Arbitrage-Free Price Ranges for nth-to-Default Swaps

Michael B. Walker[1 2]

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Abstract

The arbitrage-free range of values of the loss leg of an nth-to-default swap, and the arbitrage-free range of premium payments for such a swap, are derived for homogeneous baskets of arbitrary numbers of reference entities. Elementary arbitrage arguments are given which show that arbitrage opportunities exist if the prices lie outside of the bounds, and analyses of both a discrete-time model and a continuous-time model show that all prices within the bounds are arbitrage-free.

[1]Department of Physics, University of Toronto, Toronto, ON M5S 1A7, CANADA; email: walker@physics.utoronto.ca; telephone: (416) 978-3821
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1 Introduction

The lack of arbitrage opportunities in an efficient market places constraints on the possible prices of derivative securities. For example, the upper and lower bounds on the prices of European put and call options on a stock are a standard part of textbook discussions of these instruments (e.g. see Hull (2003)). If the option price is above the upper bound or below the lower bound, arbitrage traders can make a riskless profit.

As far as I am aware, there are very few results for arbitrage-related bounds on the prices of basket credit derivatives. Schönbucher (2003) has derived bounds for the value of the loss leg of a first-to-default swap for a basket of three reference entities and comments that even for such a small basket, the problem is not trivial. Some additional results for baskets of two and three reference entities have recently been presented in Walker (2004).

This article derives arbitrage-determined price bounds for nth-to-default swaps (ntD's). An ntD is a contract written with respect to a basket of a definite number, say N, of specified loans. Only homogeneous baskets are considered, which means that the loans in the basket have identical characteristics (e.g. notional, recovery rate, initial price) except that they are made to different obligors which can default at different times, if they do default. The buyer of an ntD contract (i.e. of ntD protection) receives the value of the loss (the notional less the recovery) at the time of the nth default, if there are at least n defaults before the expiry date of the contract. In return for this protection, the buyer of the contract pays a fixed premium at regular intervals until the nth default occurs, or until the expiry of the contract. The price bounds derived below are both for the premium rate and for the value of the losses received by the buyer of the contract. These price bounds are derived for baskets of arbitrary size N, and for all ntD's for a given N.

It is assumed that there is a liquid market for bonds and/or credit default swaps (CDS's) for each of the reference entities in basket. Section 2 establishes an upper limit for the value of an ntD by a considering reference portfolios made up of a risk-free bond and CDS's (same expiry date as the ntD) on the loans in the basket, such that the portfolio pays at least as much as the ntD in all possible outcomes. The price of this portfolio will the be an upper bound for the value of the losses paid by the ntD. A similar procedure establishes a lower bound. The lower and upper bounds, which will be derived below, are given, respectively, by

$$\begin{aligned} V_{ntD}^{min} &= \max\left[\frac{NV_{CDS} - (n-1)D(1-\rho)}{N-n+1}, 0\right], \\ V_{ntD}^{max} &= \min\left[\frac{N}{n}V_{CDS}, D(1-\rho)\right]. \end{aligned} \tag{1}$$

Here, V_{CDS} is the price of a CDS on one of the loans, and $D(1-\rho)$ is the initial risk-free investment necessary to pay off a CDS or ntD in case of the relevant default. (D is the price of a risk-free investment paying unity at the expiry date and $1-\rho$ is the loss on default, and the payoff is assumed to take place at the expiry date.) Since the reference portfolios referred to here are established by guesswork, additional work is necessary to demonstrate that there are not other portfolios which might give better bounds.

Section 3 studies the bounds in terms of a risk-neutral pricing procedure for a discrete-time one-period model for basket credit derivatives introduced in Walker (2004). The model is a simple one, with each reference entity in the basket being characterized by a variable indicating whether or not it has defaulted by the expiry date of the credit derivative. In this previous work, the risk-neutral probability measure was parameterized in terms of default-correlation parameters, since a central focus was to exhibit the risk-neutral nature of these correlations. A consequence of the risk-neutral nature of the correlations, and of the incompleteness of the model, is the fact that the price of a given basket derivative is not precisely determined by risk-neutral pricing arguments: in fact, there is a range of arbitrage-free prices for buyers and sellers in the market to choose from. Thus, although the arbitrage mechanism operates to keep the ntD price within the prescribed bounds, there is no mechanism driving the price to any particular value within the bounds. Both in principle and in practice, it is the preferences of the buyers and sellers, and their interaction in the market place, that determines at which of the prices within the bounds the ntD is eventually sold.

The results of Walker (2004) were somewhat limited. Only the first-to-default swap was analyzed in detail, and for this, since only a discrete-time model was used, only the values of the loss leg, and not for the premium rate, were obtained. Furthermore, detailed results were obtained only for baskets of two and three reference entities. The principal reason for these limitations is that it is not easy to find general results using the default-correlation parameterization for the risk-neutral probability measure which was employed in that article.

In this article, a different parameterization of the risk-neutral probability measure is introduced, which is more convenient for the problem of studying the arbitrage-determined bounds for baskets of an arbitrary number N of loans. (Although this new parameterization does not explicitly identify correlation parameters, it is important to recognize that correlations of all orders, from correlations between two reference entities up to correlations of N entities for a basket of N reference entities, are taken into account.) The end result of section 3 is the determination of the values for the loss leg of an ntD that are consistent with risk-neutral pricing, i.e. that do not allow arbitrage. It turns out that all values between the bounds established by elementary arbitrage considerations in section 2 are arbitrage-free. The bounds established in section 2 are thus the lowest upper bound and the greatest lower bound of the range of arbitrage-free prices of the loss leg of an ntD.

Section 4 describes a continuous-time model closely related to the discrete-time model studied in section 3. There are corrections to the value of the loss leg calculated in the discrete-time model resulting from the fact that the payoff occurs at default, rather than at the expiry date. A more important aspect of the continuous-time formulation is that it allows the premium payments, made periodically throughout the life of the swap, to be evaluated. Thus this section is able to establish upper and lower bounds for the break-even premium rate for an ntD in terms of the credit default swap spread. Very roughly, the upper and lower bounds on the break-even rate of a homogeneous ntD are

$$w_n^{max} \approx Ns/n, \quad w_n^{min} \leq s \tag{2}$$

where s is the credit default swap spread. For $n = 1$, $w_1^{min} = s$, whereas for $n > 1$, $w_n^{min} < s$ and can be zero. Thus, risk neutral pricing determines that the price of the first-to-default swap, w_1, must lie in the range

$$s < w_1 < Ns \tag{3}$$

(but does not indicate a preferred value of w_1 within this range). (See Eqs. 32 and 34 for more accurate, but still relatively transparent, results.)

2 Elementary arbitrage arguments

This section develops the constraints imposed by no-arbitrage conditions on the initial (time $t = 0$) values V_{ntD} of an ntD on a homogeneous basket of N risky loans, each with a notional of one dollar. It is assumed that there is a market for credit default swaps (CDS's) on each of the loans, and that the time $t = 0$ prices of the CDS's having the same expiry date (time $t = T$) as the proposed ntD's are the same for each reference entity (the assumption of homogeneity) and are called V_{CDS}. In addition, a risk-free zero-coupon bond which pays one dollar at time T and has time $t = 0$ value D (the discount factor) is assumed to be available. On the other hand, it is assumed that there is no market for ntD's or CDO's on the basket, or on any subbasket (a subbasket contains a subset of the loans in the basket) of the basket. If there were such a market (for example, for the first-to-default swap) there would then be additional assets that one could use in attempting to develop an arbitrage strategy, and the arguments of this section would have to be modified appropriately.

The ntD on the basket is assumed to pay, at the expiry date, the loss $1-\rho$ on the nth loan to default, if such an event occurs before the expiry date. (A fixed recovery rate, ρ, the same for all loans, is assumed.) Clearly the time $t = 0$ value of an ntD can not be greater than $D(1-\rho)$, the initial risk-free investment required to pay off the ntD should there be n defaults. Indeed, suppose that we do have $V_{ntD} > D(1-\rho)$. Then an astute investor would sell an ntD for V_{ntD}, put $D(1-\rho)$ into the risk-free bond, and pocket the difference. The investment in the risk-free bond is sufficient to pay off the the ntD at the expiry date, should this be required, so the investor has made an initial profit with no risk, which is not allowed in an efficient market. Thus V_{ntD} must satisfy $V_{ntD} \leq D(1-\rho)$.

Another upper bound on the price of an ntD can be found by considering the case $V_{ntD} > V_{CDS}N/n$. If this is the case, then adopt the following strategy. At time $t = 0$

- sell an ntD for V_{ntD},
- buy the fraction 1/n of a CDS on each of the N names in the basket for a total cost of $V_{CDS}N/n$,
- pocket the difference, which is positive.

At time $t = T$, assuming that n' defaults have occured,

- if $n' < n$, there is a profit from the payoff $(1-\rho)n'/n$ received from the CDS's on the n' defaulted loans,
- if $n' \geq n$ there is a nonnnegative profit from the payoff of $(1-\rho)n'/n$ from the CDS's on the n' defaulted loans less the payment of$(1-\rho)$ for the ntD.

Clearly, in order to avoid arbitrage possibilites, the price of an ntD must be less than $V_{CDS}N/n$.

To conform with future notation, the prices of a CDS, and of an ntD, will be written as

$$V_{CDS} = D(1-\rho)q, \quad V_{ntD} = D(1-\rho)F_n. \qquad (4)$$

The parameter q will be seen to be the risk-neutral probability of default before the expiry date of the loan on which the CDS is written, and F_n will be seen to be the risk-neutral probability that the nth default in the basket occurs before the expiry date. At present, however, q and F_n are simply proxies for the price of a CDS and an ntD. The above results can be grouped together by writing the upper bound on the ntD price as

$$\begin{aligned} V_{ntD}^{max} &= D(1-\rho)F_n^{max}, \\ F_n^{max} &= \min[qN/n, 1]. \end{aligned} \qquad (5)$$

Now obtain lower bounds on V_{ntD}. First note the the price V_{ntD} can not be negative. Secondly, suppose that the $t = 0$ price of an ntD satisfies $V_{ntD} < [NV_{CDS} - (n-1)D(1-\rho)]/(N-n+1)$. Then at $t = 0$

- Sell a fraction $1/(N-n+1)$ of a CDS on each of the N names in the basket for a gain of $NV_{CDS}/(N-n+1)$,
- Buy $(n-1)(1-\rho)/(N-n+1)$ risk-free bonds at a cost of $(n-1)(1-\rho)D/(N-n+1)$,
- Buy an ntD.

By assumption, this gives a positive net gain at time $t = 0$. In the case of exactly n' defaults before time $t = T$,

- if $n' < n$, pay $n'(1-\rho)/(N-n+1)$ on the CDS's and receive $(n-1)(1-\rho)/(N-n+1)$ from the bonds, for a nonnegative gain, whereas
- if $n' \geq n$ pay $n'(1-\rho)/(N-n+1)$ on the CDS's, receive $(n-1)(1-\rho)/(N-n+1)$ from the bonds and receive $(1-\rho)$ from the ntD for a nonnegative gain.

Clearly, under the above assumption, arbitrage profits can be realized. Thus, a lower bound on the price of an ntD is given by

$$\begin{aligned} V_{ntD}^{min} &= D(1-\rho)F_n^{min}, \\ F_n^{min} &= \max\left[\frac{N}{N-n+1}\left(q - \frac{n-1}{N}\right), 0\right] \end{aligned} \qquad (6)$$

A plot of F_n^{max} and F_n^{min} as functions of q is given in Fig. 1. For a given q, any value of F_n outside of these bounds gives an arbitrage opportunity and hence is not allowed. These bounds were not obtained in any systematic way, but by guessing certain trading strategies that would demonstrate them. It is thus not clear if there could be arbitrage possibilities for values of F_n lying inside the bounds shown in Fig. 1. This question will be addressed in the next section.

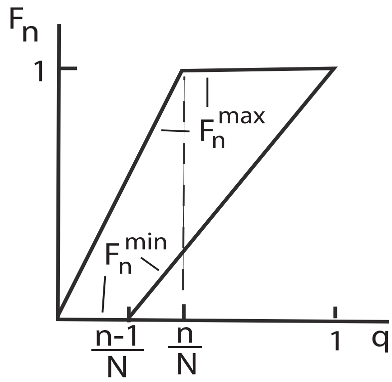

Figure 1: The maximum and minimum values of F_n are shown as a function of q. Here q and F_n give the CDS and ntD prices, as determined by the definitions $V_{CDS} = D(1-\rho)q$ and $V_{ntD} = D(1-\rho)F_n$.

3 Discrete-time model

This section describes the risk-neutral pricing of ntD's on a homogeneous basket of N risky loans. The pricing is carried out in terms of the discrete-time one-period model which was introduced in Walker (2004). The model is an incomplete market model in which the risk-neutral probability measure was characterized by risk-neutral default-correlation parameters. Because of the market incompleteness, these correlation parameters do not have definite values, but have an allowed range of values. This allowed range of values of the correlation parameters translates into an arbitrage-free range for the value of the losses of an ntD. The correlation-parameter representation of the risk-neutral measure, while intuitively appealing, was awkward to work with for baskets of arbitrary size, so that the detailed results obtained were for baskets of two and three reference entities only. Below, a different parameterization for the risk-neutral measure is introduced, and this allows results to be obtained for baskets of arbitrary numbers of reference entities. The basic theory of pricing within a single-period discrete-time model, as used in this section, is described in Pliska (1997).

Each loan i ($i = 1, \ldots, N$) in the basket under consideration is described by its time $t = 0$ price d (d is the risky discount factor) and its expiry date (time $t = T$) payoff $S_{iT}(\omega_k), k = 1, \ldots, 2^N$ (which is unity in the absence of default). The ω_k identify the different possible states of the world at time $t = T$. To characterize the possible states of the world it is convenient to introduce a default indicator λ_i for each loan i. Each loan has two possible states at the expiry date, one in which it has defaulted (for which $\lambda_i = 1$) and the other in which it has not defaulted (for which $\lambda_i = 0$) . Thus, the time T state of the world is given by $\omega = (\lambda_1, \lambda_2, ..., \lambda_N)$, and the total number of possible states of the world at time t is 2^N. The time $t = T$ value of loan i is $S_{iT}(\omega) = 1 - \rho\lambda_i$, where ρ is the common recovery rate for all loans. The risk-free discount factor D is the time $t = 0$ price of a risk-free bound paying one dollar at time T, and the discounted relative price of the ith loan is defined to be $\Delta S^*_{iT}(\omega) = DS_{iT}(\omega) - d$. A risk-neutral probability measure is any probability measure Q that satisfys the condition

$$E_Q(\Delta S^*_i) = \sum_{k=1}^{2^N} Q(\omega_k)\Delta S^*_i(\omega_k) = 0, \quad i = 1, ..., N \tag{7}$$

and has probabilities $Q(\omega_k) > 0$ for all states ω_k. Here, E_Q indicates an expected value evaluated using the probability measure Q (Q is the vector in R^{2N} which has components $Q(\omega_k)$).

Note that, since there are 2^N different time T states of the world, 2^N different probabilities $Q(\lambda_1, \ldots, \lambda_N)$ must be specified to specify the risk-neutral probability measure. Eq. 7 gives N relations between these probabilities, but this still leaves $2^N - N - 1$ independent probabilities which are undetermined. These independent probabilities can vary within certain bounds (see below) thus giving a continuous range of acceptable risk-neutral probability measures. Each risk-neutral probability measure determines an arbitrage-free price for an ntD. The problem is then to explore the range of arbitrage-free prices for ntD's determined by the allowed range of risk-neutral pricing measures.

A particular parameterization of the risk-neutral probability measure Q is now introduced beginning, for simplicity, with the case of a basket of $N = 3$

loans. The risk-neutral probabilities are now denoted by $Q(\lambda_1, \lambda_2, \lambda_3)$. First define q_2 and q_3 by

$$\begin{aligned} q_3 &= Q(111), \\ q_2 &= Q(011) = Q(101) = Q(110). \end{aligned} \tag{8}$$

Here $Q(011)$ is the probability that loan 1 does not default, and that loans 2 and 3 do default. Note that this is not the most general parameterization possible, since three different probabilities are assigned the value q_2, but it will be found to be sufficiently general to obtain the results of interest here. Now define the marginal probability of default for loan i to be q, the same for all loans. It follows from this definition that

$$Q(100) = Q(010) = Q(001) = q - 2q_2 - q_3. \tag{9}$$

Finally, the sum of all of the probabilities must be unity so that

$$Q(000) = 1 - 3q + 3q_2 + 2q_3. \tag{10}$$

The necessary condition of Eq. 7 for Q to be a risk-neutral probability measure is now satisfied, provided q is given by

$$q = (D - d)/[D(1 - \rho)]. \tag{11}$$

Thus, the model is calibrated by using Eq. 11 to determine the risk neutral probability of default for a given loan in terms of its market price d, the same for all loans.

A CDS on loan 1 pays $1 - \rho$ if loan 1 defaults and zero otherwise. It is thus represented by the time T payoff vector $X^{(1)}_{CDS}(\lambda_1\lambda_2\lambda_3) = (1 - \rho)\lambda_1$. The time $t = 0$ price of a CDS on name 1 is

$$V^{(1)}_{CDS} = DE_Q(X^{(1)}_{CDS}) = D(1 - \rho)q \tag{12}$$

The right hand side is the loss on default times the risk-neutral probability of default discounted using the risk-free discount factor. The same result is obtained for loans 2 and 3. Clearly, the market prices of the CDS's could also have been used to calibrate the model.

A basic assumption of this article is that there are no market prices for any ntD's (or CDO's) on the basket or on any subbaskets of the the basket. Thus there is no additional pricing information (over and above bond prices and CDS prices) that one might use to determine q_2 and q_3. This means that the market is incomplete and that the risk-neutral measure contains undetermined paramenters. However, it should also be noted that the risk-neutral probabilities must be positive, which places certain constraints on the possible values of the undetermined parameters q_2 and q_3. Introducing x_2 and x_3 by writing $q_2 = x_2 q$ and $q_3 = x_3 q$ allows the constraints to be written in the form

$$\begin{aligned} 2x_2 + x_3 &< 1, \\ 3x_2 + 2x_3 &< 3, \\ 3x_2 + 2x_3 &> 3 - 1/q, \\ x_2, x_3 &> 0. \end{aligned} \tag{13}$$

Any x_2 and x_3 satisfying these constraints defines an acceptable risk-neutral measure.

An ntD pays $1 - \rho$ at expiry if at least n loans have defaulted. It is thus represented by a payoff $X_{ntD}(\lambda_1\lambda_2\lambda_3)$ which is $1 - \rho$ if at least n of its $\lambda' s$ are unity, and zero otherwise. Thus,

$$V_{ntD} = DE_Q(X_{ntD}) = D(1 - \rho)F_n \tag{14}$$

where

$$\begin{aligned} F_1 &= q(3 - 3x_2 - 2x_3), \\ F_2 &= q(3x_2 + x_3), \\ F_3 &= qx_3. \end{aligned} \tag{15}$$

The quantity F_n is the risk-neutral probability that the nth default has occurred before the expiry date. Now, to find the range of prices allowed for an ntD by the risk-neutral pricing measure, one must maximize and minimize the expression for F_n from Eq. 15 with respect to the variables x_2 and x_3, subject to the constraints of Eq. 13. This linear programming problem was solved numerically, and maximum and minimum values of F_n as functions of q were found that exactly coincide with the bounds F_n^{max} and F_n^{min} determined above by elementary arbitrage arguments (see Fig. 1).

Now consider the case of a homogeneous basket of N risky loans. By analogy with Eq. 8, pick any

n loans in the basket and define $q_n, n = 2, ..., N$ to be the probability that these n have defaulted while the remaining $N-n$ loans have not defaulted. This value of q_n is taken to be the same for all $\binom{N}{n}$ ways of selecting sets of n loans. Also, the probability that a single loan defaults and the remaining $N-1$ loans do not default is the same for all loans, and is given by

$$Q(100...0) = q - \sum_{i=2}^{N} \binom{N-1}{i-1} q_i, \tag{16}$$

while the probability of no defaults is

$$Q(00...0) = 1 - Nq + \sum_{i=2}^{N} \binom{N}{i} (i-1) q_i \tag{17}$$

Also, the requirement of positive probabilities yields the constraints

$$\begin{aligned} \sum_{i=2}^{N} \binom{N-1}{i-1} x_i &< 1, \\ \sum_{i=2}^{N} \binom{N}{i} (i-1) x_i &< N, \\ \sum_{i=2}^{N} \binom{N}{i} (i-1) x_i &> N - 1/q, \\ x_i &> 0, \quad i = 2, ..., N. \end{aligned} \tag{18}$$

Finally, the probabilities that the nth default occurs before the expiry date are given by

$$\begin{aligned} F_1 &= q \left[N - \sum_{i=2}^{N} \binom{N}{i} (i-1) x_i \right], \\ F_n &= q \sum_{i=n}^{N} \binom{N}{i} x_i, \; n = 2, \ldots, N. \end{aligned} \tag{19}$$

To determine the bounds, values of N (the number of loans in the basket)and n (for nth-to-default)were are selected. Then the value of F_n was maximized and minimized (numerically) with respect to the values of $x_i, \quad i = 2, \ldots, N$, subject to the constraints. This was done as a function of q and for all n from $n = 1$ to $n = N$. Furthermore, this procedure was carried out for all sizes of baskets from $N = 1$ to $N = 30$. In all of these cases, the maximum and minimum ntD prices obtained coincide with the bounds obtained earlier by elementary arbitrage arguments. It therefore seems reasonable to conjecture that the results hold for arbitrary N.

Above, a restricted parameterization of the risk-neutral probability measure was adopted (e.g. see Eq. 8 where three distinct probabilities were given the same value). Because the maximization and minimization of the F_n with respect to the parameters determining the risk-neutral measure gave the largest a smallest possible values that were consistent with the elementary arbitrage arguments, it is clear that a more general parameterization would lead to the same result.

The conclusion of this section is therefore that, for a homogeneous basket of N risky loans, all ntD prices within the bounds determined by elementary arbitrage arguments above are arbitrage-free. (Recall that the elementary arbitrage arguments showed only that ntD prices outside the bounds were not arbitrage-free.)

4 Continuous-time model

This section also considers ntD's on a homogeneous basket of N loans, but in a continuous time model. A deterministic instantaneous risk-free interest rate $r(t)$ is assumed (t is time), so that the time $t = 0$ value of a risk-free zero-coupon bond paying one dollar at time t is

$$B(0, t) = exp(-\int_0^t r(u) du). \tag{20}$$

Also, the quantity $q(t)$ (analogous to q of the previous section) is defined to be the probability that a given bond in the basket defaults before time t. The quantity $q(t)$ is the same for all loans (the basket is assumed homogeneous) and is sometimes defined in terms of a hazard rate $h(t)$ (e.g. $q(t) = 1 - exp(-\int_0^t h(u) du)$). Furthermore, it is assumed that there is an arbitrage-free term structure for zero-coupon bonds of all maturities for each of the names in the basket. This term structure can then be used to determine the function $q(t)$ (e.g. see Hull (2003), Duffie and Singleton (1999) and Schönbucher (2003)). Thus, $q(t)$ is, by assumption, a known function determined by market prices, and a knowledge of it defines the risk-neutral measure used to price the zero-coupon risky bonds or CDS's associated with the names of the basket.

A risk-neutral probability measure appropriate for the basket will be any probability measure that preserves the $q(t)$'s as the marginal default probabilities for the default of a single loan. This is the basis for the commonly used copula approach to pricing basket credit derivatives. In this article, the probability measure is taken to have exactly the same form as in the discrete-time model of the previous section, except that now $q(t)$ and the $q_i(t), \quad i = 2, \ldots, N$ are taken to be functions of time. The probabilities $Q(\lambda_1, \ldots, \lambda_n; t)$ are therefore also functions of time, as is the probability $F_n(t)$ that the nth default occurs before time t. To find the maximum or minimum value of $F_n(t)$ at any time t, one starts with $q(t)$ and then carries out precisely the same optimization procedure as was carried out in the previous section. The bounds on F_n at a given time (i.e. for a given $q(t)$) are therefore also given by Fig. 1.

The general procedure for the risk-neutral pricing of ntD's is well documented (e.g. see Hull (2003), Hull and White (2004), Laurent and Gregory (2003) and Schönbucher (2003)). The ntD's considered in this section pay the loss relative to a notional of unity, i.e. the amount $1-\rho$ where ρ is the recovery rate for all loans, at the time of default. The time $t = 0$ value of the potential losses is therefore

$$V_{ntD} = (1-\rho)\int_0^T f_n(u)B(0,u)du \qquad (21)$$

where $f_n(t)$ is the risk-neutral probability that the nth default occurs in the time interval $(t, t+dt)$, i.e. $f_n(t) = dF_n(t)/dt$. Below, $F_n(t)$ will be taken to be a bounding value of a region of risk-neutral probabilities, and it can thus have unusual properties. For example, allowance should be made for the possiblity that $F_n(t) = 0$ (and hence $f_n(t) = 0$ also) in some range $0 < t < T_1^*$ (T_1^* can be zero) or that $F_n(t) = 1$ (and $f_n(t) = 0$) in some region $T_2^* < t < 1$ (T_2^* can be T). With these precautions, an integration by parts of Eq. 21 gives $V_{ntD} = (1-\rho)I_n$ where

$$I_n = F_n(T_2^*)B(0,T_2^*) + \int_{T_1^*}^{T_2^*} F_n(u)r(u)B(0,u)du. \qquad (22)$$

The second term represents a (usually small) adjustment to the discrete-time case due to the fact that the payoff is made at the time of default rather than at the expiry date T. In return for receiving the loss on the nth default, the buyer of ntD protection pays a premium at an annualized rate of w_n at regular intervals of δ (e.g for 4 payments per year, $\delta = 1/4$). If there are N_{pay} payments to be made, then $T = N_{pay}\delta$ and the payments are made at times $t_k = k\delta$ where $k = 1, \ldots, N_{pay}$. The payment made at the end of each period is $w_n\delta$ and the $t = 0$ value of all of the premium payments is $w_n\delta W_n$ where

$$\begin{aligned} W_n &= \sum_{k=1}^{N_{pay}} [1 - F_n(t_k)]B(0,t_k) \\ &+ \int_0^T f_n(u)\phi(u)B(0,u)du. \end{aligned} \qquad (23)$$

The factor $1 - F_n(t_k)$ is the probability that the nth default does not occur before time t_k, and the second term, which contains the factor $\phi(u)$, represents the accrued payments. (If the nth default occurs during a payment period, there is a payment made which represents the premium owing for the interval from the beginning of the period to the default time. This is the final payment.) The accrual payoff function is periodic in δ, i.e. $\phi(t+\delta) = \phi(t)$ and, for $0 < t < \delta, \quad \phi(t) = t/\delta$.

The break-even value of the annualized premium payment w_n is determined by equating the $t = 0$ values of the losses and the premium payments. This gives

$$w_n = \frac{1-\rho}{\delta}\frac{I_n}{W_n}. \qquad (24)$$

In what follows, it is useful to approximate the accrued contribution to W_n in Eq. 23 (which is in any case small) by setting the accrual payoff function $\phi(u)$ equal to its average value of 1/2. Alternatively, one can calculate the premium payments on the assumption that they are paid continuously, as in Eq. 27

		Correlation			
n	max	0.0	0.3	0.6	min
1	767	603	440	293	60
2	332	98	139	137	0
3	212	12	53	79	0
4	156	1	21	49	0
5	123	0	8	31	0
6	102	0	3	19	0
7	87	0	1	12	0
8	75	0	0	7	0
9	67	0	0	3	0
10	60	0	0	1	0

Table 1: The upper and lower bounds on the range of arbitrage-free ntD premiums, as determined in this article, are compared with representative values of ntD break-even premiums for a basket of 10 loans as determined in Hull and White (2004). Columns max and min show the upper and lower bounds, respectively. The values under correlation are taken from Hull and White (2004) and are for three different values of their default-correlation parameter.

below. With either of these assumptions, it is clear from Eqs. 22, 23 and 24 that the larger (or smaller) the value of $F_n(t)$, the larger (or smaller) the value of $w_n(T)$. However, the maximum and minimum values of $F_n(t)$, for all t, have already been obtained above by maximizing and minimizing over possible risk-neutral probabilities. The maximum and minimum values of $F_n(t)$, for all t, are simply the bounding values given above by elementary arbitrage arguments, but for $q = q(t)$.

To summarize, the upper and lower bounds on an ntD premium rate are found by using Eqs. 22 and 23 in Eq. 24, together with

$$F_n^{max}(t) = \min[Nq(t)/n, 1] \qquad (25)$$

for the upper bound, and

$$F_n^{min}(t) = \max\left[\frac{N}{N-n+1}\left(q(t) - \frac{n-1}{N}\right), 0\right] \qquad (26)$$

for the lower bound. In evaluating the upper bound, take $T_1^* = 0$ and, if $Nq(T) > n$, define T_2^* by $Nq(T_2^*) = n$; otherwise take $T_2^* = T$. For the lower bound take $T_2^* = T$ and, if $Nq(T) > n - 1$, define T_1^* by $Nq(T_1^*) = n - 1$; otherwise take $F_n(t) = 0$ for all t.

Hull and White (2004) consider a numerical example of 5-year ntD's on a basket of 10 loans. A recovery rate $\rho = 0.4$, a constant risk-free interest rate $r = 0.05$, a payment period $\delta = 0.25$, and a risk-neutral probability that a given loan defaults before time t of $q(t) = 1 - exp(-ht)$ with $h = 0.01$, were assumed. Their Table 2 shows the ntD premium rate calculated within a gaussian one-factor copula model for three different values of a correlation parameter, and these values are reproduced in Table 1. The values of the arbitrage-imposed upper and lower bounds as derived for the same problem from Eqs. 22 to 26 are also shown in Table 1. (The small accrual correction in Eq. 23 was evaluated approximately by replacing the accrual function $\phi(t)$ by its average value of 1/2.) As expected, the values obtained for selected parameters from the Hull and White (2004) model fall within the bounds found by the procedure of this article. Clearly, as already noted in Walker (2004), the buyers and sellers of ntD's (and collateralized debt obligations), have a wide range of arbitrage-free prices to choose from. and considerations other than risk-neutral pricing will play an important role in determining the price of these instruments.

5 Constant hazard rate

When the hazard rate and the risk-free interest rate are constant, particularly simple formulae for the bounds on the break-even premiums for ntD's can be arrived at. In this case $q(t) = 1 - exp(-ht)$. Further simplification results if it is assumed that the premium is paid continuously so that the time $t = 0$ value of the premium payments for an ntD is

$$V_{ntD}^{premium} = w_n \int_0^T [1 - F_n(u)]e^{-ru}du. \qquad (27)$$

The required integrals are now easily performed analytically.

For a CDS, the value of the losses is

$$V_{CDS} = (1-\rho)\int_0^T \frac{dq(u)}{du}B(0,u) \tag{28}$$

whereas the value of the premium payments, made continuously at an annualized rate of s dollars per year, is

$$V_{pay} = s\int_0^T [1-q(u)]B(0,u)du. \tag{29}$$

Equating the value of the losses to the value of the premium payments and carrying out the integrals gives $s = (1-\rho)h$. The premium rate s is also called the CDS spread. The bounds on the ntD premium rate will be expressed in terms of the CDS spread.

For the upper limit to the ntD premium rate, called w_n^{max}, define $T_2^\star$ by

$$T_2^* = \min\left[\frac{1}{h}ln\left(\frac{N}{N-n}\right), T\right] \tag{30}$$

for $n < N$, and $T_2^* = T$ for $n = N$. Then

$$\begin{aligned} w_n^{max} &= \frac{sN/n}{D}, \\ D &= \frac{N}{n} - \frac{N-n}{n}\frac{h+r}{r}\frac{1-exp(-rT_2^*)}{1-exp[-(h+r)T_2^*]}. \end{aligned} \tag{31}$$

Keeping only the first three terms in an expansion in powers of T_2^* yields a sufficiently accurate and a more transparent formula, which is

$$w_n^{max} = \frac{sN/n}{1-\frac{N-n}{2N}hT_2^*}. \tag{32}$$

For the lower limit to the premium rate, define T_1^* by

$$T_1^* = \min\left[\frac{1}{h}ln\left(\frac{N}{N-n+1}\right), T\right]. \tag{33}$$

If $T_1^* = T$, then the lower bound of ntD premium rate is $w_n^{min} = 0$. Otherwise,

$$w_n^{min} = \frac{s}{1+\frac{N-n+1}{N}\frac{h+r}{r}\frac{1-exp(-rT_1^*)}{exp[-(h+r)T_1^*]-exp[-(h+r)T]}}. \tag{34}$$

Note that $T_1^* = 0$ for $n = 1$ so that $w_1^{min} = s$.

6 Conclusions

Detailed formulae for the upper and lower arbitrage-imposed bounds on nth-to-default premium rates have been derived for all n for homogeneous baskets of an arbitrary number of reference credits. If the premium rates lie outside these bounds the possibility exists of making arbitrage profits (within the framework of the idealized model of this article). Also, all prices within the bounds are arbitrage-free. It has been assumed that there is a liquid market credit default swaps of each of the firms, as well as of risk-free bonds , so that these assets can be used to create the arbitrage possibilities.

The methods developed here can also be used to determine the arbitrage-free price ranges of collateralized debt obligations, and for the study of inhomogeneous baskets.

The existence of portfolios of assets that pay more (or less) than an nth-to-default swap in every state at expiry forces the price of the ntD to lie within the bounds determined in this article. However, there is no pressure from possible arbitrage portfolios that drives the price of an ntD towards any particular price within the bounds. Since the bounds are relatively broad(e.g. see Eqs. 32 and 34 above, as well as Table 1) arbitrage arguments (which include risk-neutral pricing), are of relatively little help in establishing a definite price for a basket credit derivative such as an ntD or a CDO.

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