1. Show using Lorentz transformations that the spacetime interval is the same in all reference frames. In the "unprimed" frame

$$\begin{aligned}(\Delta s)^2 &= c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= c^2(t_1 - t_0)^2 - (x_1 - x_0)^2 - (y_1 - y_0)^2 - (z_1 - z_0)^2\end{aligned}$$

If we assume a boost in the x direction, the Lorentz transformation gives

$$\begin{aligned}t'_1 &= \gamma(t_1 - \frac{vx_1}{c^2}), \quad x'_1 = \gamma(x_1 - vt_1), \quad y'_1 = y_1, \quad z'_1 = z_1 \\ t'_0 &= \gamma(t_0 - \frac{vx_0}{c^2}), \quad x'_0 = \gamma(x_0 - vt_0), \quad y'_0 = y_0, \quad z'_0 = z_0\end{aligned}$$

Then in the "primed" frame:

$$\begin{aligned}(\Delta t')^2 &= (\gamma(t_1 - t_0) - \frac{v\gamma}{c^2}(x_1 - x_0))^2 \\ &= \gamma^2(t_1 - t_0)^2 + \frac{v^2\gamma^2}{c^4}(x_1 - x_0)^2 - 2\frac{\gamma^2 v}{c^2}(t_1 - t_0)(x_1 - x_0)\end{aligned}$$

$$\begin{aligned}(x'_1 - x'_0)^2 &= (\gamma(x_1 - x_0) - v\gamma(t_1 - t_0))^2 \\ &= \gamma^2(x_1 - x_0)^2 + \gamma^2 v^2(t_1 - t_0)^2 - 2\gamma^2 v(x_1 - x_0)(t_1 - t_0) \\ (\Delta z')^2 &= (\Delta z)^2 \\ (\Delta y')^2 &= (\Delta y)^2.\end{aligned}$$

The spacetime interval is, in the "primed" frame:

$$\begin{aligned}(\Delta s')^2 &= c^2(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \\ &= c^2\gamma^2(t_1 - t_0)^2 + \frac{v^2\gamma^2}{c^2}(x_1 - x_0)^2 - 2\gamma^2 v(t_1 - t_0)(x_1 - x_0) \\ &\quad - \gamma^2(x_1 - x_0)^2 - \gamma^2 v^2(t_1 - t_0)^2 + 2\gamma^2 v(x_1 - x_0)(t_1 - t_0) \\ &\quad - (\Delta y)^2 - (\Delta z)^2 \\ &= c^2(\Delta t)^2\gamma^2(1 - \frac{v^2}{c^2}) - (\Delta x)^2(1 - \frac{v^2}{c^2})\gamma^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= (\Delta s)^2\end{aligned}$$

2. An object has a velocity described by

$$v(t) = at^4 + bt + c$$

Find the expression for the objects position $x(t)$ and acceleration $a(t)$, as a function of time assuming $x(t = 1) = 0$.
Acceleration is found by differentiation.

$$a(t) = \frac{dv(t)}{dt} = 4at^3 + b$$

Position is found by integrating.

$$x(t) = \int v(t) = \frac{at^5}{5} + \frac{bt^2}{2} + ct + C$$

Knowing $x(1) = 0$ we can solve for C.

$$x(1) = 0 = \frac{a}{4} + \frac{b}{2} + c + C$$
$$C = -\frac{a}{4} - \frac{b}{2} - c$$

So $x(t) = \frac{at^5}{5} + \frac{bt^2}{2} + ct - \frac{a}{4} - \frac{b}{2} - c$.

3. A boater is exiting a narrow river and perpendicularly enters a channel 100 m across that has a constant water velocity of 5 m/s parallel to the shore . If the boat can maintain an acceleration of 2 m/s^2, with a starting velocity of 1 m/s and is pointed perpendicular from the shore for the whole journey, how far downstream with respect to where he started does the boater end up when he reaches the other side?
The equations of motion are:

$$y = 2t + \frac{1}{2}t^2$$
$$x = 5t$$

Solve when $y = 100$:

$$\frac{1}{2}t^2 + 2t - 100 = 0.$$

Use the quadratic equation and take the positive root:

$$t = \frac{-2 + \sqrt{2^2 - 4(\frac{1}{2})(-100)}}{2(\frac{1}{2})} = 12.28\text{s}.$$

Plug into x:

$$x = 5(12.28) = 61.4\text{m}$$

which is the final position downstream.