Practice Problem Set 5
Solutions

1 Wolfson 22.57

The essence of this problem is to divide the annulus into small differential charge elements, find the contribution of a single element to the potential, then integrate over the distribution to find the total potential. To start, first consider the uniform surface charge density of the annulus, σ, which can be written in terms of the total charge of the annulus, Q, and the area of the annulus, A. Since the density is uniform, it can also be written in terms of a differential amount of charge, dq, and a differential amount of area of a ring on the annulus, $dA = 2\pi r dr$

$$\sigma = \frac{Q}{A} = \frac{dq}{2\pi r dr}$$

where r is the radius of the ring. Recall now Equation 22.8 from Example 22.6, which describes the potential due to a ring of charge of radius r at an arbitrary point x

$$V = \frac{kQ}{\sqrt{r^2 + x^2}}$$

Therefore the potential from a differential ring on the annulus can be written as

$$dV = k\frac{dq}{\sqrt{r^2 + x^2}} = k\frac{2\pi\sigma r}{\sqrt{r^2 + x^2}}dr$$

Notice that the integral is only dependent on radius, thus we must integrate from the outer radius of the annulus, b, to the inner radius of the annulus, a

$$V = \int_a^b dV = 2\pi k\sigma \int_a^b \frac{r}{\sqrt{r^2 + x^2}}dr = 2\pi k\sigma \left[\sqrt{r^2 + x^2}\right]_a^b$$

$$\boxed{V = 2\pi k\sigma \left[\sqrt{r^2 + b^2} - \sqrt{r^2 + a^2}\right]}$$

2 Energy Stored in a Capacitor

a)

The initial stored energy in the capacitor is given by Equation 23.3 from the textbook:

$$\boxed{U = \frac{CV^2}{2} = \frac{(5.0 \times 10^{-12}\text{F})(100\text{V})^2}{2} = 2.5 \times 10^{-8}\text{J.}}$$

b)

The following equation describes the relationship between plate separation, d, and capacitance, C, for a parallel-plate capacitor (Example 23.1 in the textbook).

$$C = \epsilon_0 \frac{A}{d}$$

where ϵ_0 is the permittivity of free space ($\epsilon_0 = 8.9 \times 10^{-12} \mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2}$) and A is the area of the plate. Therefore, when the plate separation is doubled, the capacitance is halved to 2.5pF, due to their inverse relationship. If the battery is disconnected ($V = 0$) prior to moving the plates, then this process takes place at constant charge Q. We must now define an energy equation that is not proportional to potential. Recall that potential can be defined in terms of charge (recall that this is an arbitrary charge) and capacitance $V = Q/C$. Substituting this into the above energy equation gives:

$$U = \frac{Q^2}{2C} \propto \frac{1}{C}$$

From this formula, we can see that energy is proportional to the inverse of capacitance. Thus, halving the capacitance results in a doubling of energy $\boxed{U = 5.0 \times 10^{-8} \mathrm{J}}$. The increased energy of the capacitor is accounted for by the work done in pulling the capacitor plates apart (since these plates are oppositely charged and thus attract each other).

c)

If the battery is left connected, then the capacitance is still halved, but now the process takes place at constant voltage V, instead of at a constant charge Q. Therefore our original energy equation is valid

$$U = \frac{CV^2}{2} \propto C$$

Thus halving the capacitance results in a reduction of energy by half. The new energy is $\boxed{U = 1.3 \times 10^{-8} \mathrm{J}}$. The energy lost by the capacitor is given to the battery (re-charging the battery). The work done in pulling the plates apart is also given to the battery.

3 Uniformly Charged Spherical Shell

Recall that the electric field due to a spherical conducting shell is given by

$$E = k\frac{Q}{R^2}$$

where R is the radius of the shell. Recall as well the equation for the energy of the field

$$U = \frac{\epsilon_0}{2}\int E^2 dV = U = \frac{\epsilon_0}{2}\int \left(k\frac{Q}{R^2}\right)^2 dV = \frac{\epsilon_0}{2}\int \frac{k^2Q^2}{R^4} dV$$

Since the shell is set at a radius R, the differential volume element can be turned into a differential area element: $dV = RdA$. Thus

$$U = \frac{\epsilon_0}{2}\int \frac{k^2Q^2}{R^4} RdA = \frac{\epsilon_0}{2}\frac{k^2Q^2}{R^3}\int dA = \frac{\epsilon_0}{2}\frac{k^2Q^2}{R^3} 4\pi R^2 = 2\epsilon_0 \pi \frac{k^2Q^2}{R}$$

where the intergral over the differential area element is simply the surface area of the layer. Letting $k = 1/4\pi\epsilon_0$ results in the final expression

$$\boxed{U = \frac{kQ^2}{2R}}$$