

The uncertainty principle in the presence of quantum memory

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The uncertainty principle, originally formulated by Heisenberg¹, clearly illustrates the difference between classical and quantum mechanics. The principle bounds the uncertainties about the outcomes of two incompatible measurements, such as position and momentum, on a particle. It implies that one cannot predict the outcomes for both possible choices of measurement to arbitrary precision, even if information about the preparation of the particle is available in a classical memory. However, if the particle is prepared entangled with a quantum memory, a device that might be available in the not-too-distant future², it is possible to predict the outcomes for both measurement choices precisely. Here, we extend the uncertainty principle to incorporate this case, providing a lower bound on the uncertainties, which depends on the amount of entanglement between the particle and the quantum memory. We detail the application of our result to witnessing entanglement and to quantum key distribution.

Experimental investigation of the uncertainty principle in the presence of quantum memory

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Heisenberg's uncertainty principle provides a fundamental limitation on an observer's ability to simultaneously predict the outcome when one of two measurements is performed on a quantum system. However, if the observer has access to a particle (stored in a quantum memory) which is entangled with the system, his uncertainty is generally reduced. This effect has recently been quantified by Berta et al. [Nature Physics 6, 659 (2010)] in a new, more general uncertainty relation, formulated in terms of entropies. Using entangled photon pairs, an optical delay line serving as a quantum memory and fast, active feed-forward we experimentally probe the validity of this new relation. The behaviour we find agrees with the predictions of quantum theory and satisfies the new uncertainty relation. In particular, we find lower uncertainties about the measurement outcomes than would be possible without the entangled particle. This shows not only that the reduction in uncertainty enabled by entanglement can be significant in practice, but also demonstrates the use of the inequality to witness entanglement.

Lee Rozema

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Outline

- Shannon Entropy and a reformulation of the uncertainty principle
- The Uncertainty Game and a strategy that violates the regular uncertainty principle
- ‘New’ uncertainty principle
- Applications as an entanglement witness
- The experiment

Shannon Entropy

3 Complimentary Views:

- How much information we gain when we measure X
- Our uncertainty about X before we measure it
- Minimum number of bits needed to store the result of measurement X (a random variable)

Defined as:

$$H(X) = -\sum p_x \log(p_x)$$

It is a function of the probability distribution of X

Joint entropy of two variables is a simple extension:

$$H(X, Y) = -\sum p(x, y) \log(p(x, y))$$

Deutsch's Problem with Heisenberg (PRL 1983)

- Heisenberg: $V_{\hat{A}}(|\psi\rangle)V_{\hat{B}}(|\psi\rangle) \geq \frac{1}{4} |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle|^2$

- Deutsch wanted: $u(\hat{A}, \hat{B}, |\psi\rangle) \geq \mathfrak{B}(\hat{A}, \hat{B})$

(Bound independent of the state)

- 5 years later:

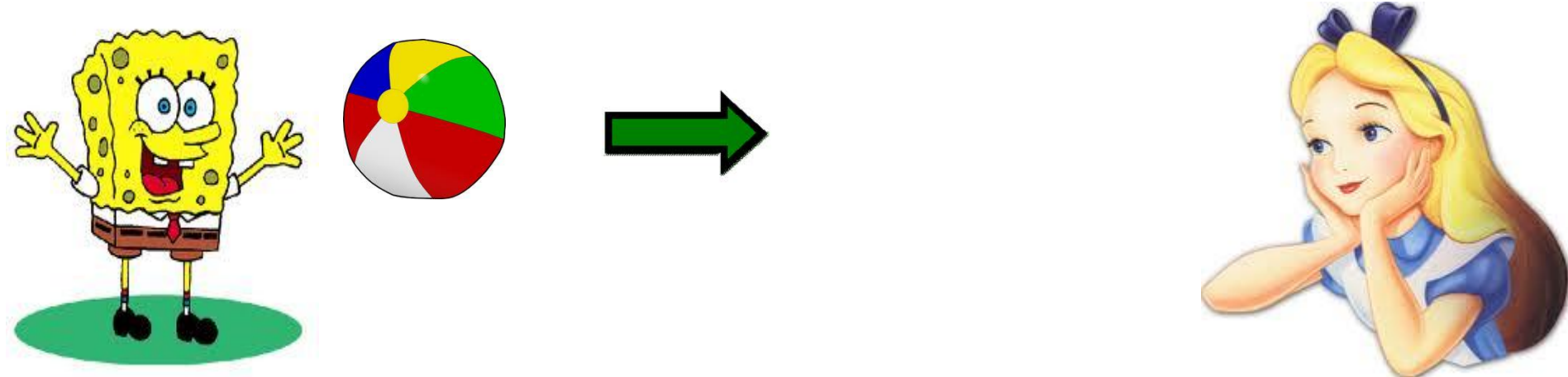
$$H(R) + H(S) \geq \log_2 \frac{1}{c}$$

Where : $R = \sum_r |r\rangle\langle r|$

$S = \sum_s |s\rangle\langle s|$

Then: $C = \max(|\langle r | s \rangle|^2)$

The Uncertainty Game

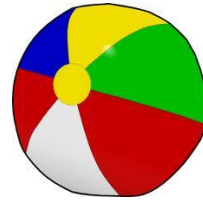


- 1) Alice and Bob agree on 2 measurements, R and S
- 2) Bob prepares $|\Psi\rangle$ and sends it to Alice
- 3) Alice measures R or S and announces which measurement she made, but not the outcome

The Goal: Bob tries to minimize his uncertainty about Alice's measurement outcome by preparing $|\Psi\rangle$

but he is bounded by $\log_2 \frac{1}{c}$

What about entanglement?



If Bob prepares $|01\rangle - |10\rangle$ and keeps one particle

Conditional Entropy

A pair of variables X, Y have $H(X, Y)$ total uncertainty before a measurement

How much uncertainty is left in X if we measure Y ?

$$H(X|Y) = H(X, Y) - H(Y)$$

Von Neuman Entropy

$$H(\rho) = -\text{Tr}(\rho \log \rho)$$

$$:= -\sum_i \lambda_i \log_2 \lambda_i$$

(λ are eigen values)

Berta et. al.'s Relation

$$H(R|B) + H(S|B) \geq \log_2 \frac{1}{c} + H(A|B)$$

$H(A|B)$ - quantifies amount of entanglement between Alice and Bob

- $H(A|B) = H(A,B) - H(B) = -\log(d) \Rightarrow$ max. entangled

$H(R|B)$ - Uncertainty in post measurement state given that we measure B

- $H(R|B) = H(\rho_{RB}) - H(\rho_B)$

$$\rho_{RB} = \sum_r p_r |\Psi_r\rangle\langle\Psi_r| \otimes \rho_B^r,$$

- $\rho_{RB} = \frac{1}{2} |0\rangle\langle 0| \otimes |1\rangle\langle 1| + \frac{1}{2} |1\rangle\langle 1| \otimes |0\rangle\langle 0|$

- $\rho_B = \frac{1}{2} |1\rangle\langle 1| + \frac{1}{2} |0\rangle\langle 0|$

Entanglement Witness

Can we tell if a state is entangled without doing full tomography?

For a given R and S we know C

$$H(R|B) + H(S|B) \geq \log_2 \frac{1}{c} + H(A|B)$$

If: $H(R|B) + H(S|B) < \log(1/c)$

Then: $H(A|B)$ is negative and A and B are entangled

Entanglement Witness

Fano's Inequality: $H(R|B) < h(q) + q \log(d)$

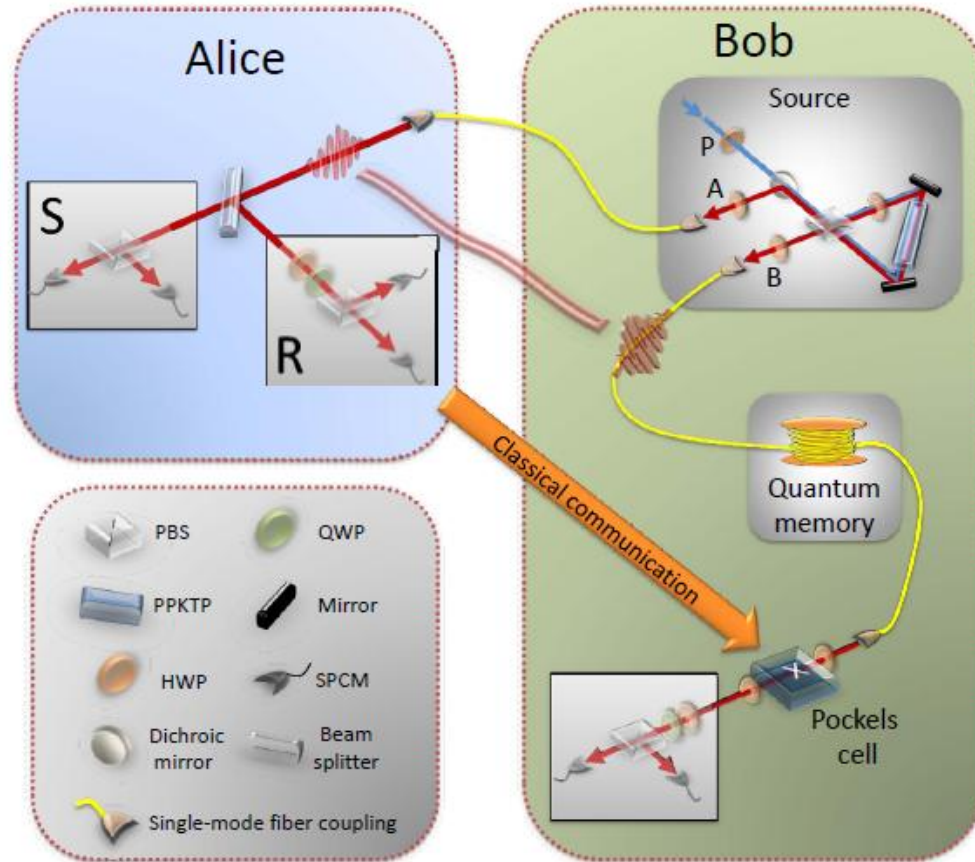
q = probability Alices outcome equals Bobs
 h = binary entropy

Can upper bound $H(R|B)$ by comparing outcomes of one measurement.

Then can check the 'entanglement witness' bound

$$H(R|B) + H(S|B) < \log(1/c)$$

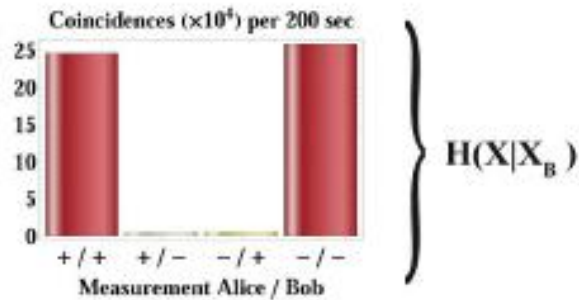
The Experiment



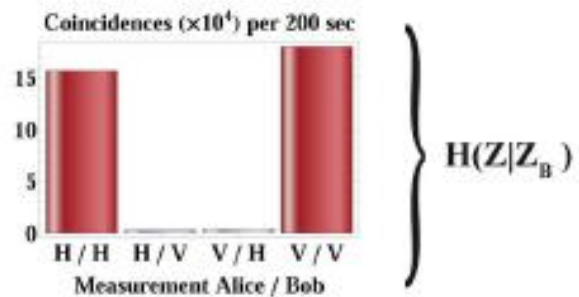
- Make states $\Rightarrow \cos(a) |HH\rangle + \sin(a) |VV\rangle$
- Alice randomly chooses S or R

To Measure Entanglement Witness

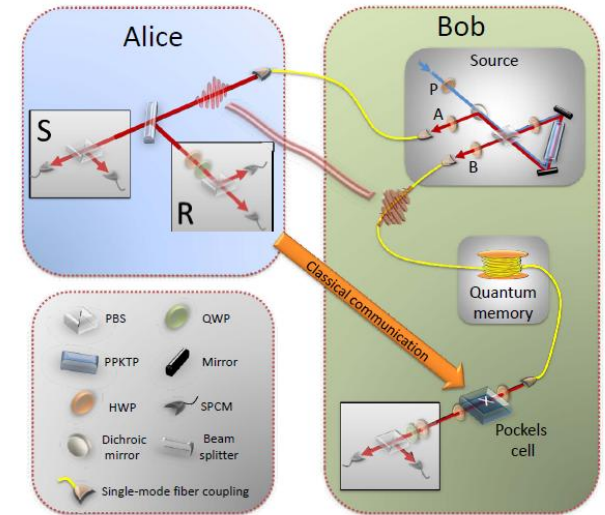
Bob measures in same basis as Alice (using Pockles Cell)



$$H(X|X_B)$$

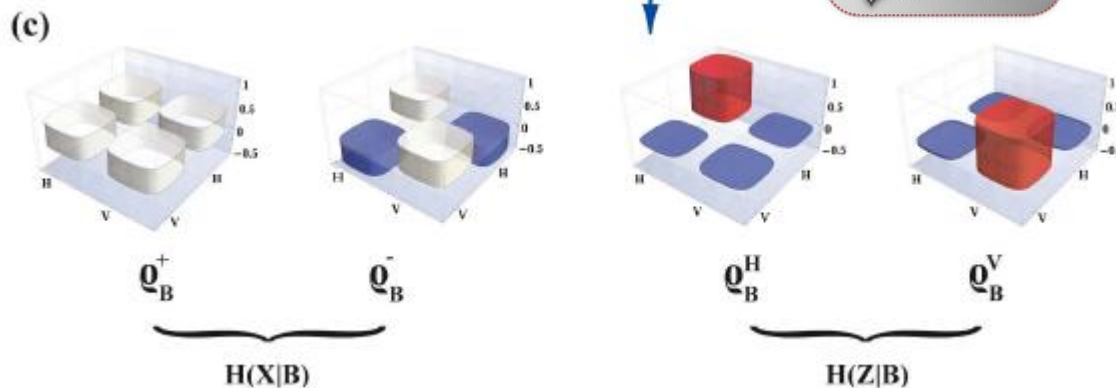
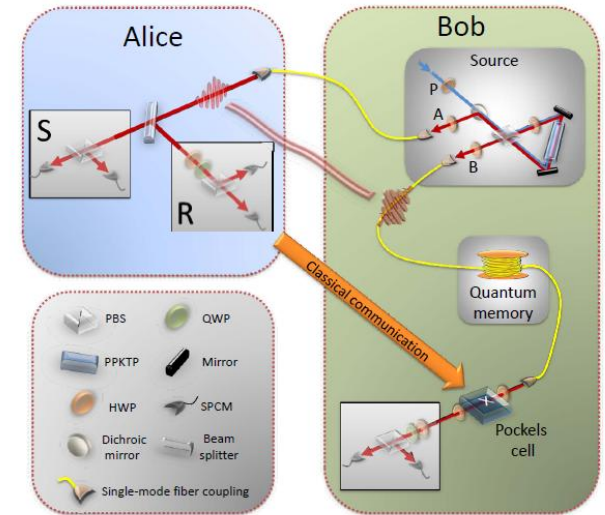


$$H(Z|Z_B)$$



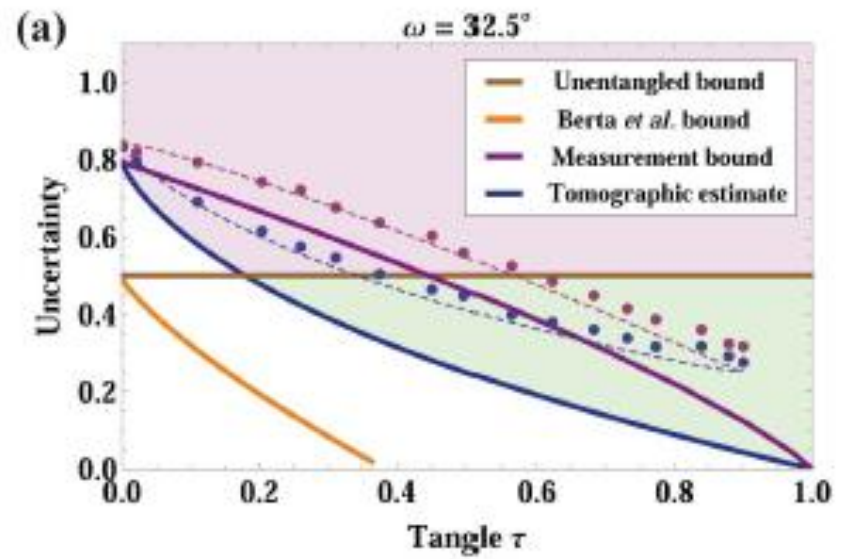
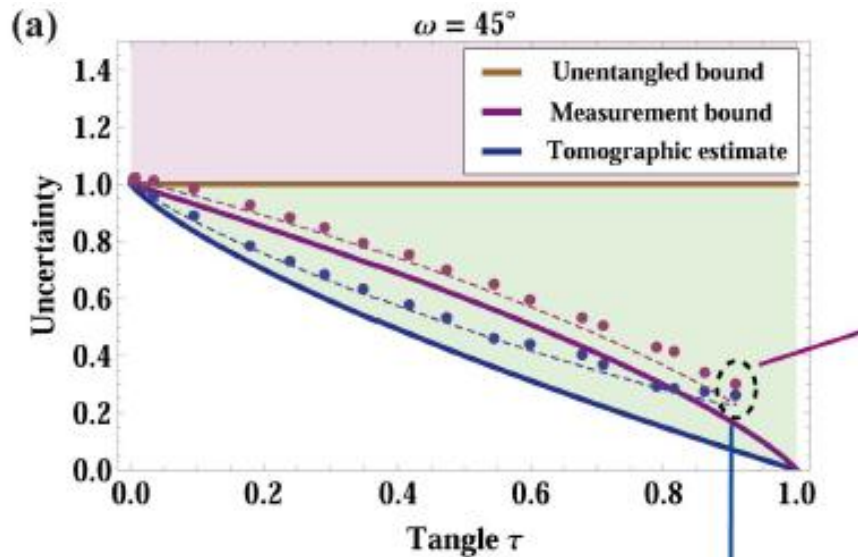
To Measure Full Inequality

Perform tomography on Bobs particle dependent on Alices out comes to get:



$$H(R|B) = H(\{p_r\}) + \sum_r p_r H(\rho_B^r) - H\left(\sum_r p_r \rho_B^r\right)$$

Results



Conclusions

- Heisenberg Uncertainty is 'incomplete' if we can entangle the particle with another particle
- New uncertainty relation has applications as an entanglement witness and to quantum cryptography