 0 index)]. Smoothness is easier to obtain in this case, as is demonstrated in Section 6.

8 Marking Tranches to Market

Consider an investor that has sold protection on a CDO tranche to a dealer, and suppose that the investor wishes to exit this position. The investor then buys protection on the same tranche from the dealer. The loss payments in the event of default cancel each other for these two positions. In the case of an equity tranche, the investor will have to pay the upfront payment on buying protection, and after this, the premium payments (which are 500 basis points in each case) will cancel each other, so the original position has been cancelled out. The mark-to-market value for an equity tranche is thus simply the upfront payment. In the case of a tranche k , with $k = 2, \dots, nTr$, the premium for the new contract $w(k, M)$ may be different from the premium $w_{old}(k, M)$ agreed to for the original contract. The present value, per unit tranche notional, of the difference of the remaining premium payments (which is the mark-to-market value of the tranche) can be calculated as

$$PV = [w(k, M) - w_{old}(k, M)]T_{eff}(k, M), \quad (18)$$

where $T_{eff}(k, M)$, an effective time to maturity, is given by Eq. 11. This is the mark-to-market value of a tranche k for $k = 2, \dots, nTr$.

It is clear that the tranche market is far from being complete (since tranches are marketed at only a few maturities, for example). This means that the constraints of no arbitrage can only impose a range of values on certain quantities, and can not impose a definite value. This is the case for the quantity T_{eff} , which gives the cost of exiting a tranche position. Because T_{eff} is linear in the quantities $f(k, t)$ and $q(t)$, it can replace the quantity 0 as the objective function in the linear programming problem of Eq. 17. This allows one to calculate the arbitrage-free range of values allowed for T_{eff} . To avoid arbitrage opportunities, the negotiated exit price for a given contract should lie in the arbitrage-free range.

As an example, suppose that the investor had sold protection on a standardized iTraxx 10-year maturity contract on 5 April 2005, and that on 21 June 2005

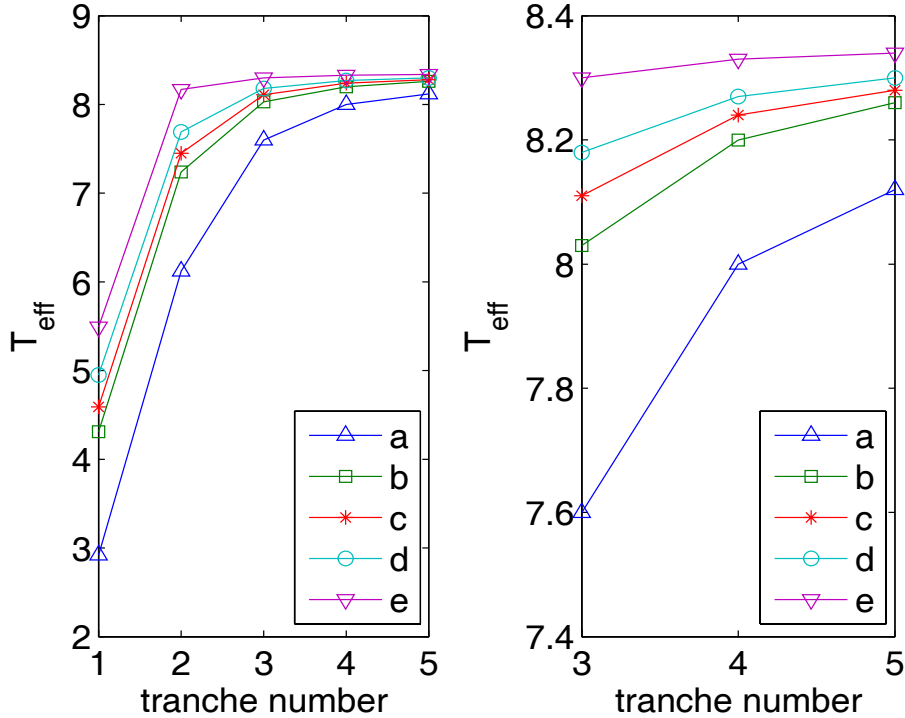


Figure 8: The bounds on the arbitrage-free range of values for the quantity T_{eff} calculated from Eq. 11 for the standardized iTraxx 10-year maturity tranches $k = 1, \dots, 5$; here k is called the tranche number. The input data are the market quotes of Table 1. The upper and lower curves (a and e) are the arbitrage-free bounds obtained assuming a risk-neutral measure constrained by only the market quotes for the 10-year maturity in Table 1. Curves b and d are similar, except all prices in the Table are used. Curve c represents the value of T_{eff} determined using the loss distribution obtained from calibration to the interpolated term and tranche structure curves. The quantity T_{eff} is smaller for lower tranche numbers because the possibility of more defaults makes the expected duration of the payments smaller.

the investor wanted to buy protection from a dealer so as to cancel the original contract. Suppose first of all that we only have the ability to calibrate to prices of a single maturity at a time. Then we would find the minimum and maximum allowed values of T_{eff} , subject to the constraint the all 10-year maturity tranche prices (and only the 10-year maturity tranche prices) in Table 1 are satisfied. These results are given by the upper and lower curves in Fig. 8. On the other hand, the ability to calibrate to all of the market prices in Table 1 gives the curves that are second from the top, and second from the bottom, in Fig. 8. The

additional constraints imposed by calibration to the market prices of all maturities significantly reduces the arbitrage-free range and hence significantly reduces the uncertainty with which T_{eff} can be determined. Finally, if one makes use of the loss distribution $F(\ell, t)$ determined above from the interpolated term and tranche structures, one finds the middle curve, which is centrally located in the arbitrage-free range, in agreement with ones expectation. The determination of this latter curve depends one ones ability to calibrate to all 110 prices of the interpolated term and tranche structures.

9 Forward-Start Contracts on the Index

This section describes some results of a simple approach to the valuation of a relatively new product on the credit derivatives market, namely the forward-start contract providing protection on a CDS index. While the valuation of forward-start CDO's on tranches requires a dynamical model, forward-start contracts on the index can be valued in terms of the static model of the present article. For these forward-starting contracts, it is essential that the risk-neutral measure be calibrated to the market prices of ordinary contracts (i.e. those starting at $t = 0$) of all maturities. Thus, the approach of this article seems ideally suited for the valuation of forward-start contracts on the index.

Consider a forward-start contract on the index that provides protection for all losses occurring after some start time T^* and before the time of maturity T . At some time t in the interval $T^* < t < T$, the cumulative losses relevant to the contract are $L_t - L_{T^*}$, where L_t is given by Eq. 1. It is the fact that these cumulative losses for the index (in contrast to the cumulative losses for a tranche - e.g. see Eq. 3) are linear in L_t and L_{T^*} that allows the forward-start contract on the index to be valued using a static model. The fair premium for this contract, $s(T^*, T)$, is found by balancing the present value of the expected losses occurring in the time interval (T^*, T) against the present value of the expected premium payments made in this time interval. The fair premium is thus determined by Eqs. 13 and 14, except that the integrals are taken over the range (T^*, T) , rather than $(0, T)$, and the sum over the premium payments is taken over payment times t_j satisfying $T^* < t_j \leq T$. Thus, this leads to the equations

$$\begin{aligned} T_{eff}^I(T^*) + T_{eff}^{*I}(T^*, T) &= T_{eff}^*(T), \\ s(T^*)T_{eff}^I(T^*) + s(T^*, T)T_{eff}^{*I}(T^*, T) &= s(T)T_{eff}^I(T), \end{aligned} \quad (19)$$

determining the forward-start premium $s(T^*, T)$ for the index and the corresponding risky duration $T_{eff}^{*I}(T^*, T)$. It follows from these equations that

$$s(T^*, T) = \frac{s(T)T_{eff}^I(T) - s(T^*)T_{eff}^I(T^*)}{T_{eff}^I(T) - T_{eff}^I(T^*)}. \quad (20)$$

This simple equation gives the forward-start premiums entirely in terms of quantities describing the ordinary (start-time $t = 0$) contracts on the index. A cor-

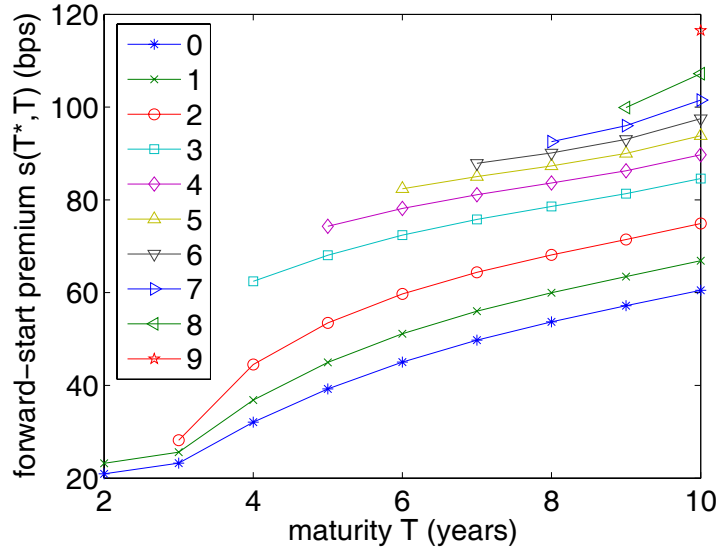


Figure 9: The premiums for forward-start contracts on the index, $s(T^*, T)$, are plotted as a function of maturity T . The different curves correspond to the different values of the start time T^* indicated (in years) in the legend. The lowest curve is for start time $T^* = 0$, and as the start time increases the curves shift progressively upwards.

responding result holds for forward credit default swaps. Eq. 20 can be used with any approach that can produce reliable values for the premiums and the risky durations of the start time $t = 0$ contracts offering protection on the index. However, it is clear that all quantities should be calculated using the same risk-neutral measure.

The risk-neutral measure that has been established in Section 6 will be used here to evaluate Eq. 20. In the example of that section, the risk-neutral measure was determined in such a way that all market prices in Table 1 were perfectly reproduced, and in addition, a set of non-marketed premiums interpolating smoothly between the marketed premiums was found. The full set of premiums reproduced by this risk-neutral measure is shown in Fig. 2. The results of using this risk-neutral measure to evaluate Eq. 20 are shown in Fig. 9.

Each of the curves in Fig. 9 shows the forward-start premium for a given start time T^* as a function of maturity T ($T > T^*$). Note that as the start time T^* increases, the curves for this example shift upwards. This can be understood by first considering the fact that the present value of the expected losses for a forward-start contract from time T^* to $T^* + 1$ plus those for a forward contract from $T^* + 1$ to T is equal to the present value of the expected losses for a contract from T^* to T . This gives a set of equations similar to Eqs. 19. From

these equations one finds the result that, for $T > T^* + 1$,

$$s(T^* + 1, T) - s(T^*, T) = \frac{T_{eff}^I(T^*, T^* + 1)[s(T^*, T) - s(T^*, T^* + 1)]}{T_{eff}^I(T^*, T) - T_{eff}^I(T^*, T^* + 1)}. \quad (21)$$

This equation shows that the curve of $s(T^* + 1, T)$ versus T is shifted upwards relative to that of $s(T^*, T)$ versus T , provided that $s(T^*, T)$ is an increasing function of T , as is the case in the examples of Fig. 9. More precisely, one notes that the shift $s(T^* + 1, T) - s(T^*, T)$ has the sign of the quantity $s(T^*, T) - s(T^*, T^* + 1)$, where $T > T^* + 1$.

As noted above, the market for ordinary (start time $t = 0$) contracts on the index is not complete, and many interpolated prices were used to tie down the risk-neutral measure that gave rise to Fig. 2. If a liquid market in forward contracts on the index were established, the market prices of these contracts would help to complete the market, and the procedures of this article would allow these prices to be used in the calibration process.

10 Conclusions

This article has described a new approach to the valuation of CDO tranches. This new approach is a macroscopic approach that has as its basis a general specification of the relevant risk-neutral measure, and does not appeal to the properties of the individual names making up the reference portfolio. This approach allows precise calibration of the risk-neutral measure to any set of tranche prices on a given reference portfolio that is arbitrage-free, thus realizing one of the main objectives of recent CDO research. The approach also determines arbitrage-free interpolated term structures and tranche structures, as well as implied loss distributions, and hence can estimate reasonable interpolated prices for bespoke tranches on the basket, can calculate accurate mark-to-market values for tranches and the index, and can calculate appropriate premiums for forward-start contracts on the index. Accurate calibration to all available market prices is essential for these applications.

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