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From Geometry to Algebra

From Curved Space to Spin Model

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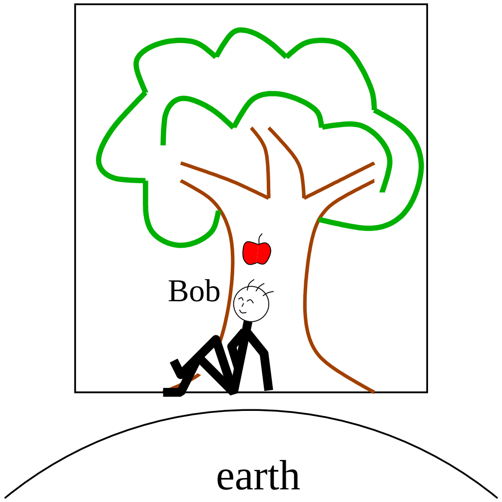
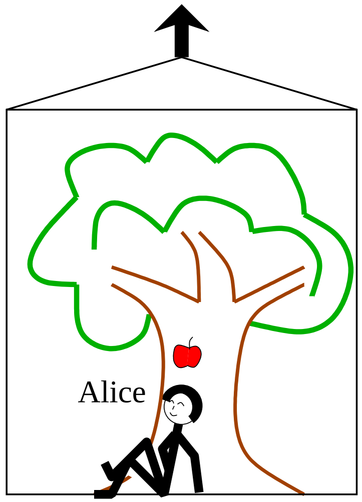
Einstein's theory of gravity

Einstein understood the true meaning of Galileo's discovery:

All matter fall with the same acceleration.



Gravity



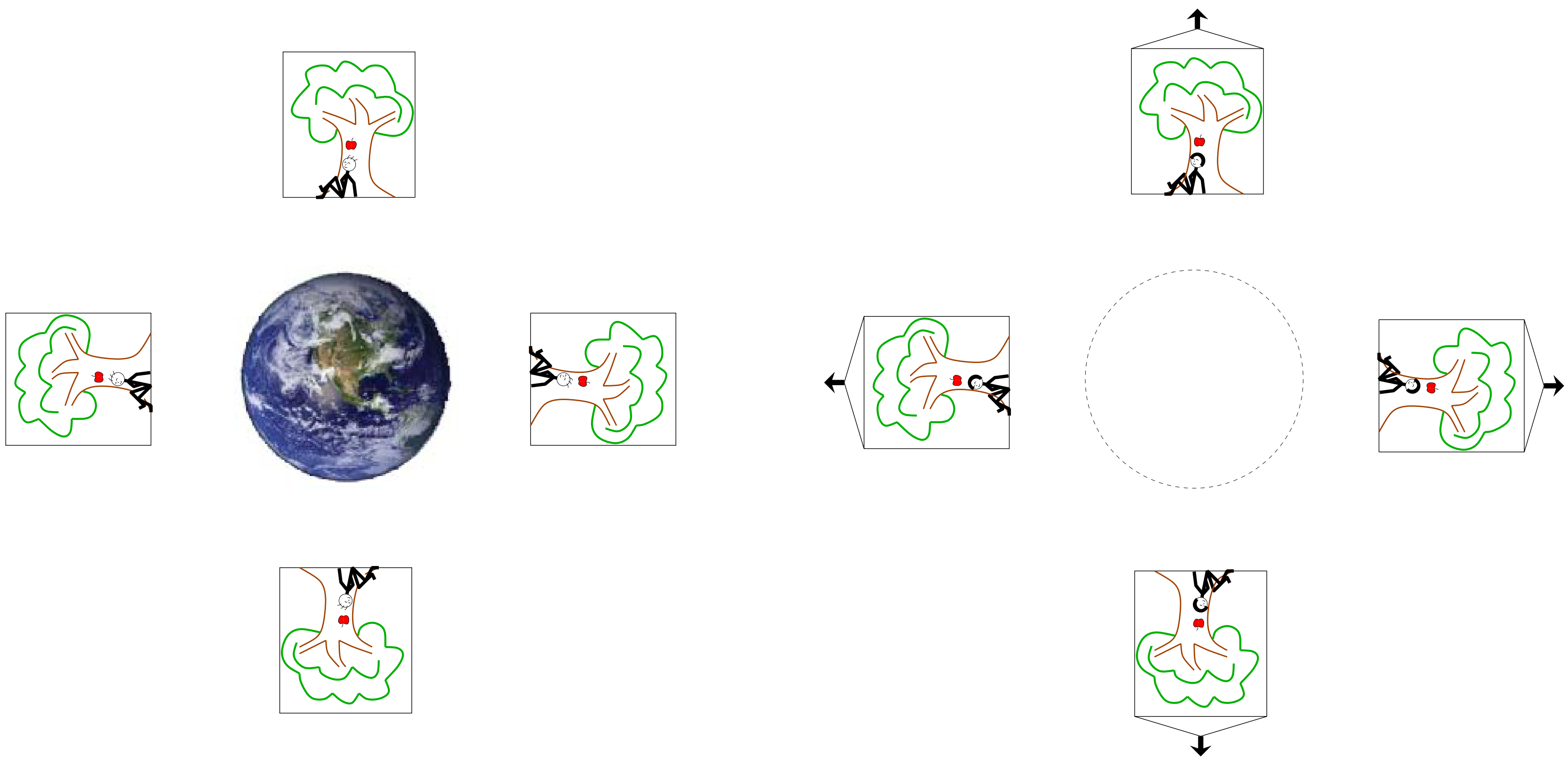
Acceleration

Non-linear coordinate trans.

$$x' = x + \frac{g}{2}t^2$$

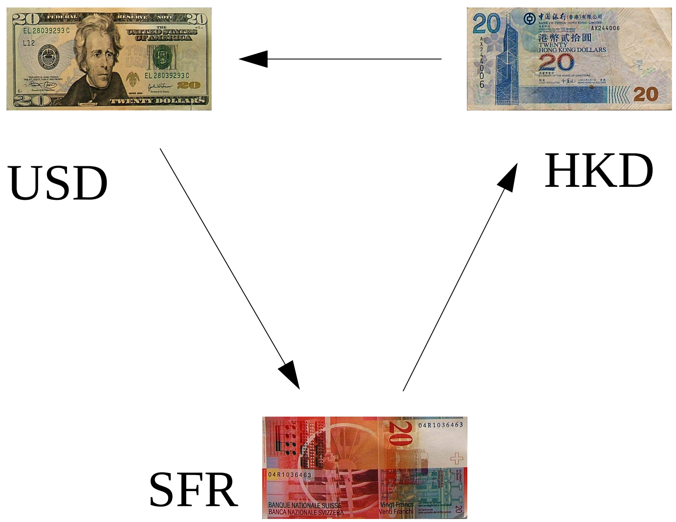
Geometry

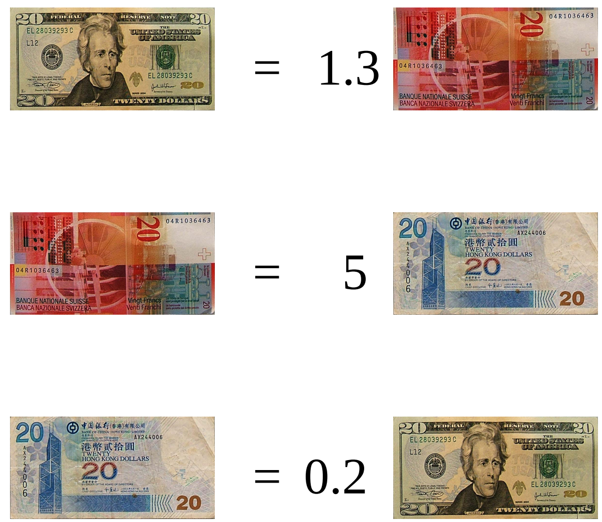
Gravity = non-linear coordinate-trans. = curved space



Weyl's geometric way to understand electromagnetism

- Relativity of coordinates $\longrightarrow$ curved space $\longrightarrow$ gravity
- Relativity of units $\longrightarrow$ distorted unit-system $\longrightarrow$ electromagnetism





$$\text{Unit}(y^\mu) = \text{Unit}(x^\mu)[1 + (y^\mu - x^\mu)A_\mu]$$

A_μ is the vector potential for electromagnetic field, which is called the *gauge* field.

- Distortion in unit system $= \partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow$ electric and magnetic fields $\mathbf{E} = \partial_t A_i - \partial_i A_0, \quad \mathbf{B} = \partial_i A_j - \partial_j A_i = \partial \times \mathbf{A}$.

Gauge theory is important in understanding the structure of photon.

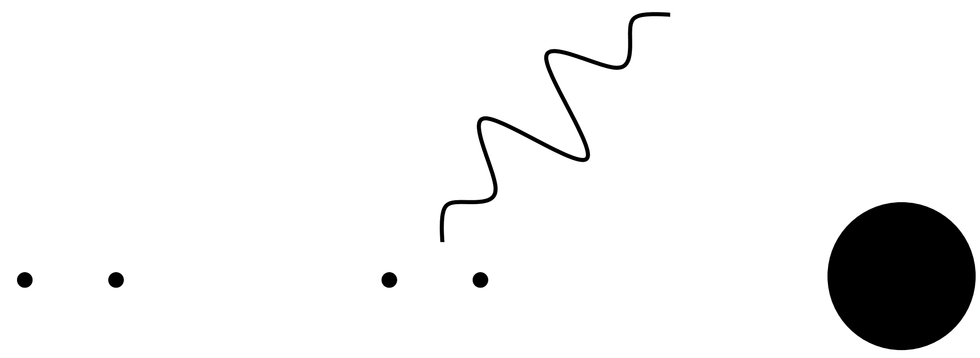
Gauge theory is also important in understanding how to make money through exchange rate (relativity of unit of money) and trading (relativity of unit of labor).

Since then, geometric way dominates our approach to understand nature

But geometric way has a fatal flaw: The foundation of geometry,

Manifold does not exist

if we combine Einstein gravity with quantum mechanics.

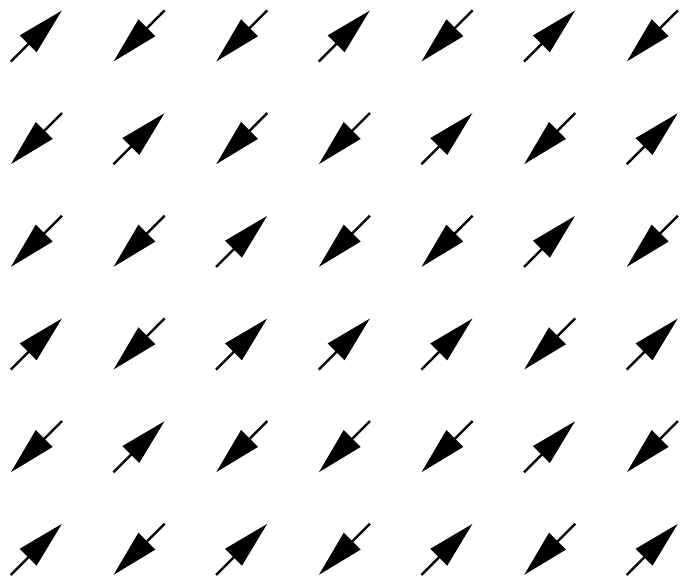


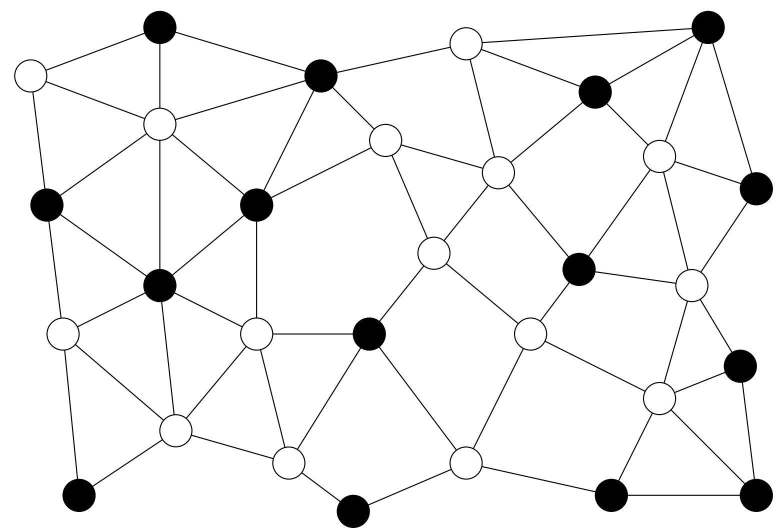
Planck length $l_P = 1.6 \times 10^{-33} cm$.

Geometry is an emergent property

This suggests that manifold and geometry may not be fundamental. The concepts of manifold/geometry may only exist at long distances. Manifold/geometry may be emergent properties from a deeper structure.

What is this deeper structure???





We try to start with lattice spin model to see if we can obtain emergent geometry at long distance.

How to study the emergence of geometry

Emergence of space:
space $\to$ curved space $\to$ propagating distortion $=$ gravitational waves $\to$ gravitons

Emergence of fiber-bundle/unit-system (gauge theory):
curved fiber-bundle/unit-system $\to$ propagating distortion $=$ light waves $\to$ photons

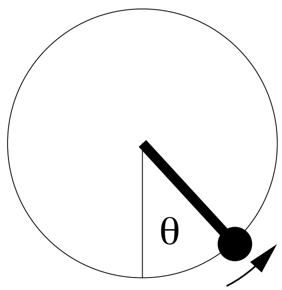
Emergence of geometry $=$ emergence of photons and gravitons

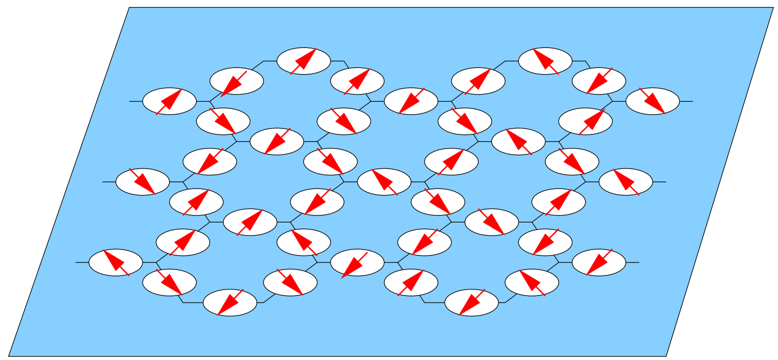
The collective motions of what quantum spin model are described by Maxwell equation and Einstein equation, which lead to phonons and gravitons.

Emergence of photons from a spin/rotor model

Rotor model on 2D-Kagome lattice $(\theta_{\mathbf{i}}, L^z_{\mathbf{i}})$:

$$H = J_1 \sum (L^z_{\mathbf{i}})^2 + J_2 \sum L^z_{\mathbf{i}} L^z_{\mathbf{j}} - J_{xy} \sum (e^{i\theta_{\mathbf{i}}} e^{-i\theta_{\mathbf{j}}} + h.c.)$$





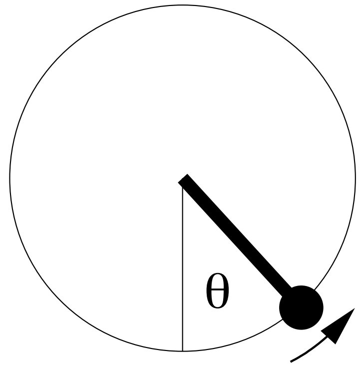
Electromagnetic wave is a partially frozen spin wave.

What is spin wave?

$$H = J_1 \sum (L^z_{\mathbf{i}})^2 + J_2 \sum L^z_{\mathbf{i}} L^z_{\mathbf{j}} - J_{xy} \sum (e^{i\theta_{\mathbf{i}}} e^{-i\theta_{\mathbf{j}}} + h.c.)$$

- Small J_1/J_{xy}, J_2/J_{xy}.
  Rotors all want to point in one direction (which can be any direction).
  $\longrightarrow$ spin wave.

Quantum rotor and quantum freeze $H = J_1(L^z)^2$



a rotor = a particle on a ring

(θ, L^z) coordinate-momentum pair. $\frac{1}{2}J_1^{-1}$ mass

At temperature T, $0 < \text{Energy} = J_1(L^z)^2 < T$,

$\Delta L^z \sim \sqrt{T/J_1}$

- Quantum rotor: $\Delta\theta \sim 1/\Delta L^z \sim \sqrt{J_1/T} \sim V_{\text{average}}/T$

- Quantum freeze: small T or large J_1 (small mass $\frac{1}{J_1}$),
  $\Delta\theta > 2\pi, \quad \Delta L^z \sim 0$
  Angular momentum L^z is quantized as integer $\to$
  constraint $L^z = 0$ and gauge invariance $\Phi_0(\theta) = \Phi_0(\theta + f)$.

Two quantum rotors

$$
\begin{aligned}
H =& J_1(L_1^z)^2 + J_1(L_2^z)^2 + 2J_2 L_1^z L_2^z \\
=& \frac{1}{2}(J_1 + J_2)(L_1^z + L_2^z)^2 + \frac{1}{2}(J_1 - J_2)(L_1^z - L_2^z)^2
\end{aligned}
$$

- $\theta_1 + \theta_2$ particle with small mass $1/(J_1 + J_2)$
  $\theta_1 - \theta_2$ particle with large mass $1/(J_1 - J_2)$
- Energy gap: for $L_1^z + L_2^z$: $\sim J_1 + J_2$, for $L_1^z - L_2^z$: $\sim J_1 - J_2$
- Partial quantum freeze when $J_1 \sim J_2$.
- Small fluctuation $\Delta(\theta_1 - \theta_2) \sim 0$, large fluctuation $\Delta(L_1^z - L_2^z) \gg 1$
  $\longrightarrow$ classical picture is valid.
- Large fluctuation $\Delta(\theta_1 + \theta_2) > 2\pi$, small fluctuation $\Delta(L_1^z + L_2^z) \sim 0$
  $\longrightarrow$
  constraint $L_1^z + L_2^z = 0$ and gauge invariance $\Phi_0(\theta_1, \theta_2) = \Phi_0(\theta_1 + f, \theta_2 +$
  $\longrightarrow$ $(\theta_1, \theta_2) \to (\theta_1 + f, \theta_2 + f)$
- Constraint $L_1^z + L_2^z = 0$ generate gauge transformation:

$$
e^{i(L_1^z + L_2^z)} : (\theta_1, \theta_2) \to (\theta_1 + f, \theta_2 + f)
$$

A lattice of quantum rotors

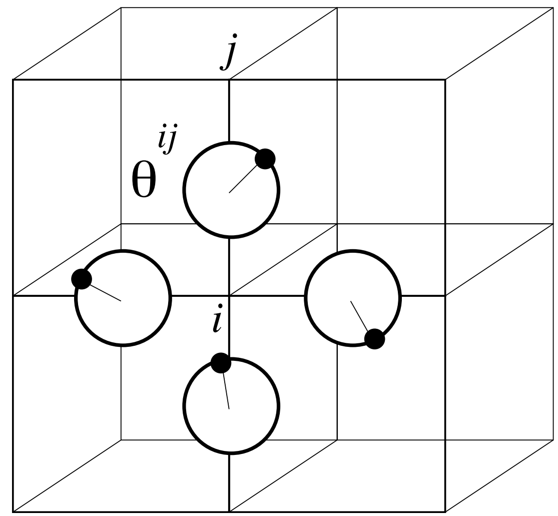
- Large $J_1/J_{xy},\ J_2/J_{xy} \rightarrow L^z_{\mathbf{i}}$ more certain and $\theta_{\mathbf{i}}$ more uncertain $\rightarrow$ quantum freeze $\rightarrow$ gapped $L^z_{\mathbf{i}} = 0$ state.
- Large $(J_1 + J_2)/J_{xy}$ and small $(J_1 - J_2)/J_{xy}$ $\rightarrow$ partial quantum freeze
- $e^{\pm i\theta_{\mathbf{i}}}$ $\rightarrow$ conversion factor between units.
  $\theta_1 - \theta_2 + \theta_3 - \theta_4 + \theta_5 - \theta_6 \neq 0$ $\rightarrow$ distortion of unit system.
- Partially quantum-frozen spin waves satisfy Maxwell equation $\rightarrow$ emergent light

A little Math: Emergence of photons

Each link: one rotor (θ^{ij}, L_{ij}).

$$\mathcal{L} = \sum_{\text{links}} L_{ij}\dot{\theta}^{ij} - U \sum_{\text{vert}} (\sum_{\text{star}} L_{ij})^2$$
$$- J \sum_{\text{links}} (L_{ij})^2 - g \sum_{\text{all-sq}} \prod_{\text{1-sq}} e^{i\theta^{ij}}$$
$$\sim \mathbf{E} \cdot \dot{\mathbf{A}} - \mathbf{E}^2 - (\partial_i A_j - \partial_j A_i)^2 + \mathbf{A}^4 + \cdots$$

where $\theta^{ij} \to \mathbf{A}$, $L_{ij} \to \mathbf{E}$,



- Three modes: helicity $h = 0, \pm 1$

  Two trans. $h = \pm 1$ modes:

  $\delta \mathbf{A} \sim \delta\theta^{ij} \ll 2\pi, \quad \delta\mathbf{E} \gg 1$

  Classical picture valid.
- The long. $h = 0$ mode: (f, π)

  $\mathbf{A} = \partial f, \quad \pi = \partial \cdot \mathbf{E}$

  $\delta\pi \approx 0, \quad \delta f \to \infty$

  Very quantum.

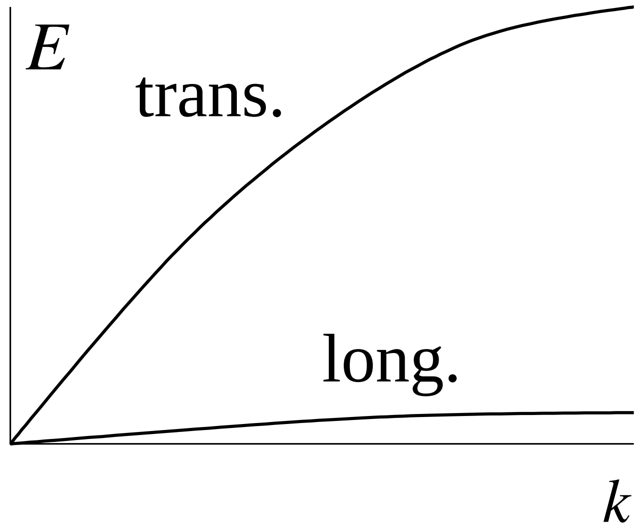
  Classical picture not valid.
- π is discrete on lattice $\to$

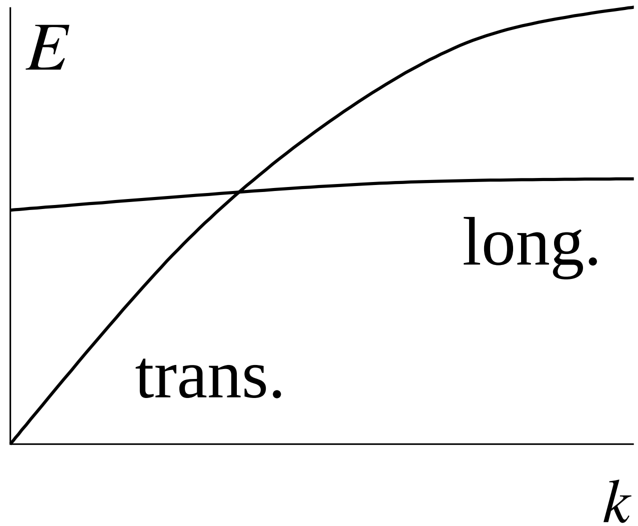
  Soft quantum modes =gap
- Weak fluctuations of π $\to$ constraint

  $\pi = \partial \cdot \mathbf{E} = 0$

  Strong fluctuations of f $\to$ gauge trans.

  $\mathbf{A} \to \mathbf{A} + \partial f$



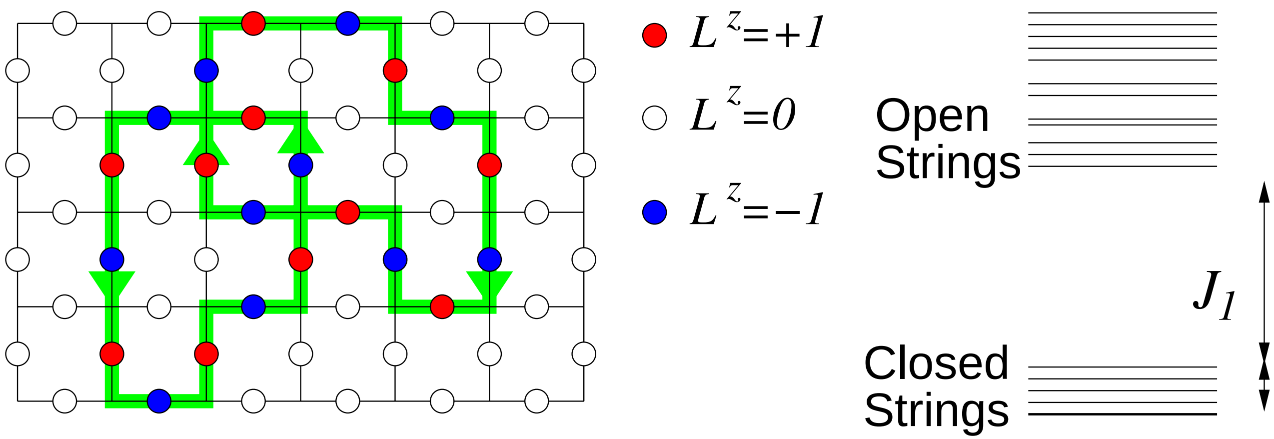


$\mathcal{L}$ + Constraint and gauge trans. = Maxwell's theory of electricity and magnetism.

Light wave = collective motion of condensed string-net

$$\mathcal{H} = U \sum_{\text{vert}} (\sum_{\text{star}} L_{ij})^2 + J \sum_{\text{links}} (L_{ij})^2 + g \sum_{\text{all-sq}} (e^{i(\theta_1 - \theta_2 + \theta_3 - \theta_4)} + h.c)$$

When $J = g = 0$, the no string state and closed string states all have zero energy:



No string state: $|0_z 0_z 0_z ...\rangle$ Closed-string state: loops of $L^z = \pm 1$

Emergence of gravitons

Each vertex: three rotors

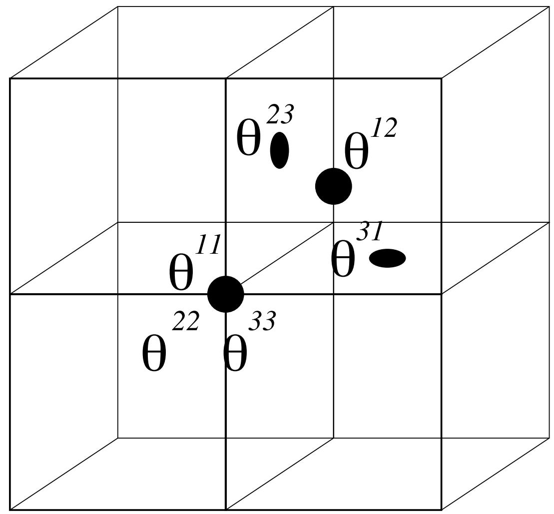
$(\theta^{aa}, L_{aa}), aa = 11, 22, 33.$

Each face: one rotor

$(\theta^{ab}, L_{ab}),\ ab = 12, 23, 31.$

$\mathcal{L} = \sum L_{ab}\dot{\theta}^{ab} -$ Complicated H

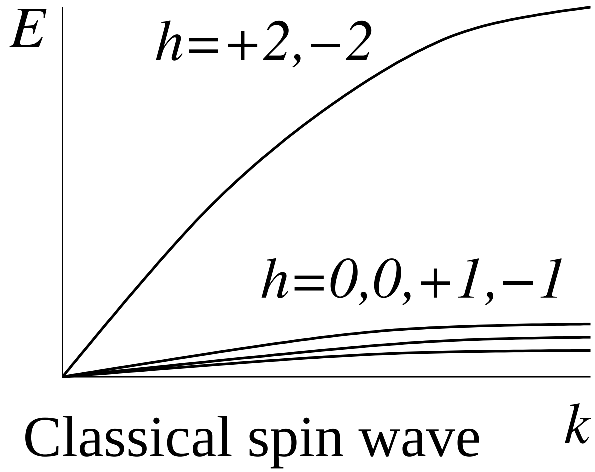
Total six modes (spin waves) with helicity $0, 0, \pm 1, \pm 2$



$$\mathcal{L} = L_{ab}\dot{\theta}^{ab} - \left[(L_{ab})^2 - \frac{(L_{aa})^2}{2}\right] - \theta^{ab}R^{ab} - (\partial_a L_{ab})^2 - (R^{aa})^2 + \cdots$$

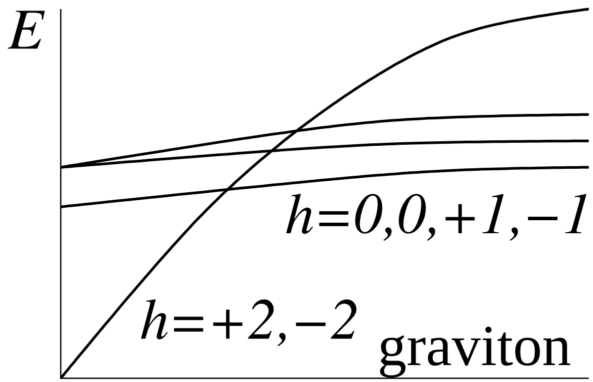
where $R^{ab} = \epsilon^{ahc}\epsilon^{bdg}\partial_h\partial_d\theta^{gc}$.

The helicity ± 2 modes are classical and the classical picture is valid.



Classical spin wave

- $h = 0, \pm 1$ modes are described by (θ^a, L_a):
  $\theta^{ab} = \partial_a \theta^b + \partial_b \theta^a$, $L_a = \partial_b L_{ab}$
  Quantum fluctuations: $\delta L_a = 0$, $\quad \delta\theta^a = \infty$
  L_a is discrete $\to$ gap.
  Constraint and gauge transformation:
  $L_a = \partial_b L_{ab} = 0$, $\theta^{ab} \to \theta^{ab} + \partial_a \theta^b + \partial_b \theta^a$
- A $h = 0$ mode is described by (θ, L):
  $L_{ab} = (\delta_{ab}\partial^2 - \partial_a\partial_b)L$, $\theta = (\delta_{ab}\partial^2 - \partial_a\partial_b)\theta^{ab} = R^{aa}$
  Quantum fluctuations: $\delta\theta = 0$, $\quad \delta L = \infty$
  To have gap, θ^{ab} discretized and L_{ab} compactified:
  $L \sim L + n_G$, $\Delta\theta^{ab} = 2\pi/n_G$
  Constraint and gauge transformation:
  $(\delta_{ab}\partial^2 - \partial_a\partial_b)\theta^{ab} = 0$, $L_{ab} \to L_{ab} + (\delta_{ab}\partial^2 - \partial_a\partial_b)L$



Partial quantum freeze

Low energy effective theory

Symmetric tensor field theory (θ^{ij}, L_{ij})

$$\mathcal{L} = L_{ij}\partial_0\theta^{ij} - J_1 L_{ij}L_{ij} - J_2 L_{ii}L_{jj} \\ - g_1\partial_k\theta^{ij}\partial_k\theta^{ij} - g_2\partial_i\theta^{ij}\partial_k\theta^{kj} - g_3\partial_i\theta^{ij}\partial_j\theta^{kk}.$$

Helicity modes $h = 0, 0, \pm1, \pm2$.

The vector constraint

$$\partial_i L_{ij} = 0$$

which generates gauge transformations

$$\theta^{ij} \to W\theta^{ij}W^\dagger = \theta^{ij} + \partial_i f_j + \partial_j f_i$$

The gauge invariant field is a symmetric tensor field

$$R^{ij} = R^{ji} = \epsilon^{imk}\epsilon^{jln}\partial_m\partial_l\theta^{nk}$$

Gauge inv. Lagrangian

$$\mathcal{L} = L_{ij}\partial_0\theta^{ij} - \alpha(L_{ij})^2 - \beta(L_{ii})^2 - \gamma(R^{ij})^2 - \lambda(R^{ii})^2$$

The constraints remove $h = 0, \pm1$ modes. $h = 0, \pm2$ has $\omega \sim k^2$.

The scaler constraint

$$R^{ii} = (\delta_{ij}\partial^2 - \partial_i\partial_j)\theta^{ij} = 0$$

and the corresponding gauge trans.

$$L_{ij} \to L_{ij} - (\delta_{ij}\partial^2 - \partial_i\partial_j)f$$

Gauge inv. Lagrangian density

$$\mathcal{L} = L_{ij}\partial_0\theta^{ij} - -\frac{J}{2}[(L_{ij})^2 - \frac{1}{2}(L^i_i)^2] - \frac{g}{2}(R_{ij})^2$$

Only $h = \pm 2$ modes with $\omega \sim k^4$.

Gauge inv. Lagrangian $L = \int d^x \mathcal{L}$:

$$\mathcal{L} = L_{ij}\partial_0\theta^{ij} - -\frac{J}{2}[(L_{ij})^2 - \frac{1}{2}(L^i_i)^2] - \frac{g}{2}\theta^{ij}R^{ij}$$

Only $h = \pm 2$ modes with $\omega \sim k$.

$\mathcal{L}$ + Constraint and gauge trans. = linearized Einstein gravity with $\theta^{ij} \sim g^{ij} - \delta^{ij}$

Geometry emerges from Algebra

Lattice model = Algebra, photons/gravitons = Geometry
Photons/gravitons emerge from lattice model $\to$
Geometry emerge from Algebra

Local bosonic/spin model provides a unified origin of:
(a) Gauge interaction (electromagnetism)
(b) Gravity (linearized, so far)
(c) Fermi statistics

Gauge interaction, gravity, and Fermi statistics are properties of our vacuum $\to$
Ether (our vacuum) = A local bosonic/spin model