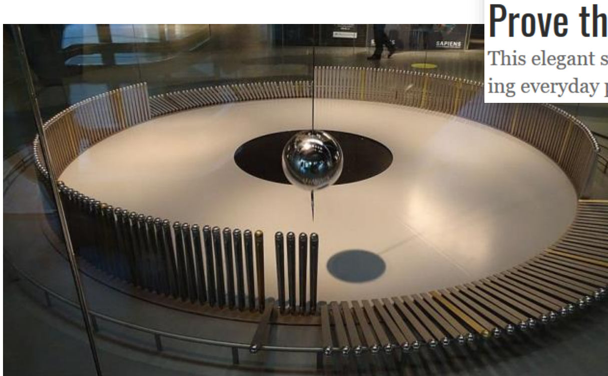
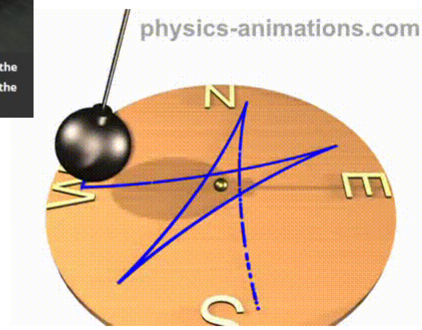
PHY131H1F - Hour 30

How Does Foucault's Pendulum Prove the Earth Rotates?

This elegant scientific demonstration has been delighting everyday people for nearly 200 years


A Foucault-inspired pendulum apparatus at the CosmoCaixa museum in Barcelona, Spain. As the path of the pendulum shifts due to Earth's rotation, the bob will gradually knock over all of the vertical rods around the circle's circumference. (Wikimedia Commons)



https://www.smithsonianmag.com/smithsonian-institution/how-does-foucaults-pendulum-prove-earth-rotates-180968024/

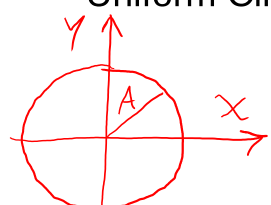
Today:

10.3 Dynamics of Simple Harmonic Motion

10.4 Energy in Simple Harmonic Motion

10.5 The Simple Pendulum

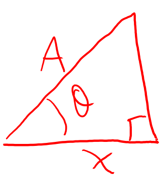
Uniform Circular Motion



radius $= A$, constant ω

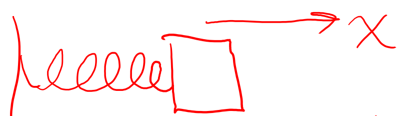
$\theta_f = \theta_i + \omega t$, $\theta_i = 0$

$\omega = \frac{2\pi}{T} \Rightarrow \theta = \frac{2\pi}{T} t$



$x = A\cos\left(\frac{2\pi}{T} t\right)$

Simple Harmonic Motion



$a_x = -\left(\frac{4\pi^2}{T}\right) x$

"Trial Solution":

$x = A\cos\left(\frac{2\pi}{T} t\right)$

where $A =$ constant.

Mathematically, S.H.M. is identical to one component of uniform circular motion!

What are v_{max} and a_{max}?

- If the position function is given by:

$$x = A\cos\left(\frac{2\pi}{T}t\right)$$

$x_{max} = A$

- Then the velocity and acceleration functions are:

$$v_x = -\left(\frac{2\pi}{T}\right)A\sin\left(\frac{2\pi}{T}t\right)$$

$$a_x = -\left(\frac{2\pi}{T}\right)^2 A\cos\left(\frac{2\pi}{T}t\right)$$

$v_{max} = \frac{2\pi A}{T}$

$a_{max} = \frac{4\pi^2 A}{T^2}$

- A is the amplitude of the vibration; T is the period of the vibration.

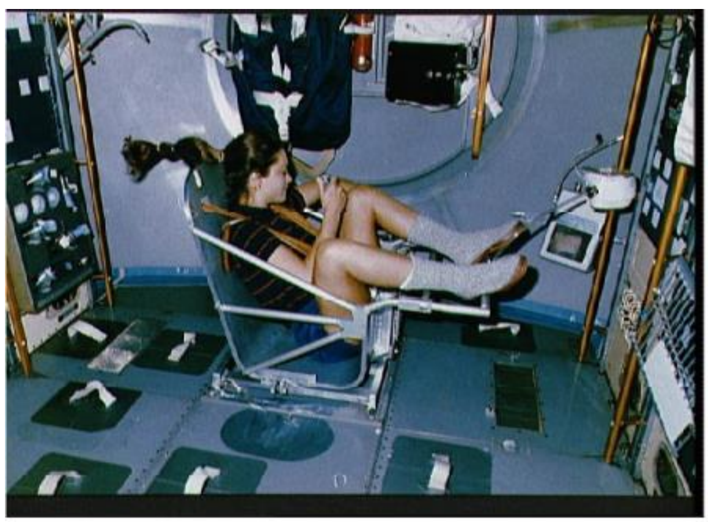
Learning Catalytics question

The Body Mass Measurement Device chair (mass = 32 kg) has a vibrational period of 1.2 s when empty. When an astronaut sits on the chair, what will be the vibrational period?

A. More than 1.2 s

B. Less than 1.2 s

C. 1.2 s

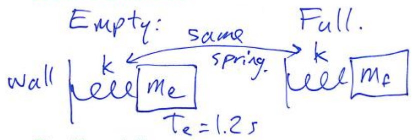


Astronaut Tamara Jernigan (Shuttle Columbia during STS-40, 5-14 June 1991) is weighed into space. This is the first type of "chair pose space." As the chair moves forward and backward, a calculation of the weight counter how astronaut retards the movement of the chair.

[Doc cam notes]

- The Body Mass Measurement Device chair (mass m_e = 32 kg) has a vibrational period of 1.2 s when empty. When an astronaut sits on the chair, the period changes to 2.1 s. [redacted] Determine the mass of the astronaut.

Sketch and translate



m_e = mass of empty chair.
m_f = mass of full chair
$m_f = m_e + m_a$

Simplify and diagram

Use $T_e = 2\pi\sqrt{\frac{m_e}{k}}$, solve for k

m_a = mass of astronaut = ?

$T_f = 2\pi\sqrt{\frac{m_f}{k}}$, solve for m_f.

Use $m_a = m_f - m_e$

Represent mathematically

$$\frac{T_e^2}{4\pi^2} = \frac{m_e}{k}, \quad k = \frac{m_e 4\pi^2}{T_e^2} = \frac{32(4)\pi^2}{1.2^2} = 877.3 \text{ N/m}$$

$$\frac{T_f^2}{4\pi^2} = \frac{m_f}{k}, \quad m_f = \frac{T_f^2}{4\pi^2}\cdot k = \frac{2.1^2}{4\pi^2}\cdot 877.3$$

Solve and Evaluate

$m_f = 98$ kg. $\quad m_a = 98 - 32 = 66$ kg.

2.2 pounds/kg: $m_a \approx 150$ pounds. ✓ ok.

What is the total energy of a mass vibrating on a spring, as a function of time?

Sketch and translate

Assume horizontal spring. Set Amplitude = A.

$$x = A\cos\left(\frac{2\pi}{T}t\right)$$

From Aid Sheet: $v = -\frac{2\pi A}{T}\sin\left(\frac{2\pi}{T}t\right)$

Simplify and diagram

$$E = K + U_s$$
$$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Represent mathematically

$$E = \frac{1}{2}m\left[\frac{-2\pi A}{T}\sin\left(\frac{2\pi}{T}t\right)\right]^2 + \frac{1}{2}k\left[A\cos\left(\frac{2\pi}{T}t\right)\right]^2$$

↖ Recall: $T = 2\pi\sqrt{\frac{m}{k}}$

$$E = \frac{m}{2}\left[\frac{-2\pi}{2\pi}\sqrt{\frac{k}{m}}A\sin\left(\frac{2\pi}{T}\right)\right]^2 + \frac{k}{2}\left[A\cos\left(\frac{2\pi}{T}t\right)\right]^2$$

$$E = \frac{m}{2}\left(\frac{k}{m}\right)A^2\left[\sin\left(\frac{2\pi}{T}t\right)\right]^2 + \frac{k}{2}A^2\left[\cos\left(\frac{2\pi}{T}t\right)\right]^2$$

Solve and Evaluate

$$E = \frac{1}{2}kA^2\left[\sin^2\left(\frac{2\pi t}{T}\right) + \cos^2\left(\frac{2\pi t}{T}\right)\right] \quad (= 1)$$

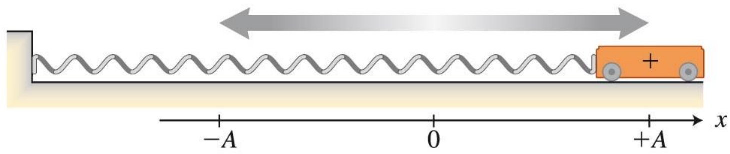
Recall Trig Identity:

$$\sin^2\theta + \cos^2\theta = 1$$

$$E = \frac{1}{2}kA^2$$

↑ !! Constant!

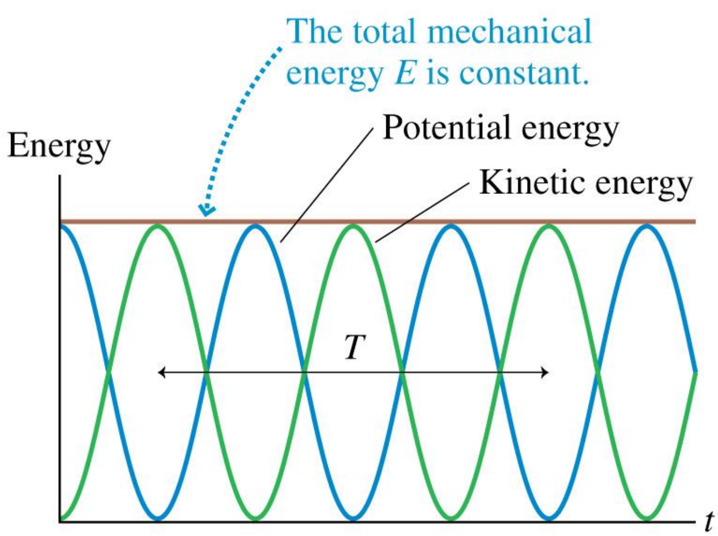
This is a surprise, since K & U_s are constantly varying, but their sum is constant.



Energy of a mass on a spring

Clock reading t	Displacement	Elastic potential energy U_s	Kinetic energy K	Total energy U_{tot}
$\frac{1}{2}T$	$-A$	$\frac{1}{2}kA^2$	0	$U_{tot} = \frac{1}{2}kA^2$
$\frac{1}{4}T$	0	0	$\frac{1}{2}mv_{max}{}^2$	$U_{tot} = \frac{1}{2}mv_{max}{}^2$
$\frac{3}{4}T$	0	0	$\frac{1}{2}mv_{max}{}^2$	
0	A	$\frac{1}{2}kA^2$	0	$U_{tot} = \frac{1}{2}kA^2$
T	A	$\frac{1}{2}kA^2$	0	

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}m(v_{max})^2 \quad \text{(conservation of energy)}$$



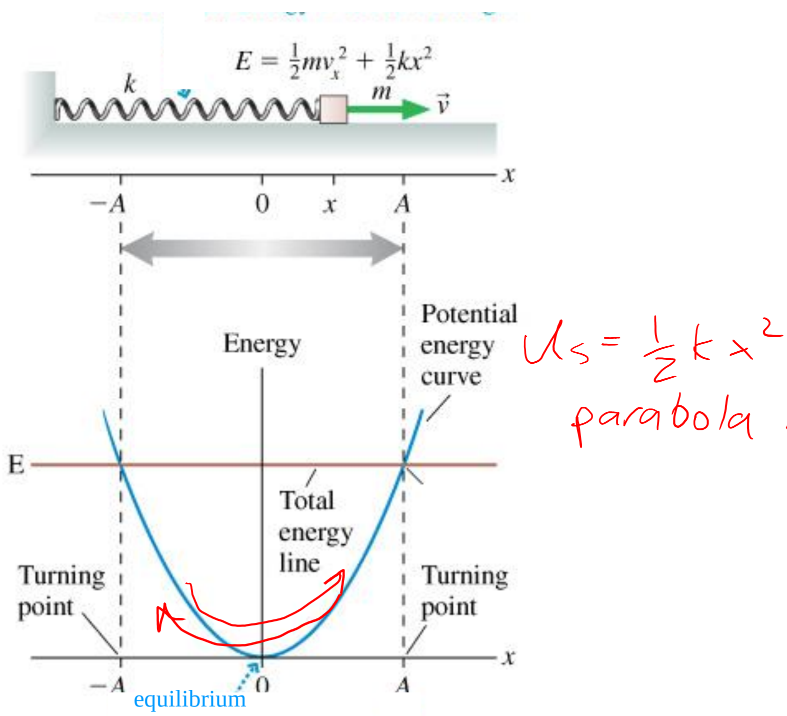
Relationship between the amplitude of the vibration and the cart's maximum speed

- The equation $U = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ can be rearranged to give:

$$v_{max} = \sqrt{\frac{k}{m}}A$$

- This makes sense conceptually:
  - When the mass of the cart is large, it should move slowly.
  - If the spring is stiff, the cart will move more rapidly.

> **TIP** In the above discussion we neglected the interactions of the system with the surface of the track and with the air. These would both do negative work on the system and gradually decrease its energy, eventually bringing the vibrating system to rest.



Learning Catalytics question

By what factor must we increase the amplitude of vibration of an object and the end of a spring in order to double its **maximum speed** during a vibration?

A. $\sqrt{2}$
B. 2
C. 4

$$v_{max} = \frac{2\pi}{T} A \quad , T = \text{fixed}$$

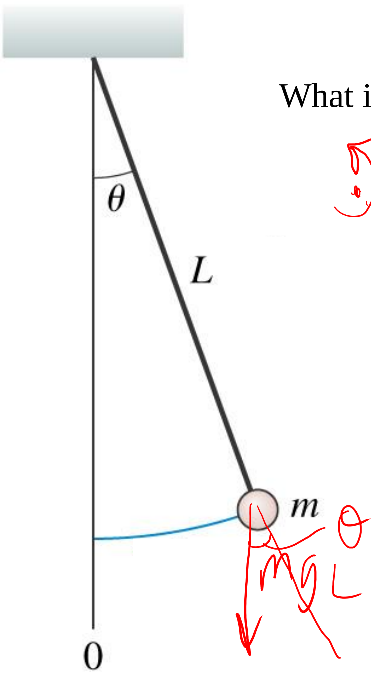
$$T = 2\pi\sqrt{\frac{m}{k}}$$

Learning Catalytics question

By what factor must we increase the amplitude of vibration of an object and the end of a spring in order to double the **total energy** of the system?

A. $\sqrt{2}$
B. 2
C. 4

$$E = \frac{1}{2} k A^2$$



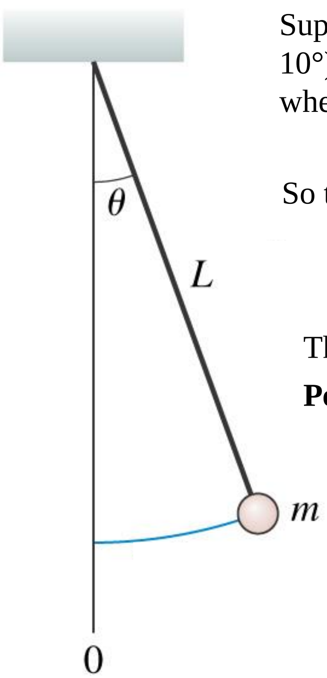
What is the net torque on this pendulum?

Force diagram:

Torque from tension is zero (acts toward rotation axis).

$\Sigma\tau$ is all from gravity.

$$\Sigma\tau = mgL\sin\theta$$



Suppose we restrict a pendulum's oscillations to small angles (< 10°). Then we may use the **small angle approximation** $\sin\theta \approx \theta$, where θ is measured in radians. The net torque on the mass is

$$\Sigma\tau = I\alpha = -mgL\theta$$

So the simple harmonic motion equation for θ as a function of time is:

$$\alpha = -\frac{mgL}{I}\theta$$

The solution to this is $\theta = A\cos\left(\frac{2\pi}{T}t\right)$, where A is a constant, and the **Period** of oscillations (in seconds) is:

$$T = 2\pi\sqrt{\frac{I}{mgL}}$$

But the rotational inertia of a point mass m a distance L from the rotation axis is $I = mL^2$, so

$$T = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}$$

Learning Catalytics Question

Two pendula have the same length, but different mass. The force of gravity, $F=mg$, is larger for the larger mass. Which will have the longer period?

A. the larger mass

B. the smaller mass

C. neither

$T = 2\pi\sqrt{\frac{L}{g}}$

mass doesn't matter!

Mass on Spring versus Pendulum

	Mass on a Spring	Pendulum
Condition for S.H.M.	Small oscillations (Hooke's Law is obeyed)	Small angles
Period	$T = 2\pi\sqrt{\frac{m}{k}}$	$T = 2\pi\sqrt{\frac{L}{g}}$

[Demonstration]

Learning Catalytics Question

A person swings on a swing. When the person sits still, the swing oscillates back and forth at its natural frequency. If, instead, the person stands on the swing, the natural frequency of the swing is

A. greater

B. the same

C. smaller

sitting L_1

standing L_2

$L_2 < L_1 \Rightarrow T = 2\pi\sqrt{L/g}$ less

$f = \frac{1}{T}$ higher