# Self-Organized Critical Forest-Fire Model

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A forest-fire model is introduced which contains a lightning probability $f$. This leads to a self-organized critical state in the limit $f \to 0$ provided that the time scales of tree growth and burning down of forest clusters are separated. We derive scaling laws and calculate all critical exponents. The values of the critical exponents are confirmed by computer simulations. For a two-dimensional system, we show that the forest density in the critical state assumes its minimum possible value, i.e., that energy dissipation is maximum.

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Recently, Bak, Chen, and Tang [1] introduced a forest-fire model which they assumed to show self-organized critical behavior. Their forest-fire model is a probabilistic cellular automaton defined on a $d$-dimensional hypercubic lattice with $L^d$ sites. In the beginning each site is occupied by either a tree or a burning tree or it is empty. The state of the system is parallely updated by the following rules: (i) A burning tree becomes an empty site. (ii) A green tree becomes a burning tree if at least one of its nearest neighbors is burning. (iii) At an empty site a tree grows with probability $p$.

If the system size is larger than the correlation length of the fire, the model assumes a steady state with finite fire density. Bak, Chen, and Tang found that the fire-fire correlation function obeys a power law in two and three dimensions. They concluded that the system is critical in the limit $p \to 0$ where the fire correlation length diverges. Bak coined the expression "self-organized criticality" to describe the behavior of extended dissipative systems which assume a critical steady state independent of the initial state and without tuning of parameters to a special value. The most prominent example of a self-organized critical system is the so-called sandpile model [2].

Grassberger and Kantz [3], as well as Mossner, Drossel, and Schwabl [4], performed computer simulations of the forest-fire model with values of $p$ smaller than those used by Bak, Chen, and Tang. These simulations show that the forest-fire model is not critical but instead it becomes more and more deterministic with decreasing $p$ and develops regular, spiral-shaped fire fronts. The size of these spirals, as well as the distance between them, is of the order $1/p$. The temporal fire-fire correlation function oscillates regularly with a period proportional to $1/p$. By contrast, a critical system should contain fire fronts of all sizes up to the correlation length, and their temporal correlation function should show a power-law spectrum of frequencies.

The reason that there are no fire fronts smaller than $1/p$ is the following: Trees that are next neighbors belong to the same forest cluster. A tree only catches fire when one of its neighbors burns. So a small forest cluster cannot be ignited; therefore it grows until it becomes part of a burning cluster. Since the fire burns constantly in the steady state, a burning forest cluster must be so large that trees grow at one end while the fire burns the other end; i.e., the diameter of a burning forest cluster is proportional to $1/p$, and consequently the size of a fire front is also proportional to $1/p$.

The model becomes critical when a mechanism is included that allows for small forest clusters to burn also. We therefore introduce a "lightning parameter" $f$ and a fourth rule: (iv) A tree without a burning nearest neighbor becomes a burning tree during one time step with probability $f$. In order to understand how criticality arises in this extended forest-fire model, let us first simplify dynamics and assume that a whole forest cluster is burned down instantaneously, i.e., during one time step when one of its trees is struck by lightning. In this case dynamics are invariant (except for a change of the time scale) when $f$ and $p$ are multiplied by the same factor. Then there is only one relevant parameter $f/p$ in the system. Let $\bar{\rho}$ be the mean overall forest density in the system in the steady state. The average number of lightning strokes in the system during $t$ time steps is $tf\bar{\rho}L^d$. The average number of trees growing in the system during $t$ time steps is $tp(1-\bar{\rho})L^d$. Consequently, the average number of trees destroyed by a lightning stroke is

$$\bar{s} = (f/p)^{-1}(1-\bar{\rho})/\bar{\rho}\,. \qquad (1)$$

In order to avoid finite-size effects, the number of sites $L^d$ must be chosen much larger than the largest forest cluster. For any finite value of $f/p$, the value of $\bar{s}$ is then independent of $L$. Equation (1) represents a power law $\bar{s} \propto (f/p)^{-1}$ for small values of $f/p$ if $\lim_{f/p \to 0}\bar{\rho} < 1$. This is the case for $d \geq 2$ as we conclude from the following consideration: If the mean forest density $\bar{\rho}$ were near 1 for small values of $f/p$ in $d \geq 2$ dimensions, the largest forest cluster would contain a finite percentage of all trees in the system, and the average number of trees burned by a lightening stroke would diverge in the limit $L \to \infty$ in contradiction to (1). We therefore expect a critical point in the limit $f/p \to 0$. The special case $d = 1$ will be treated later.

Let us now return to the real forest-fire model where a forest cluster is not burned down instantaneously but during a finite time interval $T(s)$ which depends on the number of trees $s$ in the forest cluster and will be determined below. Given a value of $f/p$, the forest dynamics of the system are essentially the same as in the case of instantaneous burning down provided that $p$ is so small that a forest cluster is burned down before trees grow at its edge, i.e., if $1/p \gg T(\bar{s})$. Equation (1) remains valid, and we expect critical behavior in the limit $f/p \rightarrow 0$ with $p \ll T^{-1}(\bar{s})$. Figure 1 shows the system in the steady state with $f/p=0.1$.

In order to show that the system really is critical in this limit, we calculate the scaling relations and the critical exponents which follow from the assumption of scale invariance. We then confirm the values of the critical exponents by computer simulations. The radius $R(s)$ of a forest cluster is the square root of the mean quadratic distance of the cluster members from their center of mass. We define the correlation length $\xi$ by $\xi=R(\bar{s})$, and the exponents $\nu$ and $z$ by $\xi \propto (f/p)^{-\nu}$ and $T(\bar{s}) \propto \xi^z$. The condition that forest clusters burn down rapidly then reads $p \ll (f/p)^{\nu z}$. Let $N(s)ds$ denote the number of clusters consisting of $s$ trees in a system of a given size. For brevity we introduce the moments $m_n = \int_1^\infty s^n N(s)ds$ of this non-normalized distribution function. Equation (1) implies that $N(s)$ does not decay faster than a power law in the limit $f/p \rightarrow 0$. Since the density of trees in the system is finite, $N(s)$ decays at least as fast as $s^{-2}$. The cluster distribution $N(s)$ therefore obeys a power law in the limit $f/p \rightarrow 0$. Under the scaling transformation $\mathbf{x} \rightarrow \mathbf{x}/b$, $f/p \rightarrow b^{1/\nu} f/p$, the normalized cluster distribution $N(s)ds/m_0$ and the function $R(s)$ must be invariant. This implies the following scaling laws for $N(s)$ and $R(s)$:

$$N(s) \propto s^{-\tau} \times \begin{cases} \mathcal{C}(s/s_{\max}), & \tau > 2, \\ \mathcal{C}(s/s_{\max}) \ln^{-1}(s_{\max}), & \tau = 2, \end{cases}$$
$$R(s) \propto s^{1/\mu} \tilde{\mathcal{C}}(s/s_{\max}), \tag{2}$$

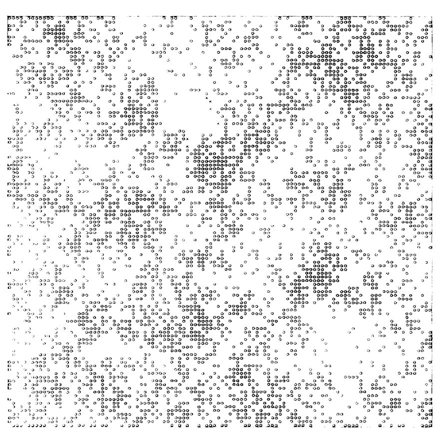

FIG. 1. Steady state of the forest-fire model for $f/p=0.1$ and $d=2$. Trees are represented by circles, and burning trees by crosses.

with $s_{\max} \propto (f/p)^{-\lambda}$. Here, $\lambda$ and $\tau$ are new critical exponents, and the cutoff functions $\mathcal{C}(x)$ and $\tilde{\mathcal{C}}(x)$ decrease monotonically from the value 1 for $x \ll 1$ to 0 for $x \gg 1$. The logarithmic correction to $N(s)$ for $\tau=2$ guarantees a finite mean forest density in the limit $f/p \rightarrow 0$. The exponent $\mu$ is the fractal dimension of a forest cluster. Equation (2) and the definition of the correlation length lead to the following scaling relations:

$$\lambda = \nu\mu, \quad d = \mu(\tau - 1). \tag{3}$$

Since $\mu \le d$, the exponent $\tau$ must obey $\tau \ge 2$, as already stated above. The mean number of trees destroyed by a lightning stroke is

$$\bar{s} = m_2/m_1 \propto \begin{cases} s_{\max}^{3-\tau}, & 3 > \tau > 2, \\ s_{\max}/\ln(s_{\max}), & \tau = 2. \end{cases} \tag{4}$$

We already derived [Eq. (1)] $\bar{s} \propto p/f$. Since there is only one diverging length scale in a critical system, one has $\bar{s} \propto s_{\max}$. Together with Eqs. (3) and (4), this leads to the following values of the critical exponents [5]:

$$\lambda = 1, \quad \tau = 2, \quad \mu = d, \quad \nu = 1/d. \tag{5}$$

Since $\tau=2$, the power law for $s_{\max}$ noted after Eq. (2) acquires a logarithmic correction factor and now reads

$$s_{\max} \propto (f/p)^{-1} \ln(s_{\max}) \propto (p/f) \ln(p/f).$$

Even for $d=1$, the scaling laws and critical exponents derived are valid since the factor $\bar{\rho}/(1-\bar{\rho})$ entering Eq. (1) diverges only logarithmically, as we conclude from the following considerations: The number of forest clusters in the system is constant in the steady state. It increases by 1 when an isolated tree grows, and it decreases by 1 when lightning strikes a tree. Consequently $Lp(1-\bar{\rho}) - 2pm_0 = fL\bar{\rho}$. In addition, we have the following two relations that are valid in any dimension: $L^d \bar{\rho} = m_1$ and $(p/f)L^d(1-\bar{\rho}) = m_2$. From these equations we finally obtain $\bar{\rho}/(1-\bar{\rho}) \propto \ln(s_{\max})$. The scaling law for $N(s)$ again contains a logarithmic correction factor $\ln^{-1}(s_{\max})$; the power law for $s_{\max}$, however, is exact in one dimension.

The distribution $n(T)$ of the lifetime of a fire also obeys a power law: The probability that lightning strikes a cluster of size $s$ is proportional to $sN(s) \propto s^{-1}$. The cluster burns down in $T(s) \propto R(s) \propto s^{1/d}$ time steps which implies $n(T) \propto T^{-\alpha}$ with $\alpha = 1$. The mean lifetime of a fire is proportional to $T(\bar{s}) \propto \xi \propto (f/p)^{-1/d}$ which leads to the dynamical critical exponent $z=1$.

Knowing that the number of burning trees in a forest cluster $t$ time steps after a lightning stroke is proportional to $t^{d-1}$, the Fourier transform of the fire-fire correlation function can be shown to obey a power law $\propto f^{-2d} + O(f^{-2d+2}, \ldots, f^{-2})$ for small frequencies $f$. In the sandpile model where the number of active sites is assumed to be constant for the duration of an avalanche, the corresponding power law is $\propto f^{-2}$ in one and two dimensions [6].

Our computer simulations for $d=1,2,3$ confirm the values of the critical exponents calculated above. For $d=2$ and $f/p=1/70$, the cluster size distribution and the cluster radius as functions of cluster size are shown in Fig. 2. The slopes of the plots yield the exponents $\tau=2$ and $\nu=1/2$.

We also determined the mean forest density $\bar{\rho}$ in the steady state. For large values of $f/p$, the mean forest density is small. It increases with decreasing $f/p$ and approaches a constant value in the limit $f/p\to 0$. We determined this value numerically and obtained $\bar{\rho}\simeq 0.39$ and $\bar{\rho}\simeq 0.21$ for $d=2$ and 3, respectively. In the following, we show that the critical value $\bar{\rho}=0.39$ in two dimensions is determined by an *extremum principle*: $\bar{\rho}$ assumes its minimum possible value, i.e., the fire destroys as much forest as it can. Consider an area of the forest which is very large compared to the lattice constant but very small compared to the correlation length. The forest in this area will grow until a fire passes through it which has started somewhere in the forest around it (since the area is small compared to the mean size of a forest cluster and $f/p$ is near 0, the lightning nearly never strikes it directly). Let $\Delta T$ denote the mean time interval between two such fire fronts. Its value depends neither on the exact size of the area nor on the precise value of $f/p$. The average forest density just before the fire enters the area is $\rho_{max}$, and is $\rho_{min}$ just after the fire has passed. If the value of $\Delta T$ were different from the actual value, the values for $\rho_{max}$ and $\rho_{min}$ and therefore the mean forest density $\bar{\rho}$ would also be different. If $\Delta T$ were very large, $\rho_{max}$ would be near 1, and $\rho_{min}$ near 0. With decreasing $\Delta T$, the value of $\rho_{max}$ would decrease, and the value of $\rho_{min}$ would therefore increase. For very small values of $\Delta T$, the difference between $\rho_{max}$ and $\rho_{min}$ would also be very small, and the value of $\rho_{max}$ would be just above the percolation threshold 0.59 for site percolation while the value of $\rho_{min}$ would be just below it. $\Delta T$ and $\bar{\rho}$ are related to $\rho_{max}$ and $\rho_{min}$ by the equations

$$p\Delta T=\ln\left(\frac{1-\rho_{min}}{1-\rho_{max}}\right),\quad \bar{\rho}=1-\frac{\rho_{max}-\rho_{min}}{p\Delta T}.$$

We still need a third relation between these four parameters in order to determine all possible values of $\bar{\rho}$ and therefore its minimum. Having in mind that the tree distribution just before the fire enters the area is stochastic since trees grow stochastically, we determined $\rho_{min}$ as a function of $\rho_{max}$ numerically by randomly filling a lattice with trees up to the density $\rho_{max}$ for different values of $\rho_{max}>0.59$, and by counting the number of trees that belong to the largest forest cluster. This forest cluster connects the edges of the lattice and would be destroyed by a fire sweeping the lattice. From the number of trees that do not belong to this cluster we obtained the density $\rho_{min}$. Using the above relations, we plot $\bar{\rho}$ as a function of $\rho_{max}$ (Fig. 3). The minimum value of $\bar{\rho}$ is $\bar{\rho}\simeq 0.39$, and the corresponding value of $\Delta T$ is $\Delta T\simeq 0.91/p$. The minimum value of the mean forest density is just the value observed in the critical state of our forest-fire model. We conclude that the critical state is organized in such a way that the number of growing trees and of burned trees is maximum, i.e., that energy dissipation in the system is maximum. We expect that the extremum principle holds also in higher dimensions.

Finally, we would like to comment on the physical significance of the two conditions $f/p\to 0$ and $p\ll(f/p)^{\nu z}$, leading to the critical behavior of the forest-fire model. Rewriting them in the form

$$(f/p)^{-1/d}\ll p^{-1}\ll f^{-1},$$

we see that they describe a double separation of time scales: The time in which a forest cluster burns down is much shorter than the time in which a tree grows, which again is much shorter than the time between two lightning occurrences at the same site. Separation of time scales is quite frequent in nature, while the tuning of parameters to a certain finite value in nature only takes place accidentally. Thus, the forest-fire model is critical over a wide range of parameter values, and we expect that its critical state should also be robust with respect to slight modifications of the model rules, e.g., another lattice symmetry or fire spreading to next-nearest neighbors.

Let us compare the critical behavior of the forest-fire model to the sandpile model: Here too, a separation of time scales is required: Sand must be added slowly com-

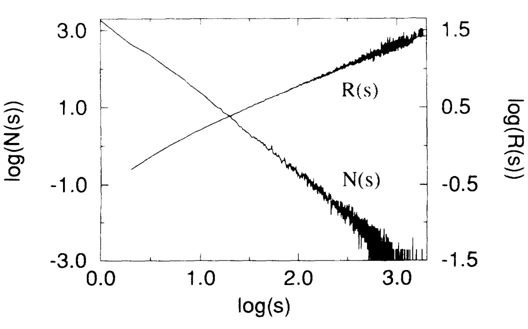


FIG. 2. Mean number of clusters and mean cluster radius as functions of the cluster size for $f/p=1/70$ and $d=2$.

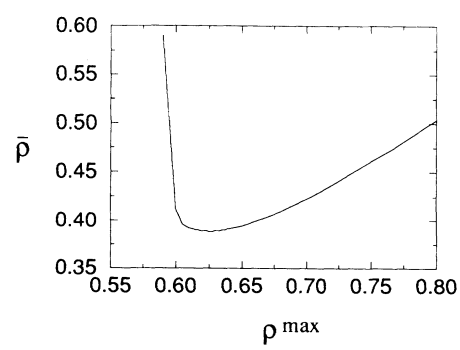


FIG. 3. Mean forest density as a function of maximum forest density for $d=2$.

pared to the lifetime of an avalanche. This corresponds to our condition that a forest cluster burns down rapidly. The power-law distribution of the avalanches in the sandpile model is a consequence of the local conservation of sand particles. In our forest-fire model, the power-law distribution of forest clusters is a consequence of a second separation of time scales $p^{-1} \ll f^{-1}$, which guarantees that a large amount of energy is deposited in the system between two lightning occurrences and consequently a large number of trees is destroyed by a lightning stroke.

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