Practice Problem Set 4 - Solutions

1. Wolfson 21.74

The hint alludes to the fact that two interpenetrating charge densities, one ρ and the other $-\rho$, are electrically neutral and equivalent—in terms of static field calculations—to having no charge at all.[1] Then, the superposition principle can be used to find the net electric field from the individual fields of each interpenetrating part.

Our building blocks are uniformly charged balls, and we are interested in the electric fields inside such balls. Gauss' law can be used to easily find this electric field; this is done in Example 21.3 of the textbook. There, it is concluded that the field inside the ball is

$$\vec{E} = \underbrace{Q}_{\frac{4}{3}\pi R^3 \rho} \frac{\vec{r}}{4\pi\epsilon_0 R^3} = \frac{\rho\vec{r}}{3\pi\epsilon_0}, \tag{1}$$

where $\vec{r}$ is the position measured from the center of the sphere.

Hence, we approach the problem by considering two interpenetrating spheres: one centered at $\vec{0}$ with uniform charge density ρ and the other centered at $\vec{a}$ (with $|\vec{a}| = R/2$) with uniform charge density $-\rho$. As we just argued, the first produces an electric field

$$\vec{E}_1 = \frac{\rho\vec{r}}{3\epsilon_0} \qquad (|\vec{r}| \leq R) \tag{2}$$

and the second produces an electric field

$$\vec{E}_2 = \frac{(-\rho)(\vec{r} - \vec{a})}{3\epsilon_0} \qquad \left(|\vec{r}| \leq \frac{R}{2}\right). \tag{3}$$

The net field in the region common to both spheres is

$$\vec{E}_{\text{net}} = \vec{E}_1 + \vec{E}_2 = \frac{\rho\vec{a}}{3\epsilon_0}. \tag{4}$$

[1] In some cases, electrons and protons in neutral matter can be thought of as a real-world example of this situation.

This field has magnitude

$$|\vec{E}_{\text{net}}| = \frac{\rho|\vec{a}|}{3\epsilon_0} = \frac{\rho R}{6\epsilon_0}, \tag{5}$$

and its direction is given by the vector $\vec{a}$, which extends from the center of the large sphere to that of the small sphere.

2. Flatland question

In order to apply Gauss' law to solve this problem, we must realize that in 2D, "surfaces" that enclose charge are simply closed curves, and surface integrals are simply line integrals:

$$\oint \mathrm{d}\vec{A} \cdot \vec{E} \rightarrow \oint_C \mathrm{d}l \; \hat{n} \cdot \vec{E}, \tag{6}$$

where $\mathrm{d}l$ is the length element of the curve C and $\hat{n}$ is the outer-normal vector to the curve.

(a) By applying Gauss' law, we will find that Coulomb's law *is not* the same in 2D as it is in 3D.

The problem at hand has circular, or *polar* symmetry, which is analogous to spherical symmetry in 3D. Polar symmetry requires the field to point purely in the radial direction, and requires its magnitude to depend only on on the radial coordinate. We consider a Gaussian circle of radius r centered at the point charge (Fig. 1).

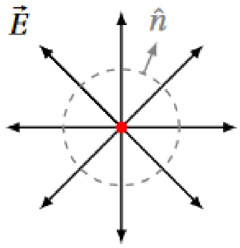

Figure 1: Point charge (in red) in 2D with field lines (solid lines) emanating from it. The Gaussian circle (dashed) of radius r and the outer-normal vector are shown as well.

The flux through the Gaussian cricle is

$$\Phi = \oint_C \mathrm{d}l\ \hat{n}\cdot\vec{E} = |\vec{E}| \oint_C \mathrm{d}l = 2\pi r|\vec{E}|, \tag{7}$$

where $\hat{n}\cdot\vec{E} = |\vec{E}|$ because $\hat{n}$ and $\vec{E}$ are everywhere parallel, and where $|\vec{E}|$ is uniform over the Gaussian circle of radius r. Applying Gauss' theorem, we get

$$\Phi = \frac{1}{\epsilon_0} q \tag{8a}$$

$$2\pi r|\vec{E}| = \frac{1}{\epsilon_0} q \tag{8b}$$

$$|\vec{E}| = \frac{q}{2\pi\epsilon_0 r}. \tag{8c}$$

(b) The charge configuration at hand has line symmetry, meaning that the charge density depends only on the distance from an infinite line. This implies that the electric field is purely in the direction perpendicular to the line, and is opposite on either side of the line.

We choose as our Gaussian surface a rectangle whose width (of length L) is parallel to the line of charge and whose height (of length $2y$) is perpendicular to the line of charge (Fig. 2).

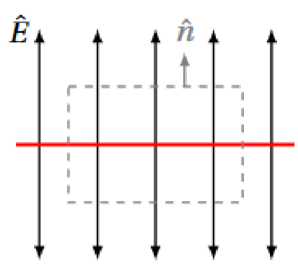

Figure 2: Line charge (in red) in 2D with field lines (solid lines) emanating from it. The Gaussian rectangle (dashed) of radius of height $2y$ and width L as well as the outer-normal vector are also shown.

The flux through this rectangle is given by

$$\Phi = \oint_C \mathrm{d}l \; \hat{n} \cdot \vec{E} = 2L|\vec{E}|, \tag{9}$$

so Gauss' theorem yields

$$\Phi = \frac{1}{\epsilon_0} q_{\text{enclosed}} \tag{10a}$$

$$2L|\vec{E}| = \frac{1}{\epsilon_0} \lambda L \tag{10b}$$

$$|\vec{E}| = \frac{\lambda}{2\epsilon_0} \tag{10c}$$

(c) We split up the charged rod into inifinitesimal charges $\mathrm{d}q = \lambda \mathrm{d}x$, each producing a field given by the modified Coulomb's law from part **(a)** (Fig. 3).

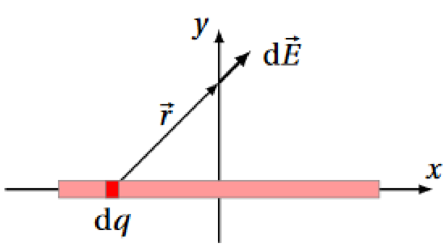

Figure 3: Electric field $\mathrm{d}\vec{E}$ at position $(0, y)$ due to an infinitesimal point charge $\mathrm{d}q$ at position $(x, 0)$. We integrate over the whole x axis to find the total field at the point.

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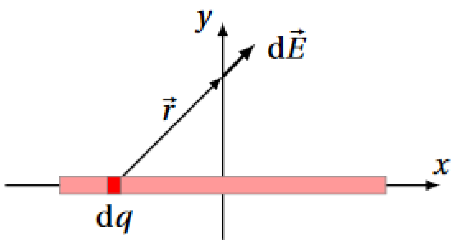

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Since the charge distribution is symmetric about the y axis, the x component of the electric field is everywhere zero. We find the y component via integration. The field at some $y > 0$ is given by

$$|\vec{E}| = E_y = \int \frac{\mathrm{d}q}{2\pi\epsilon_0 r} \hat{r}_y \tag{11a}$$

$$= \int_{-\infty}^{\infty} \frac{\lambda \mathrm{d}x}{2\pi\epsilon_0 r} \frac{|y|}{r} \tag{11b}$$

$$= \frac{\lambda |y|}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{r^2} \tag{11c}$$

$$= \frac{\lambda |y|}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x^2 + y^2}. \tag{11d}$$

We do this integral via the change of variables

$x = |y|u$, so $\mathrm{d}x = |y|\,\mathrm{d}u$.

$$|\vec{E}| = \frac{\lambda|y|}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\mathrm{d}u\,|y|\frac{1}{y^2u^2+y^2} \tag{11e}$$

$$= \frac{\lambda y^2}{2\pi\epsilon_0}\frac{1}{y^2}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{1}{u^2+1} \tag{11f}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left(\arctan u\right)\Big|_{-\infty}^{\infty}. \tag{11g}$$

Since $\arctan(\pm\infty) \to \pm\pi/2$, we finally find

$$|\vec{E}| = \frac{\lambda}{2\epsilon_0}, \tag{12}$$

in agreement with part **(b)**.

3. Gauss's Law For Gravitation Question

(a) The point of this exercise is to realize that the same steps leading to Gauss' law for electric fields could lead to a gravitational Gauss' law as well—the only difference would be in the symbols and their meanings.[2]

For convenience, we rewrite Gauss' law (for electric fields) in terms of Coulomb's constant:

$$\oint_S \mathrm{d}\vec{A}\cdot\vec{E} = 4\pi k q_{\text{enclosed}}. \tag{15}$$

[2]Clearly, a significant difference between gravity and electricity is that gravity is always attractive. This difference does not cause any problems in the derivation of the gravitational Gauss' law.

Comparing the two field laws in Eq. 13, we establish the following correspondences:

$$\vec{E} \leftrightarrow -\vec{g} \tag{16a}$$
$$k \leftrightarrow G \tag{16b}$$
$$q \leftrightarrow m. \tag{16c}$$

This allows us to immediately write down the gravitational Gauss' law:

$$\oint_S \mathrm{d}\vec{A}\cdot\vec{g} = -4\pi G m_{\text{enclosed}}. \tag{17}$$

(b) We apply the exact same strategy as we did for the uniformly *charged* ball.

Briefly, since the problem has spherical symmetry, we know that the gravitational field points purely in the radial direction and depends only on the radial coordinate. Hence, we can use the gravitational Gauss' law to find the magnitude of the field at a radius r.

We choose a spherical Gaussian surface of radius r concentric with the Earth. Anticipating that the gravitational field $\vec{g}$ points in the direction opposite to the outer-normal, the gravitational flux through the surface is

$$\oint_S \underbrace{\mathrm{d}\vec{A}\cdot\vec{g}}_{-\mathrm{d}A\,|\vec{g}|} = -|\vec{g}|\oint_S \mathrm{d}A \tag{18a}$$
$$= -4\pi r^2|\vec{g}|, \tag{18b}$$

and the mass enclosed by the surface is

$$m_{\text{enclosed}} = \rho\left(\frac{4}{3}\pi r^3\right), \tag{19}$$

so the gravitational Gauss' law dictates that

$$\oint_S \mathrm{d}\vec{A}\cdot\vec{g} = -4\pi G m_{\text{enclosed}} \tag{20a}$$
$$-4\pi r^2|\vec{g}| = -4\pi G\rho\frac{4}{3}\pi r^3 \tag{20b}$$
$$|\vec{g}| = \frac{4}{3}\pi G\rho r. \tag{20c}$$

Of course, this is the same behaviour as for the electric field in a uniformly charged sphere, subject to the replacements of Eq. 16.