

Notes for the second lecture on “The growth and morphology of icicles”

This pdf contains a few slides that were not shown in the actual lecture, for lack of time.

This pdf contains live links which are underlined like this.

For data and movies, see the *Icicle Atlas*
https://www.physics.utoronto.ca/Icicle_Atlas

All papers, theses, reports etc on this topic are available at:
https://www.physics.utoronto.ca/nonlinear/papers_icicles.html

YouTube playlist.

Flickr images and movies of icicles.

Flickr images of the icicle machine.

Do not hesitate to contact me with any questions, ideas or whatever
Stephen Morris patternphysics@gmail.com or sw.morris@utoronto.ca
<https://www.physics.utoronto.ca/nonlinear/>

Part 2: Ripples



Icicle ripples



Icicle ripples



[Flickr photo by J. Hiebert](#)



Ice stalagmite

Ice stalagmites
in a cave

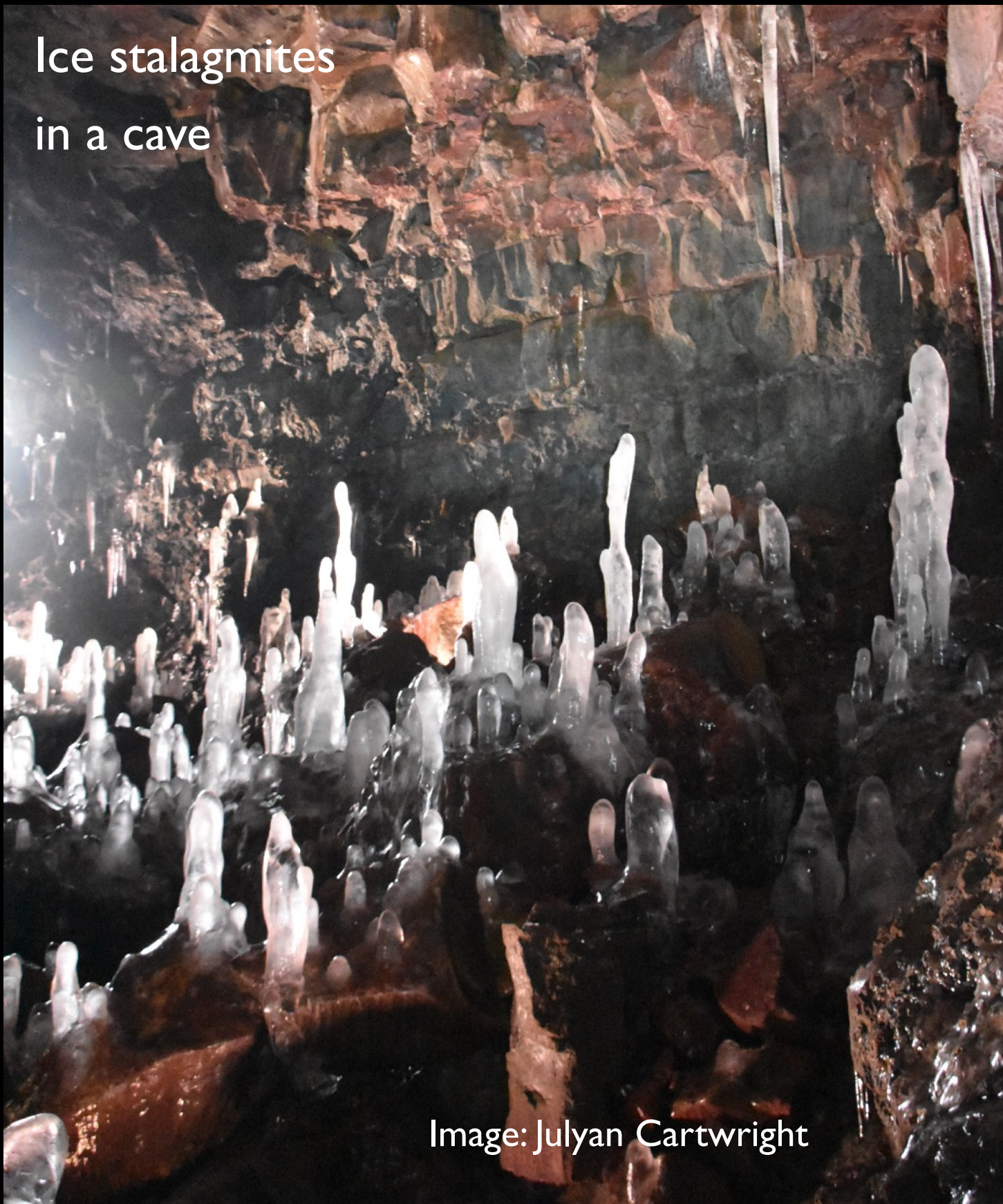


Image: Julyan Cartwright

An early
experiment in
which the
funnel heater
failed





Icicle ripples

“Michelin Man” ring-like ridges are often seen on the surface of natural icicles.

Ripples are always observed to have a wavelength very close to 1 cm, independent of flow rate, undercooling etc.

Ripples *not* observed on distilled water icicles. They require small levels of impurities to form.

This was not realized until Chen’s experiments.

A. S-H. Chen and S.W. Morris,
New J. Phys., **15**, 103012 (2013).

Existing theory does not take any account of impurities.

N. Ogawa and Y. Furukawa, *PRE*, **66**, 041202 (2002).
K. Ueno, *Phys. Fluids* **19**, 093602 (2007),
and several other papers.



Icicle ripples

- T. Terada, “Stripe patterns in Nature” (1933). *Observation*.
- T. Tozuka, *Kagaku*, **8**, 218 (1938). *Observation*.
- S. Matsuda, *MSc thesis* (1997). *Experiment*.
- N. Ogawa, Y. Furukawa, *Phys. Rev. E*, **66**, 041202 (2002). *Theory*.
- K. Ueno et al., *7 theory papers*.

Torahiko Terada



Icicle ripples

T. Terada, “*Stripe patterns in Nature*” (1933).



以上のものとは少し違った部類のものであるが、氷柱や鐘乳石が簡単な円錐形または紡錘形となる代わりに、どうかすると、表面に週期的の皺を生じ、その縦断面の輪郭が波形となることがある。この原因についてもあまりよく知る人がないようである。この場合にもやはり表面を流下する液体の運動にある週期性がある、それがまた同時に氷結と融解、あるいは析出と沈着との週期性を支配するものである、とまでは想像しても悪くないであろう。しかしこの場合にも熱的対流が関係するか、それとも、単に流水層の渦層の器械的不安定によるものであるかは、今後の詳細な実験的研究によってのみ決定さるべきであろう。

A somewhat different type of phenomenon from those mentioned above [waves on sand dunes, sinter deposits around geysers, etc.] is when icicles and stalactites, instead of taking simple cylindrical or conical shapes, form periodic wrinkles on their surfaces, with their longitudinal profiles forming waves. No one seems to understand much about the causes of those waves ... [The mechanism] can only be determined through in-depth experimental research.

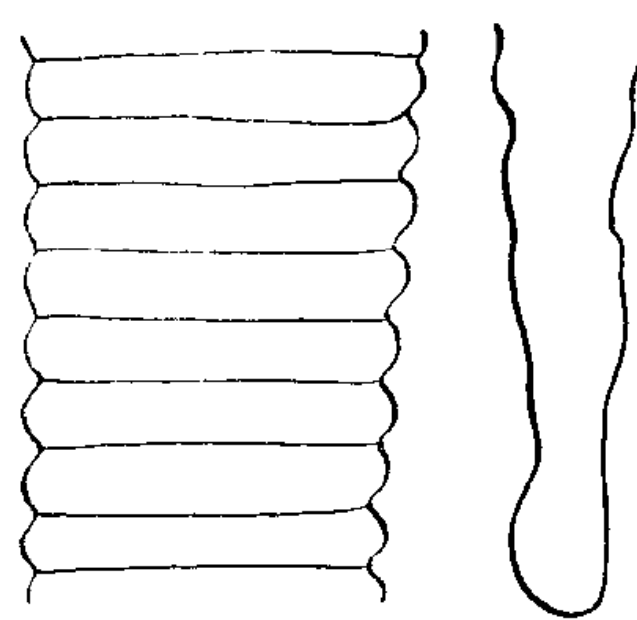
Translation by Tom Gally, University of Tokyo

Icicle ripples

- T. Terada, “Stripe patterns in Nature” (1933). *Observation*.
- T. Tozuka, *Kagaku*, **8**, 218 (1938). *Observation*.
- S. Matsuda, *MSc thesis* (1997). *Experiment*.
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- K. Ueno et al., 7 theory papers.

218 Tozuka 1938

科 學 第 8



波頭の 間隔と一致させ、それを定規にして方々へ當てがつて見ると大體、何處でも五つの山と一致するから平均しては等しい間隔になつて居ると見てよい。マッチの軸は 5 cm であるから、この一波の幅は約 8 mm 位に當る。

第 2 圖

第 3 圖

小さい細い“つらら”は第 3 圖の様に不規則な凹凸がある許りである。それが發達するに従つて上述の様な規則正しい縞が生ずるのであるらしい。何故だか僕にはわからない。

日本醫科大學生理學教室

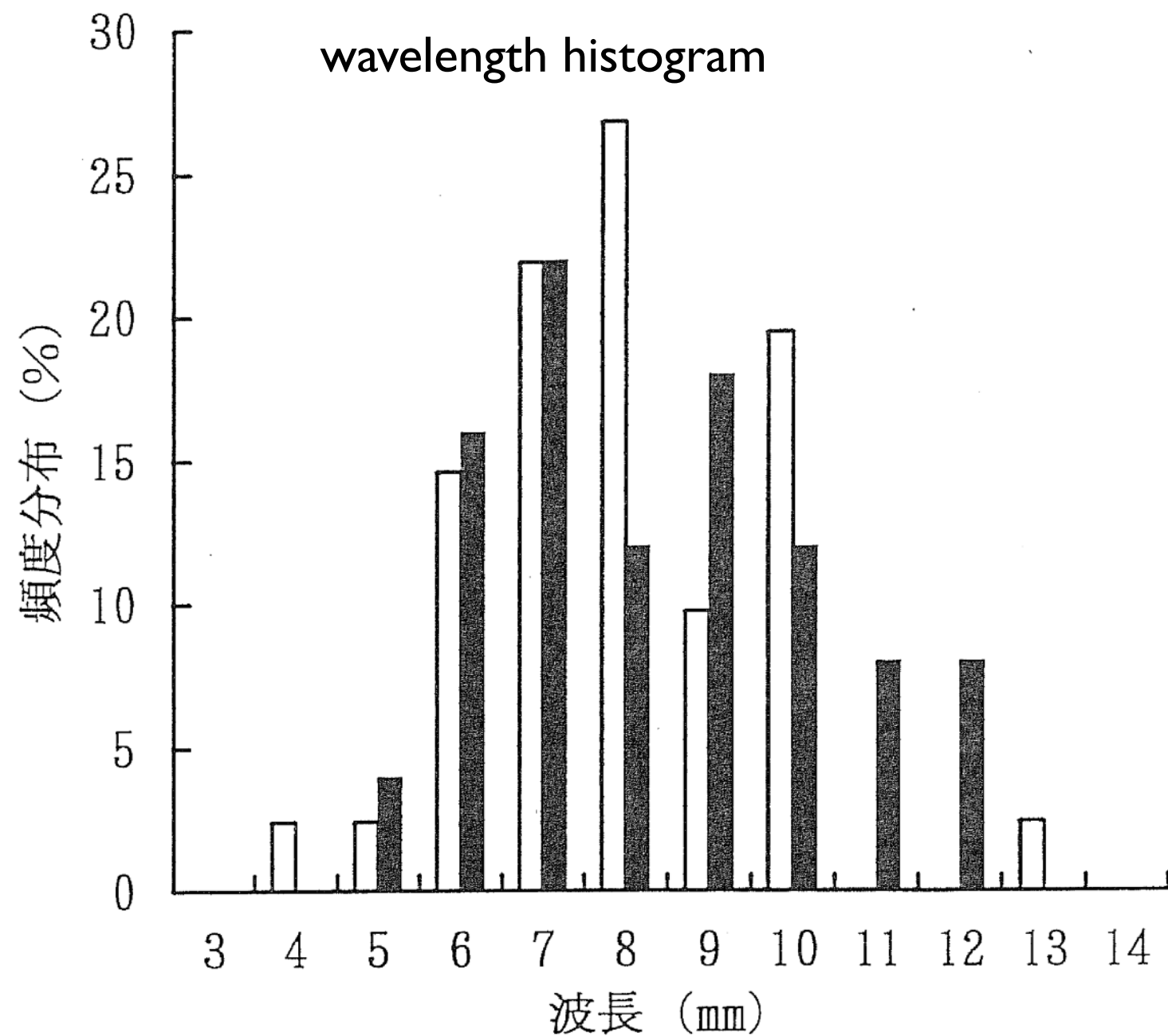
戸 塚 武 彦

Matsuda M.Sc. thesis 1997.

He used tap water.



5cm



■ 自然界のつらら 平均波長 8.5mm, N= 82

□ 槌の表面 (90°) 平均波長 8.2mm, N=100

Cave ripples: *crenulations*

Ripples are also
common in cave
formations

C. Camporeale
and L. Ridolfi,
Phys. Rev. Lett., **108**,
238501 (2012).



Icicle ripples

In order to study the effect of concentration and impurity species, we systematically doped distilled water with the following:

Species	formula	Mol. weight g/Mol	Charge	van 't Hoff factor	Number of icicles	Author(s)
Sodium Chloride	NaCl	58.4	1	2	129	Chen, Ladan
Sodium fluorescein (dye)	Na ₂ C ₂₀ H ₁₂ O ₅	376.3	2	3	49	Ladan
Sodium sulfate	Na ₂ SO ₄	142.04	2	3	17	Ladan
D-Glucose	C ₆ H ₁₂ O ₆	180.2	0	1	5	Ladan
Cupric sulfate	CuSO ₄	159.6	2	2	2	Ladan
Polyethylene glycol	PEG3350	3350	0	1	2	Ladan
Ferrous sulfate	FeSO ₄	151.9	2	2	1	Ladan
Triton X 100 Surfactant	C ₁₄ H ₂₂ O(C ₂ H ₄ O) _n	647	0	1	20	Chen, Ladan
					225	

Molal concentrations count all the dissociated ions.

We used sodium fluorescein dye both as an impurity and to visualize the flow of water. Some runs also had TX100 surfactant added to change their surface tension.

A. S-H. Chen and S.W. Morris, *New J. Phys.*, **15**, 103012 (2013).

J. Ladan and S.W. Morris, *New J. Phys.*, **23**, 123017 (2021).

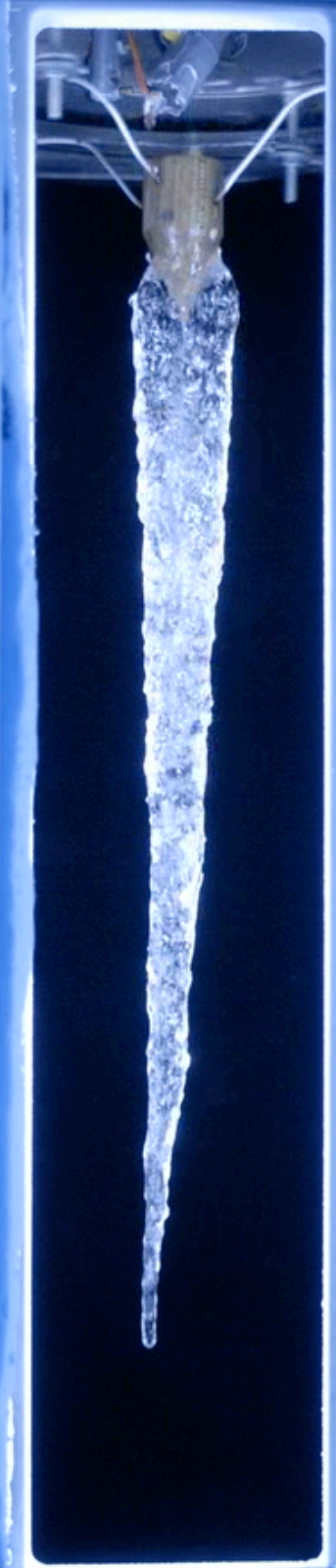
J. Ladan and S.W. Morris, *Phys. Rev. E*, **106**, 054211 (2022).



distilled water

40 mg NaCl per kg H₂O

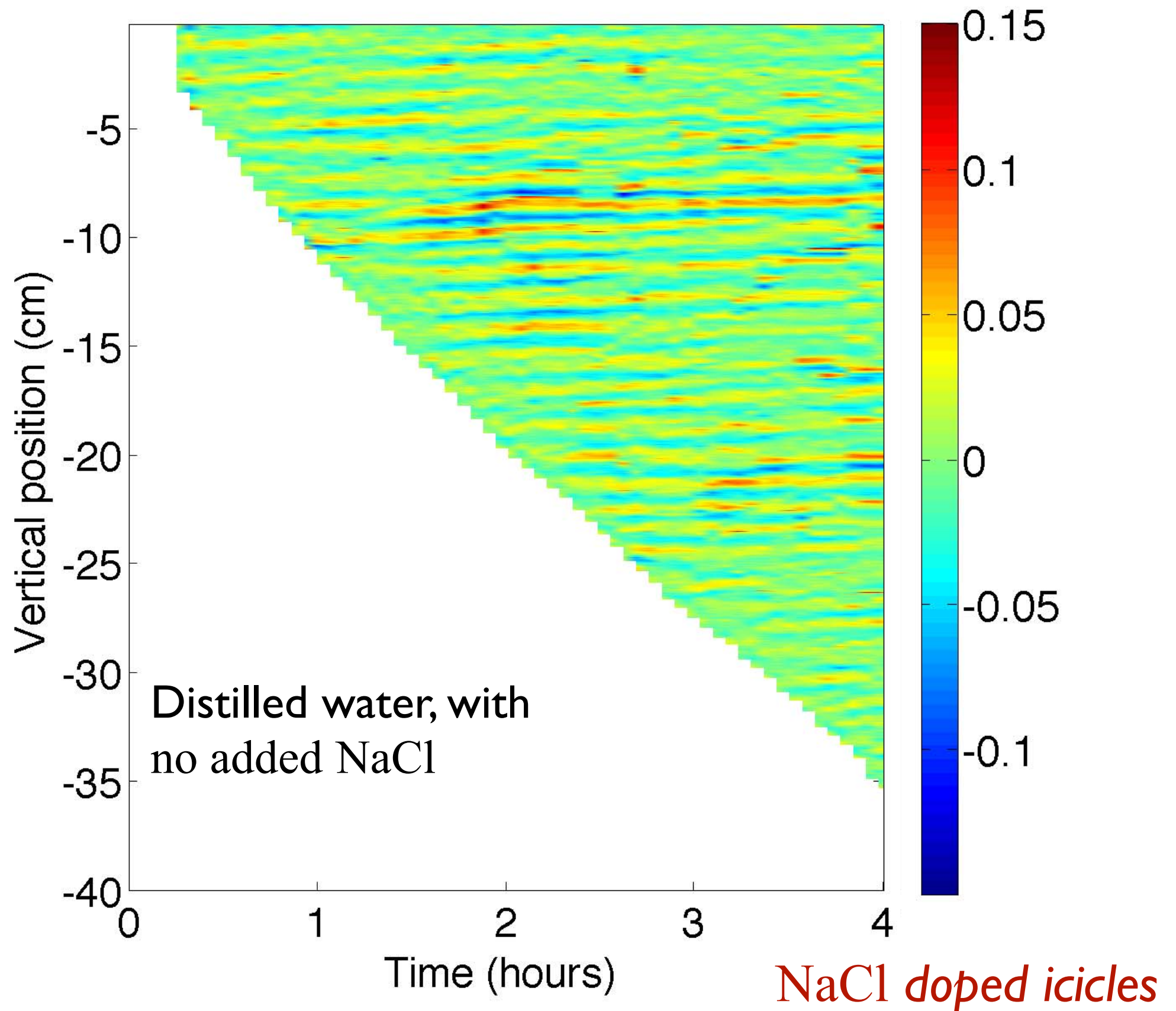
1.28 g NaCl per kg H₂O



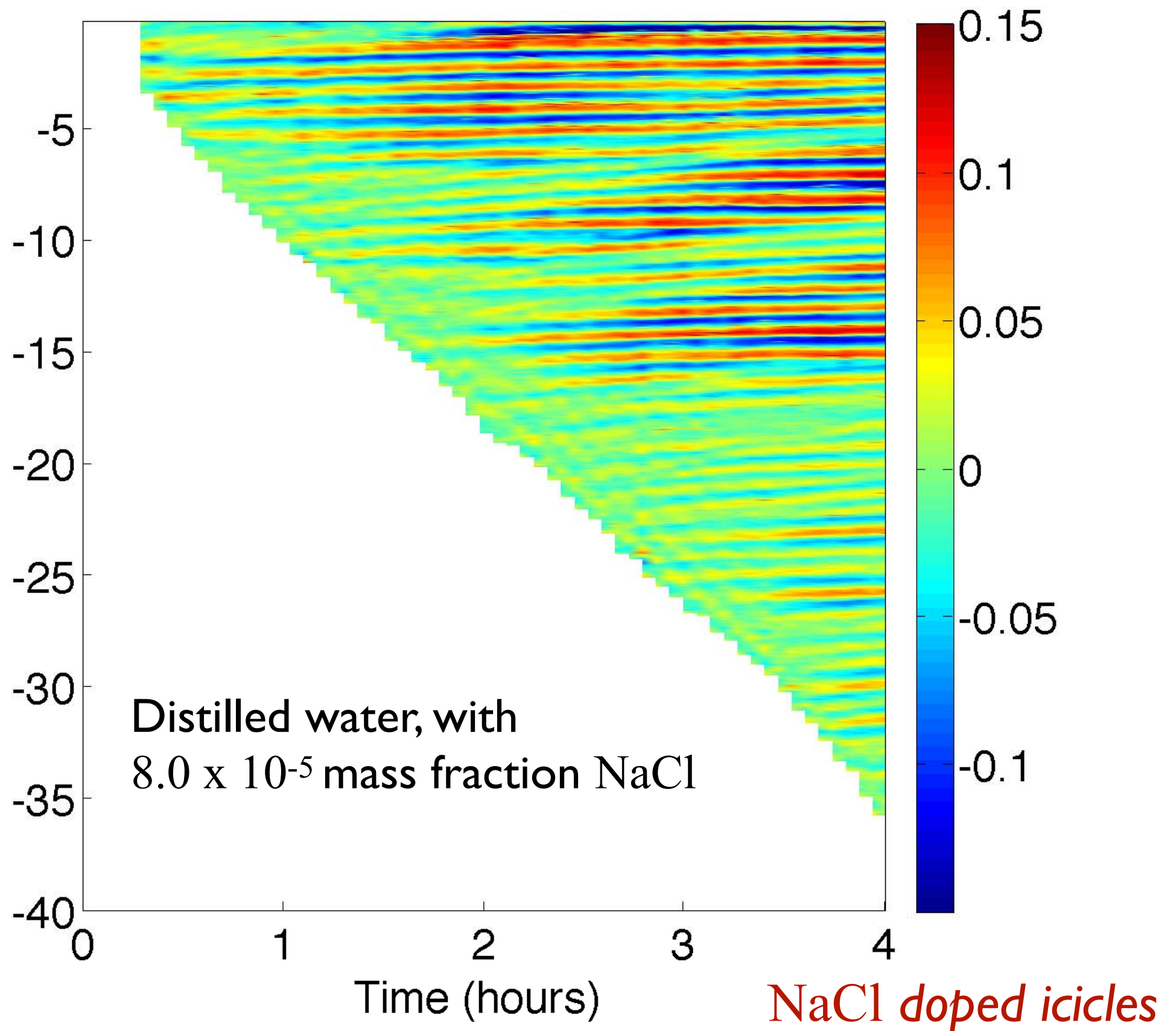
-12.3 deg C, 2.0 g/min

Much less than Sea water: ~ 35 g NaCl per kg H₂O

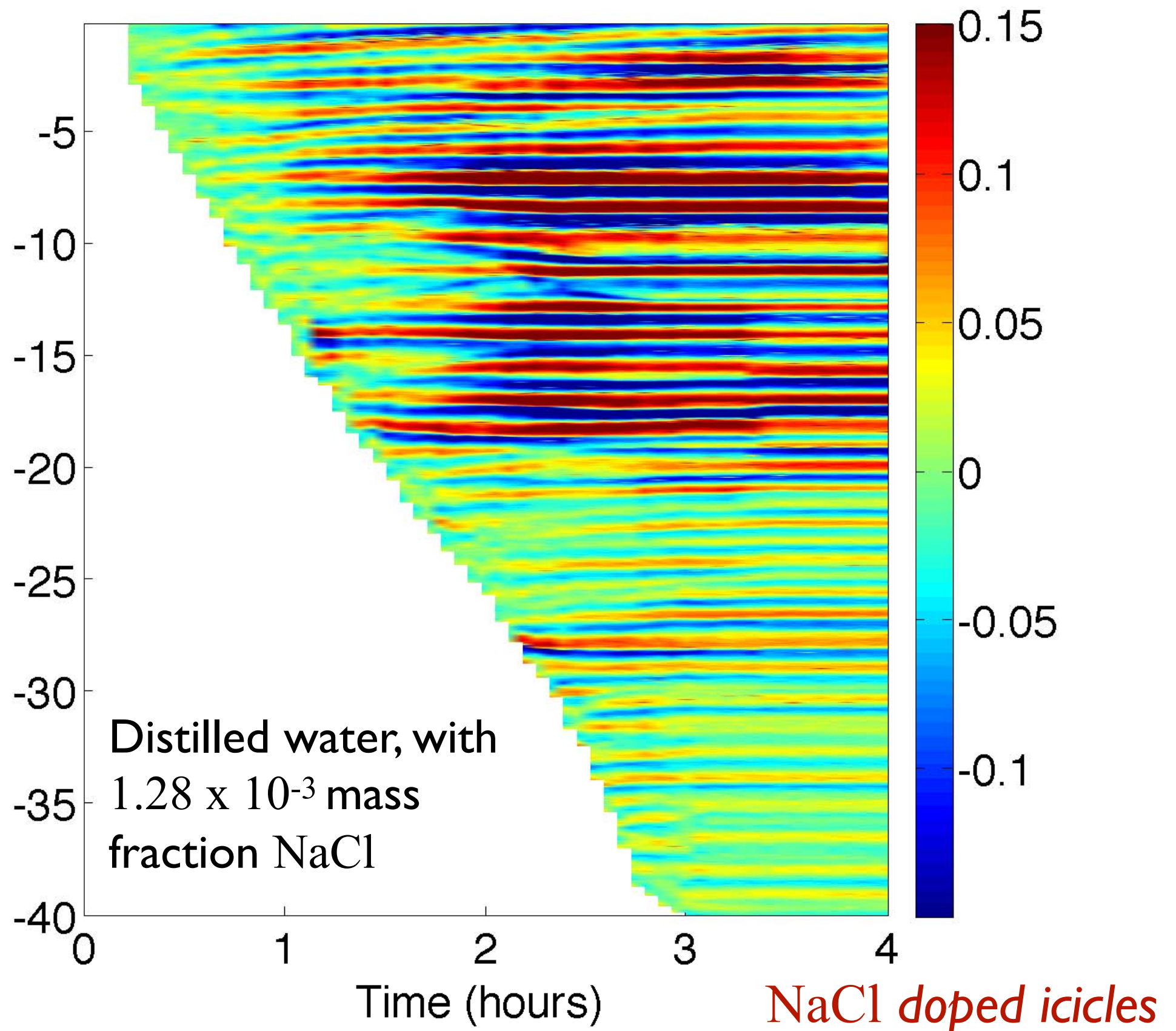
Icicle topography vs time



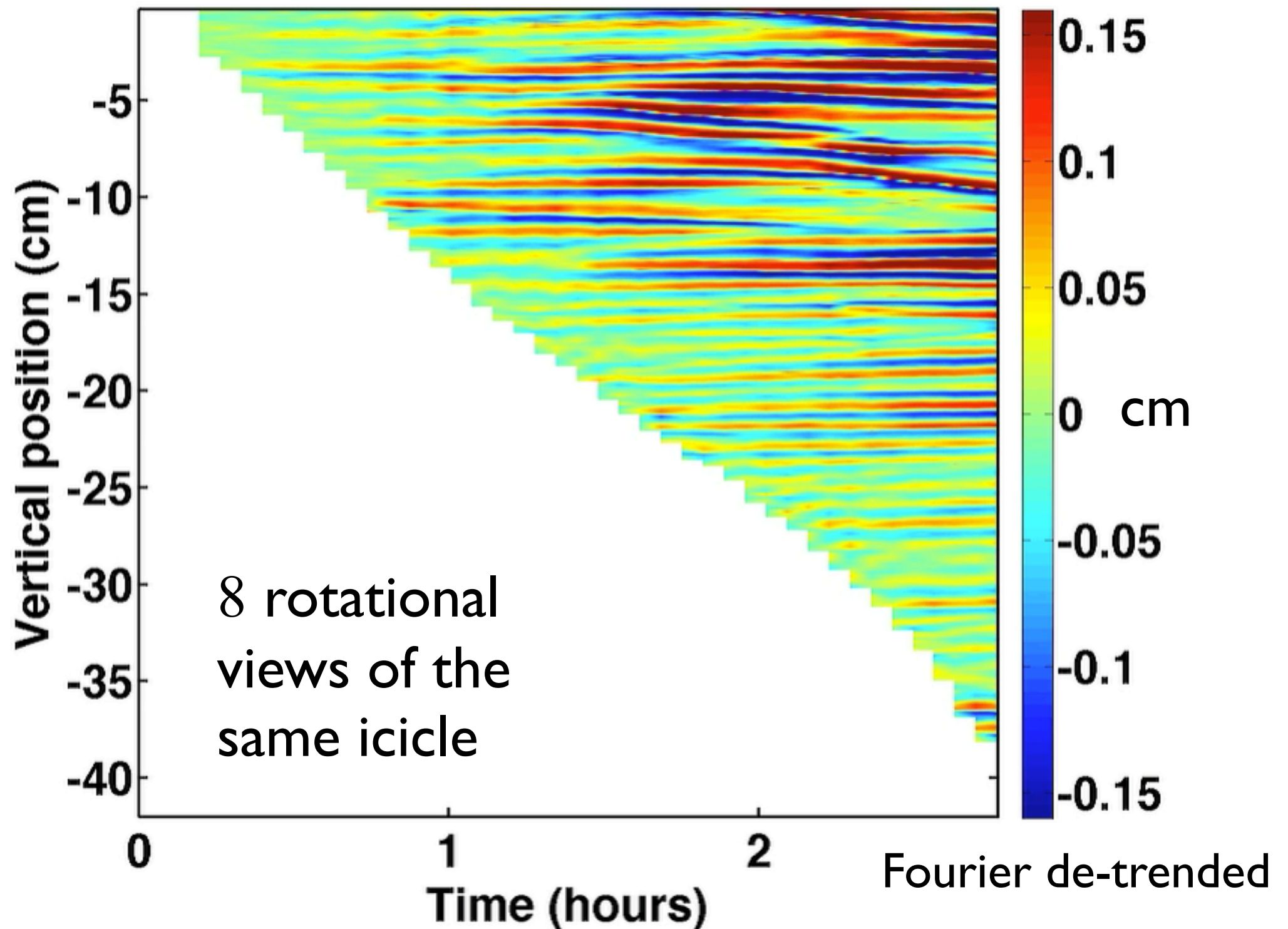
Icicle topography vs time



Icicle topography vs time



Icicle topography vs time



Distilled water, with
 1.28×10^{-3} mass
fraction NaCl

NaCl doped icicles

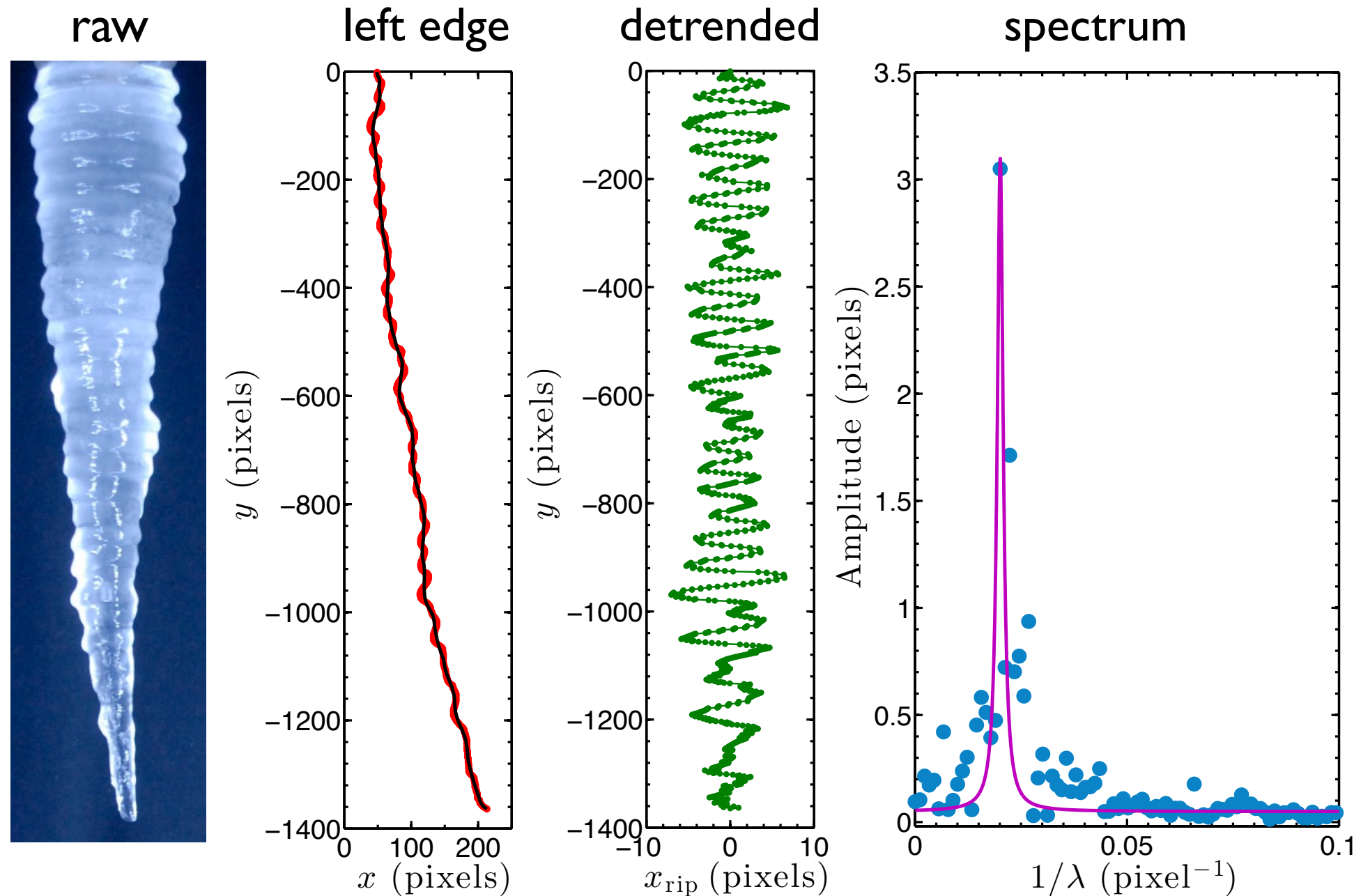
Measuring ripples

NaCl doped icicles

Chen measured icicle ripple amplitudes and wavelengths using a Fourier technique.

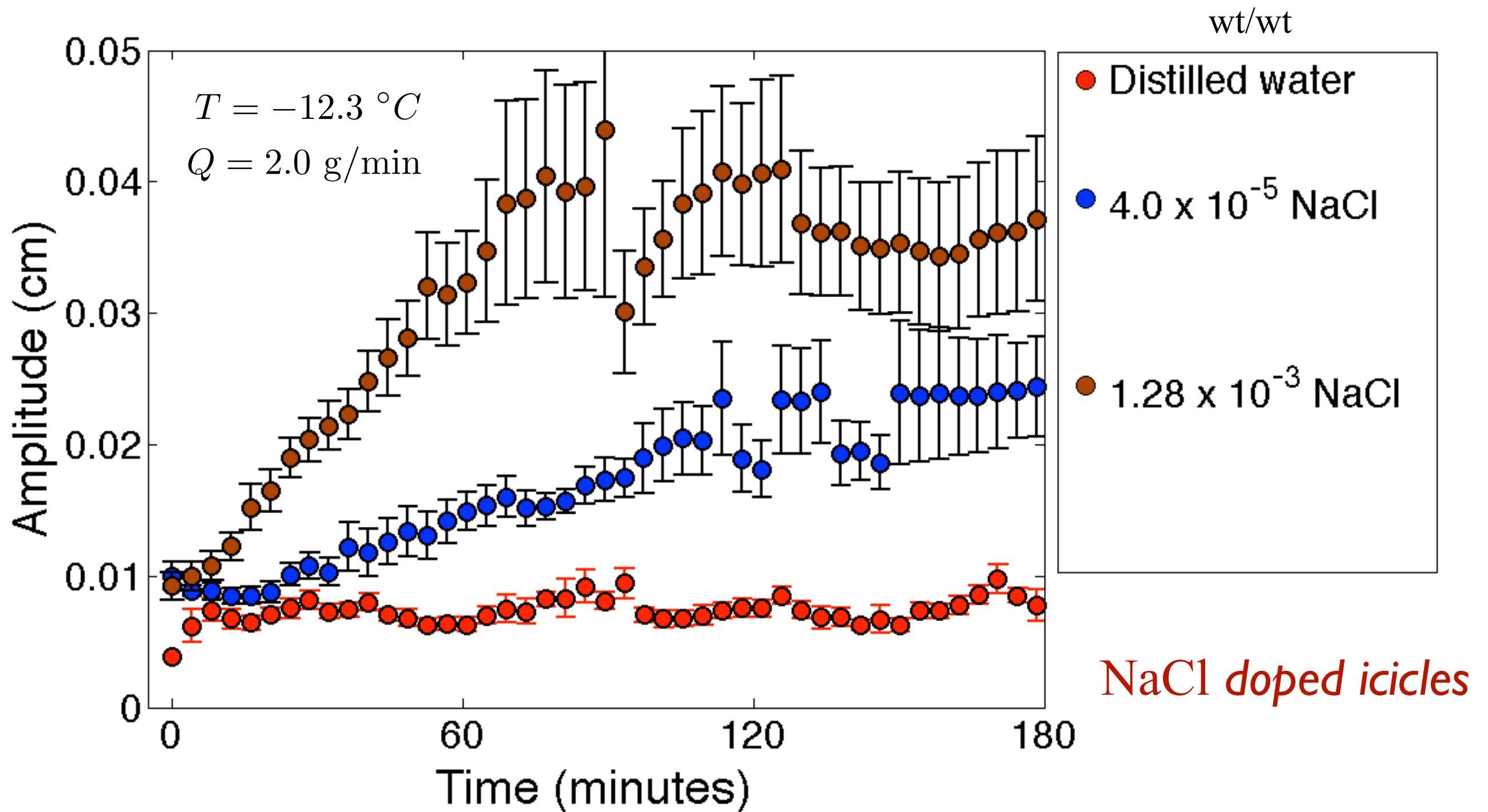
Pictures 8x per rotation give 16 edges every 8 minutes.

Amplitude is noisy, so rotational averages are used.



Ripple wavelength is more tightly constrained than amplitude.

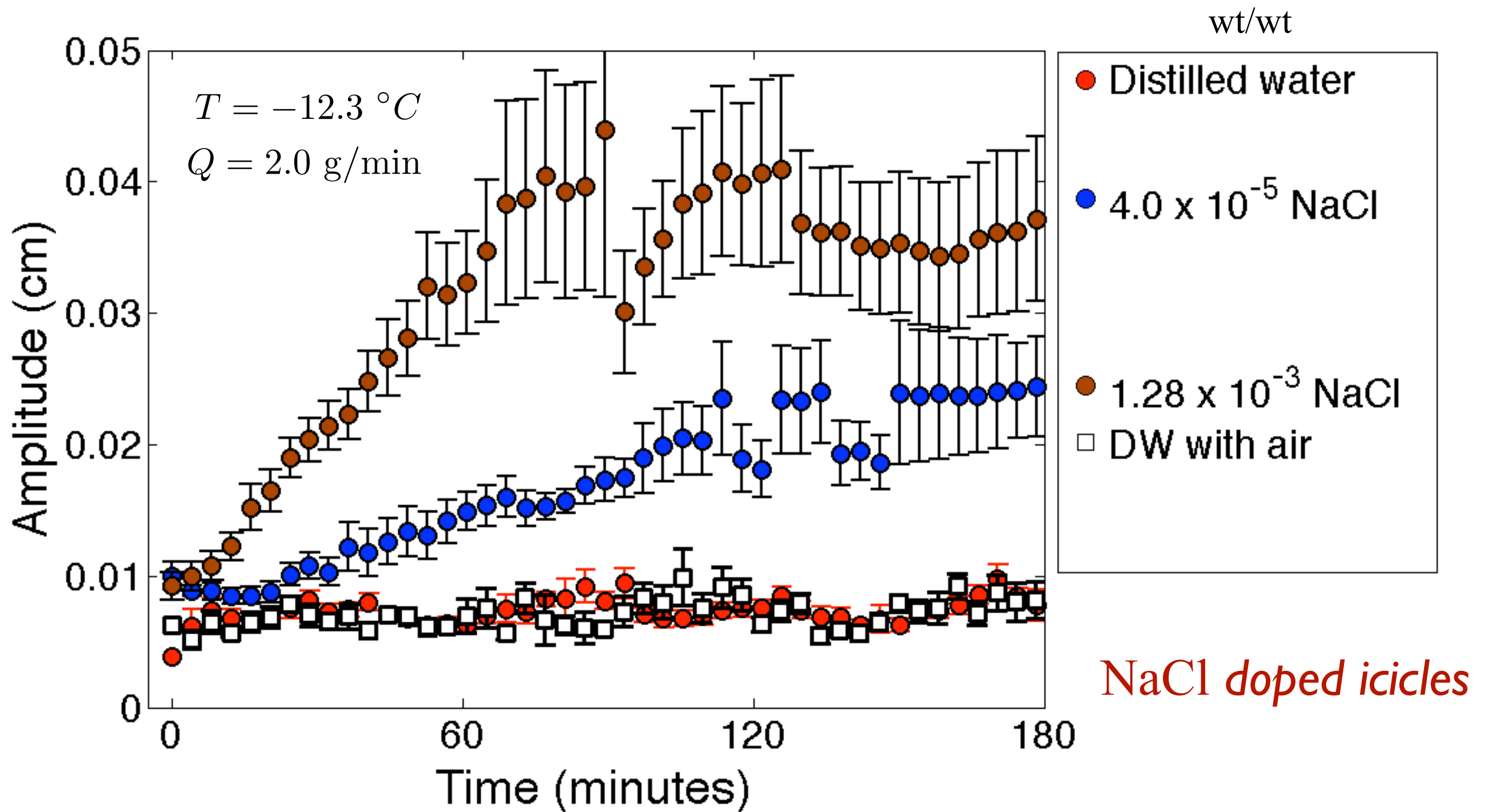
Growth and saturation of ripple amplitude vs time



Amplitudes averaged over 1 rotation of icicle and over repeated runs.

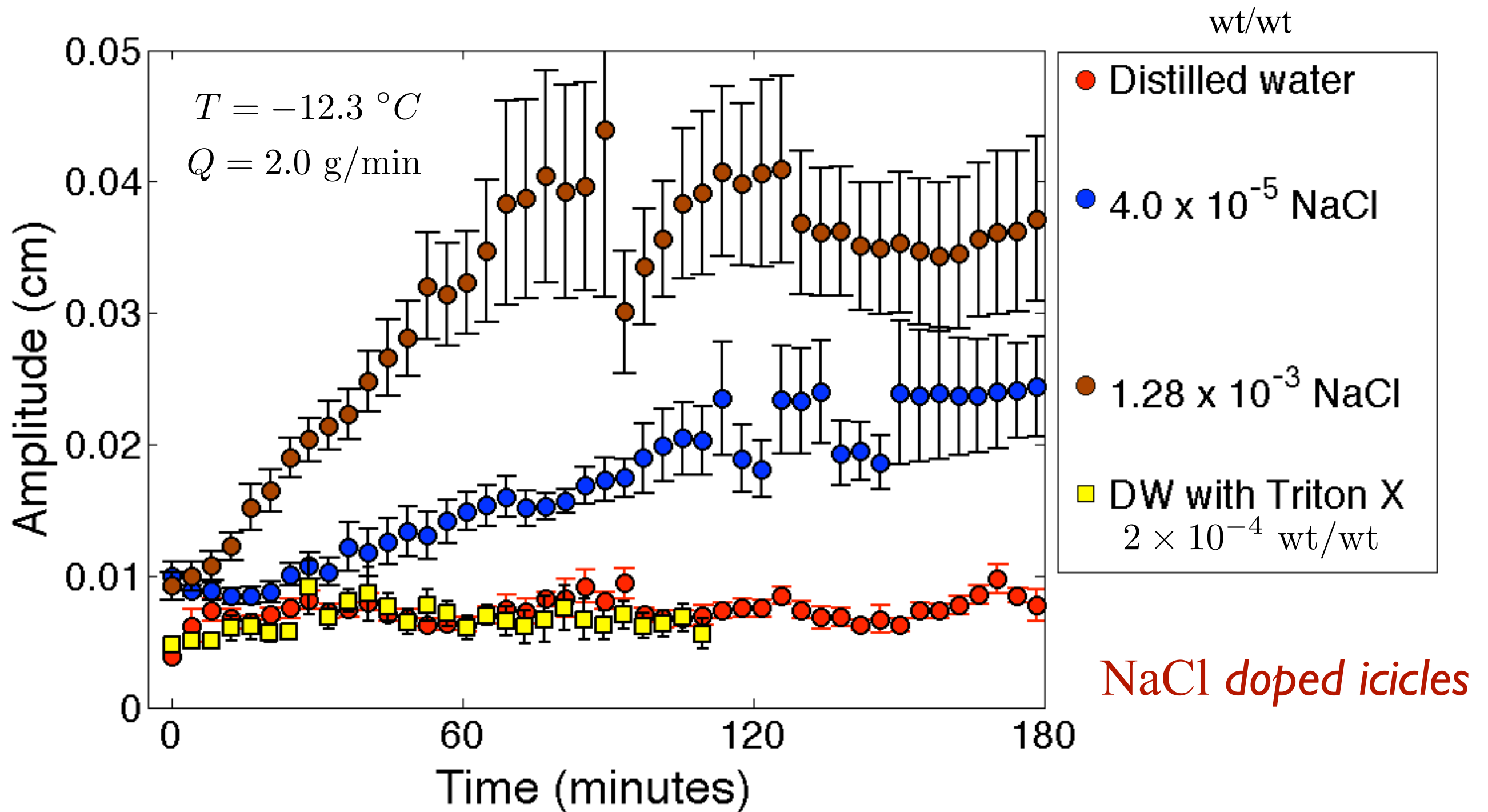
$t = 0$ when icicle reaches 10 cm, topography data from top 10 cm only.

Growth and saturation of ripple amplitude vs time



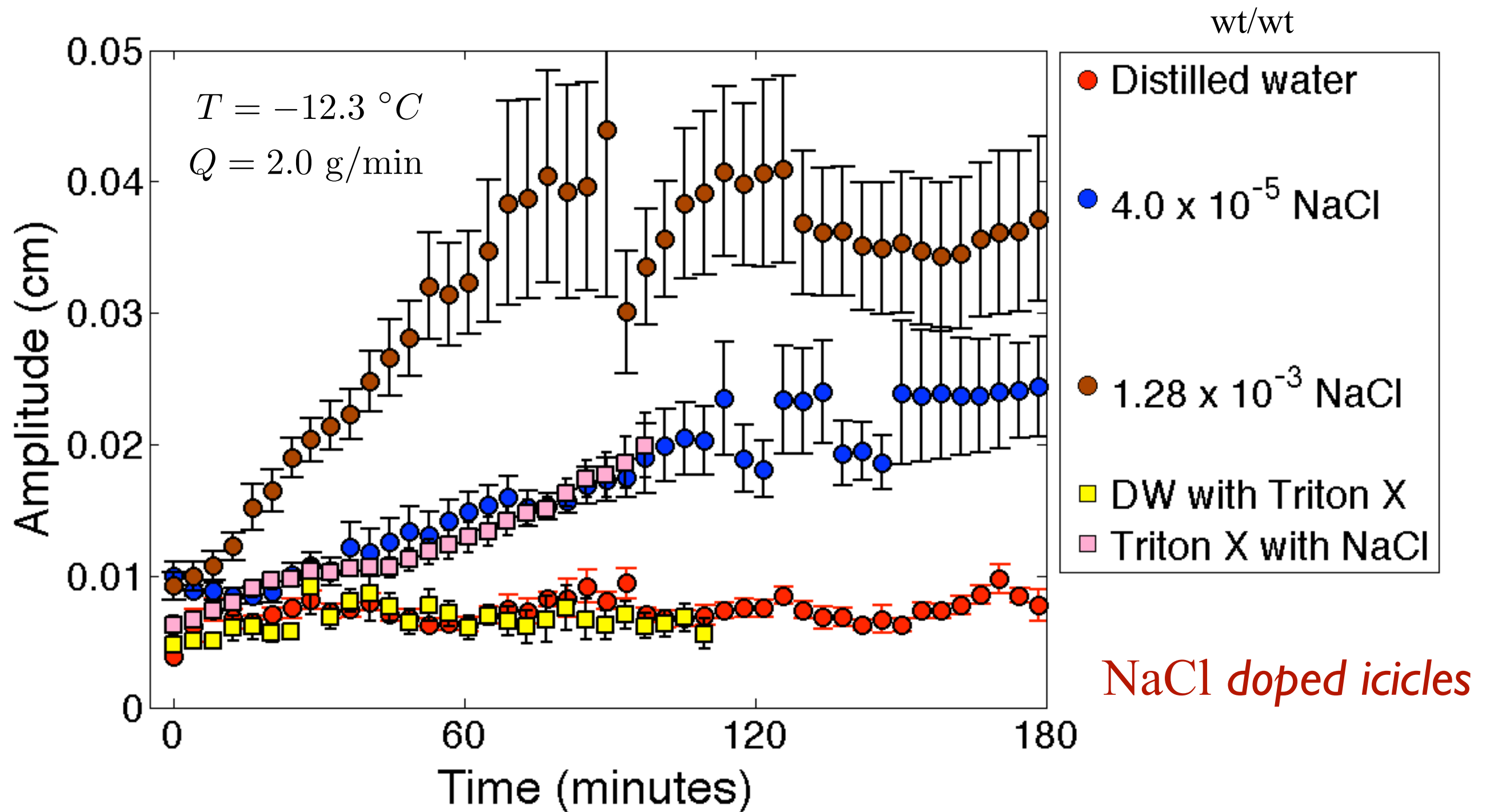
Bubbling air through the distilled water for a long time does **not** produce ripples. All samples are likely saturated with dissolved air.

Growth and saturation of ripple amplitude vs time



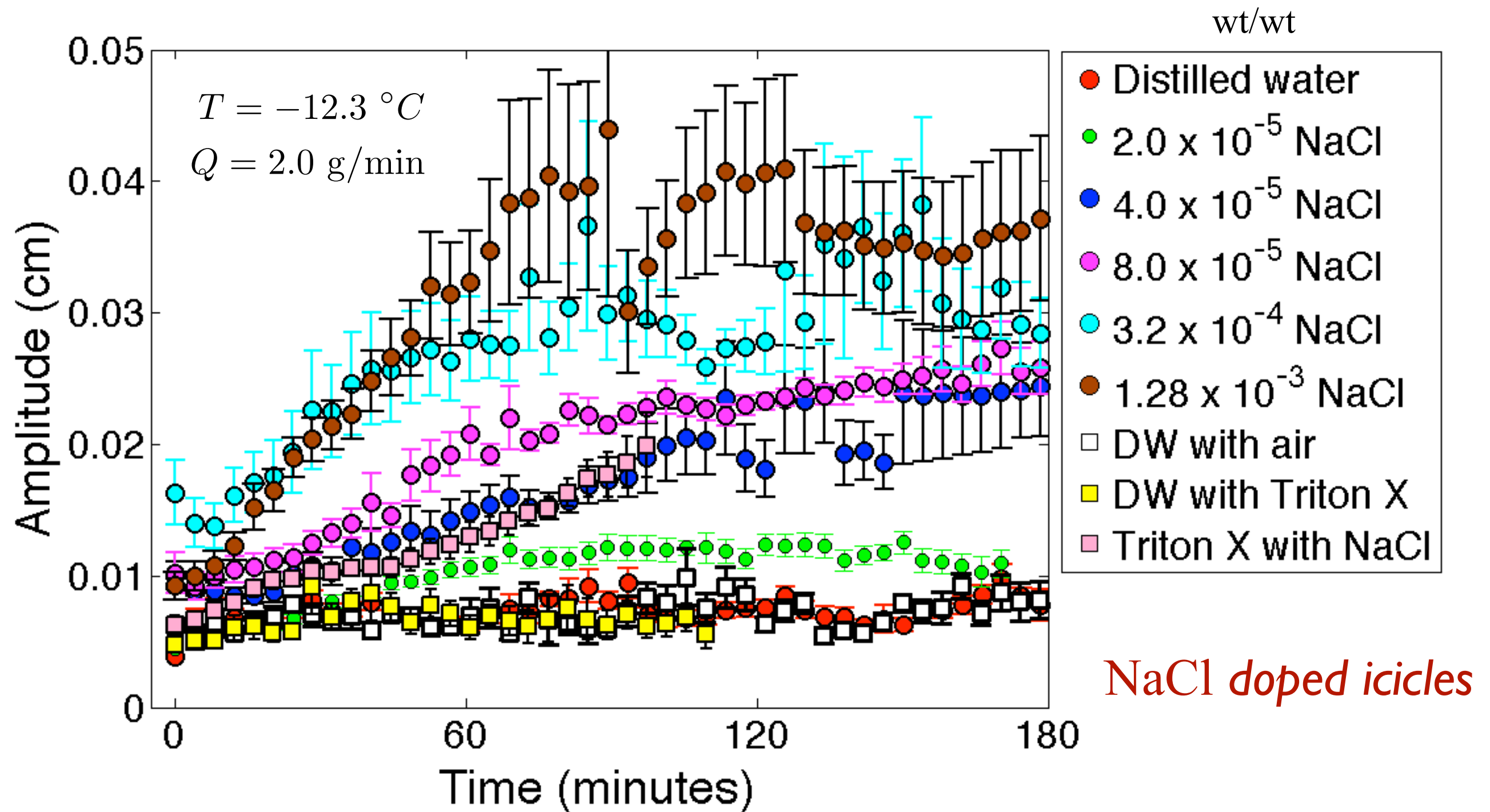
Adding 2×10^{-4} wt/wt of a strong surfactant, Triton X, to distilled water does **not** produce ripples, even though the surface tension is reduced by 45%.

Growth and saturation of ripple amplitude vs time



Adding a small concentration of salt *and* the Triton X, **does** produce ripples, but somewhat smaller than those with just the salt.

Growth and saturation of ripple amplitude vs time



Both ripple amplitude and growth rate increase with salt concentration, but surfactant and dissolved air do not produce ripples by themselves.

Composition of natural icicles

We collected about a dozen ripply “wild type” natural icicles and measured their conductivity. Their impurity levels are consistent with ripply laboratory icicles.

This icicle:

Typical lab salty icicle:

Ripple threshold:

Distilled water:

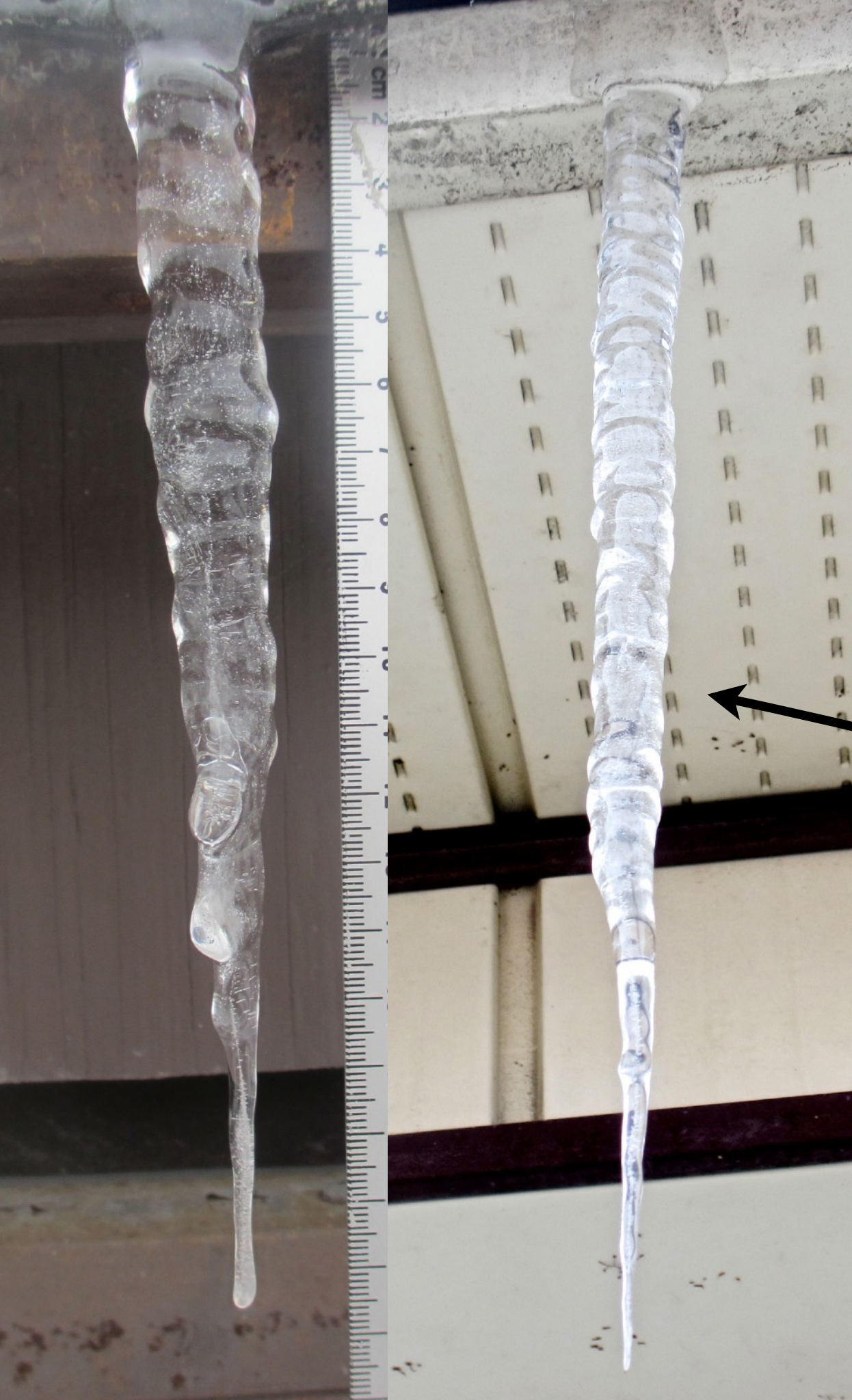
Conductivity

$19\mu\text{S}/\text{cm}$

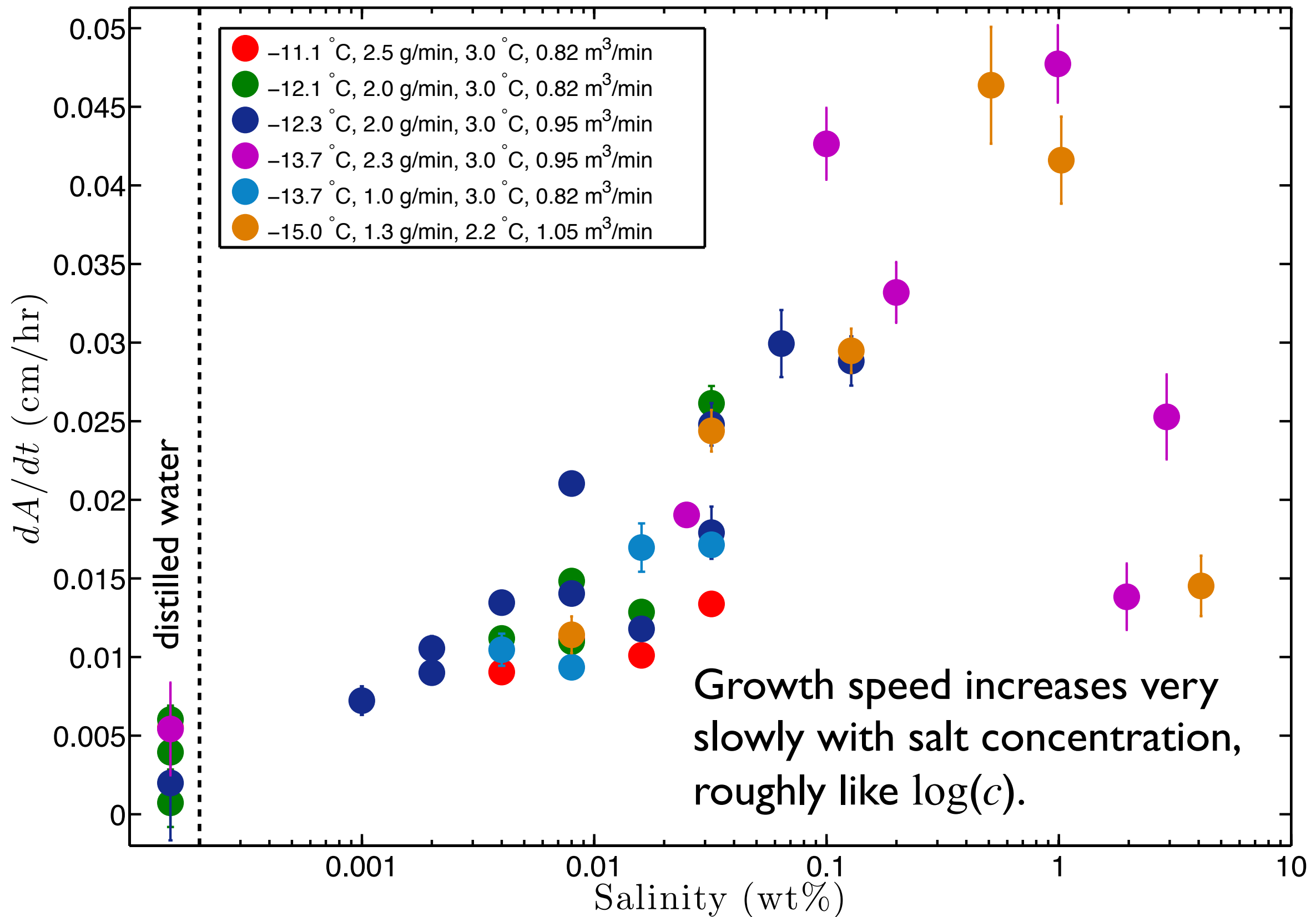
$200\mu\text{S}/\text{cm}$

$7\mu\text{S}/\text{cm}$

$2\mu\text{S}/\text{cm}$



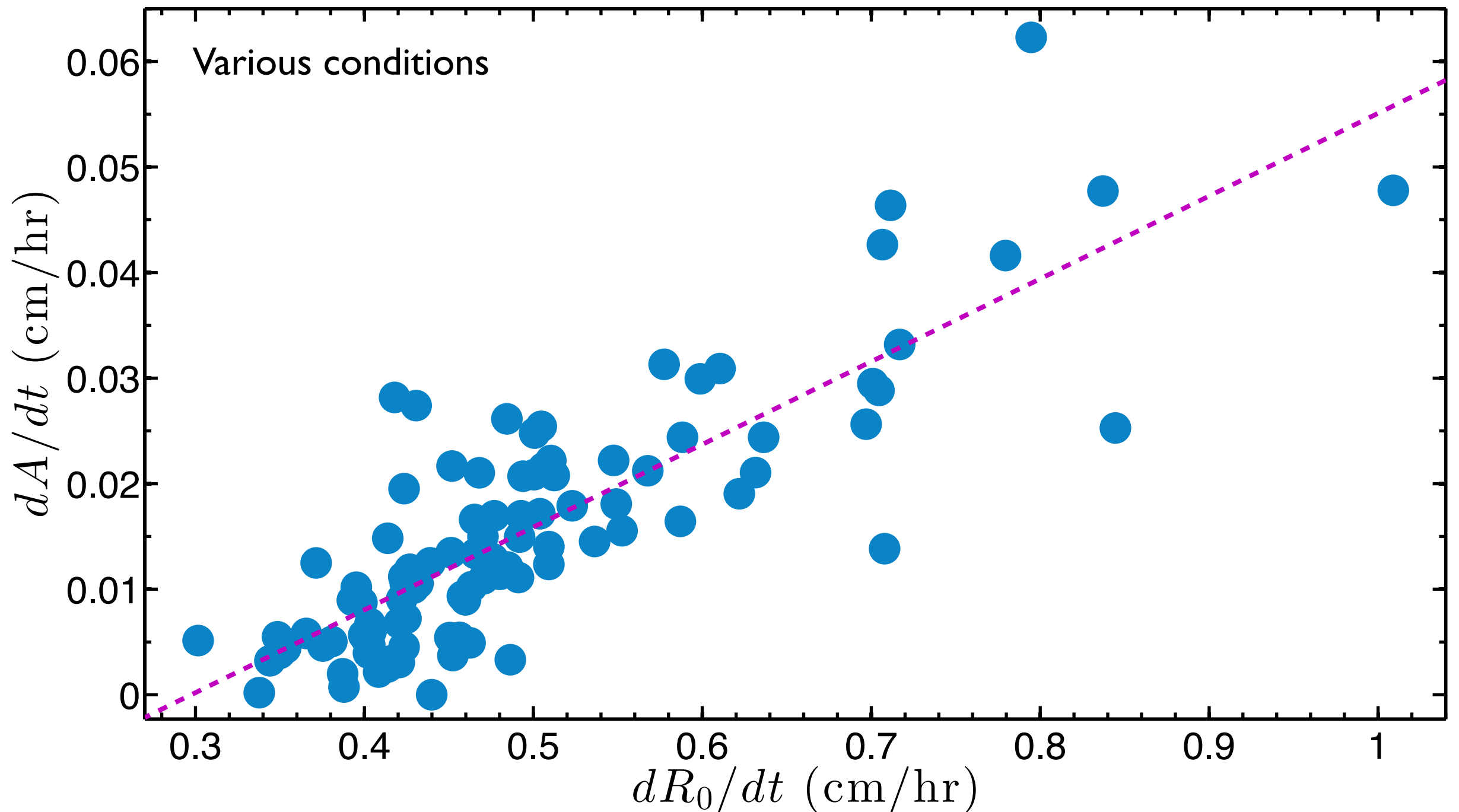
Growth speed of ripples vs concentration



Averaged over time after 1 hour of growth, top 10 cm

NaCl doped icicles

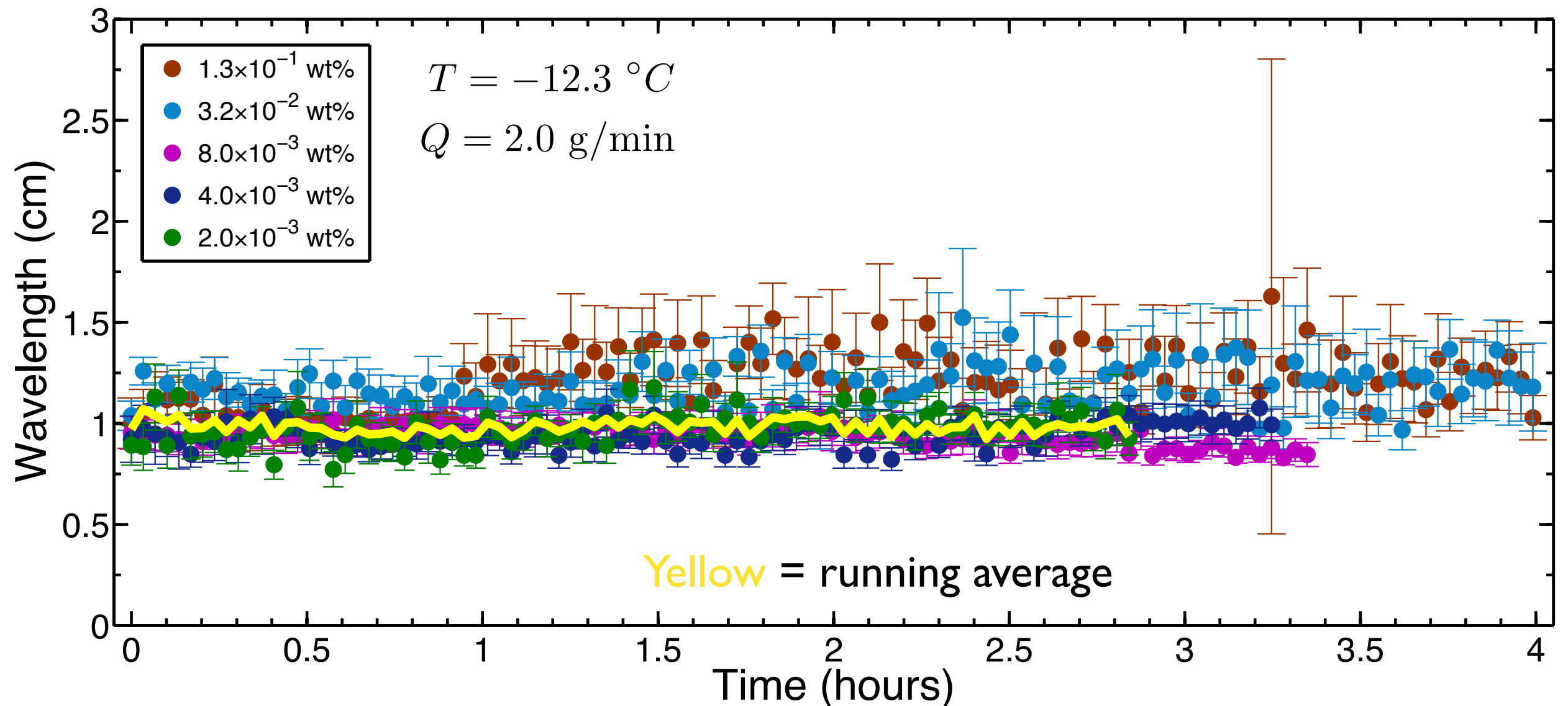
Growth speed of ripples vs radial growth speed



Ripple amplitude growth speed is correlated to radial growth speed.

NaCl doped icicles

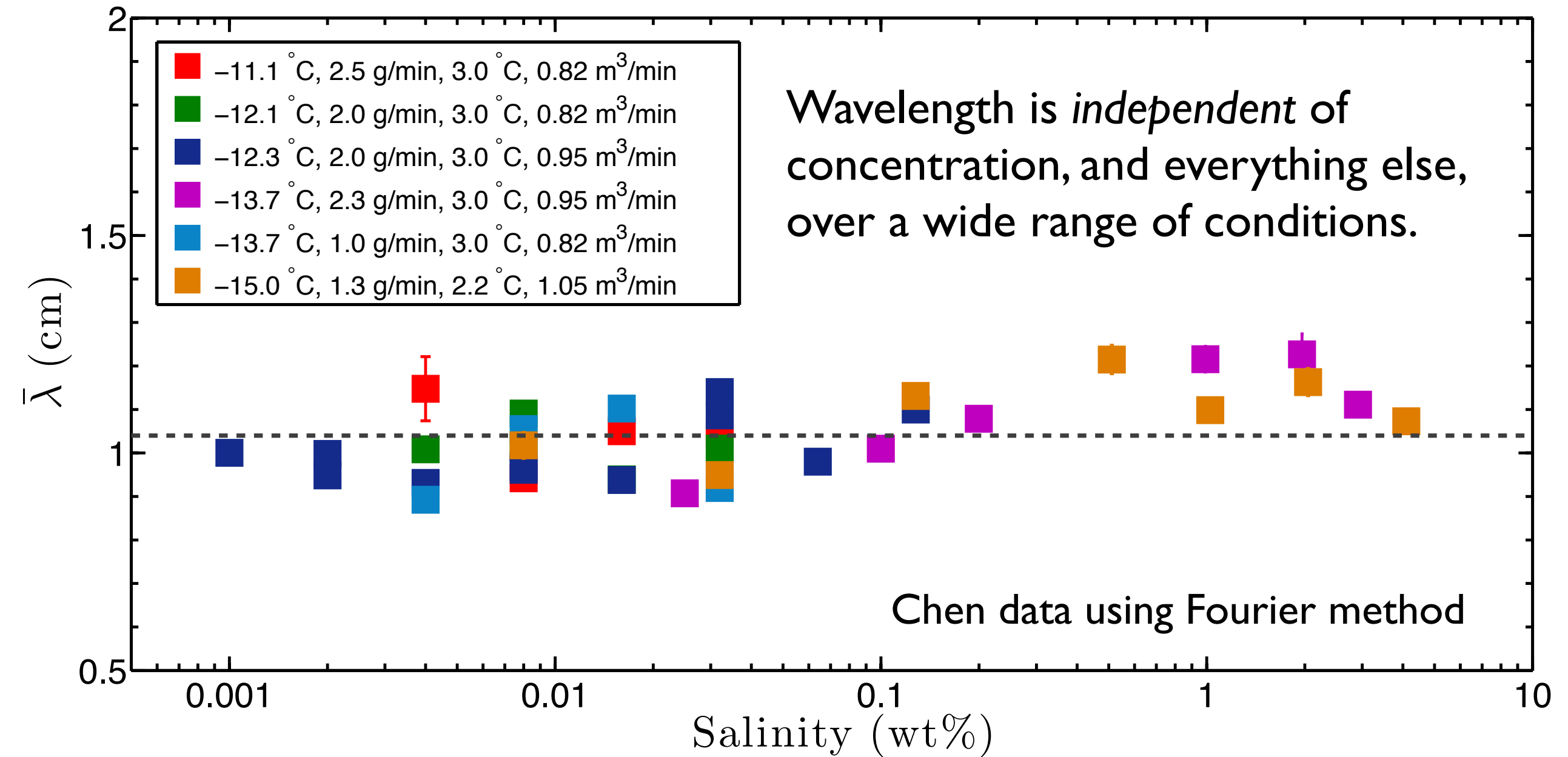
Wavelength of ripples during growth



The wavelength does not change as the ripples grow in amplitude

NaCl doped icicles

Wavelength of ripples vs concentration

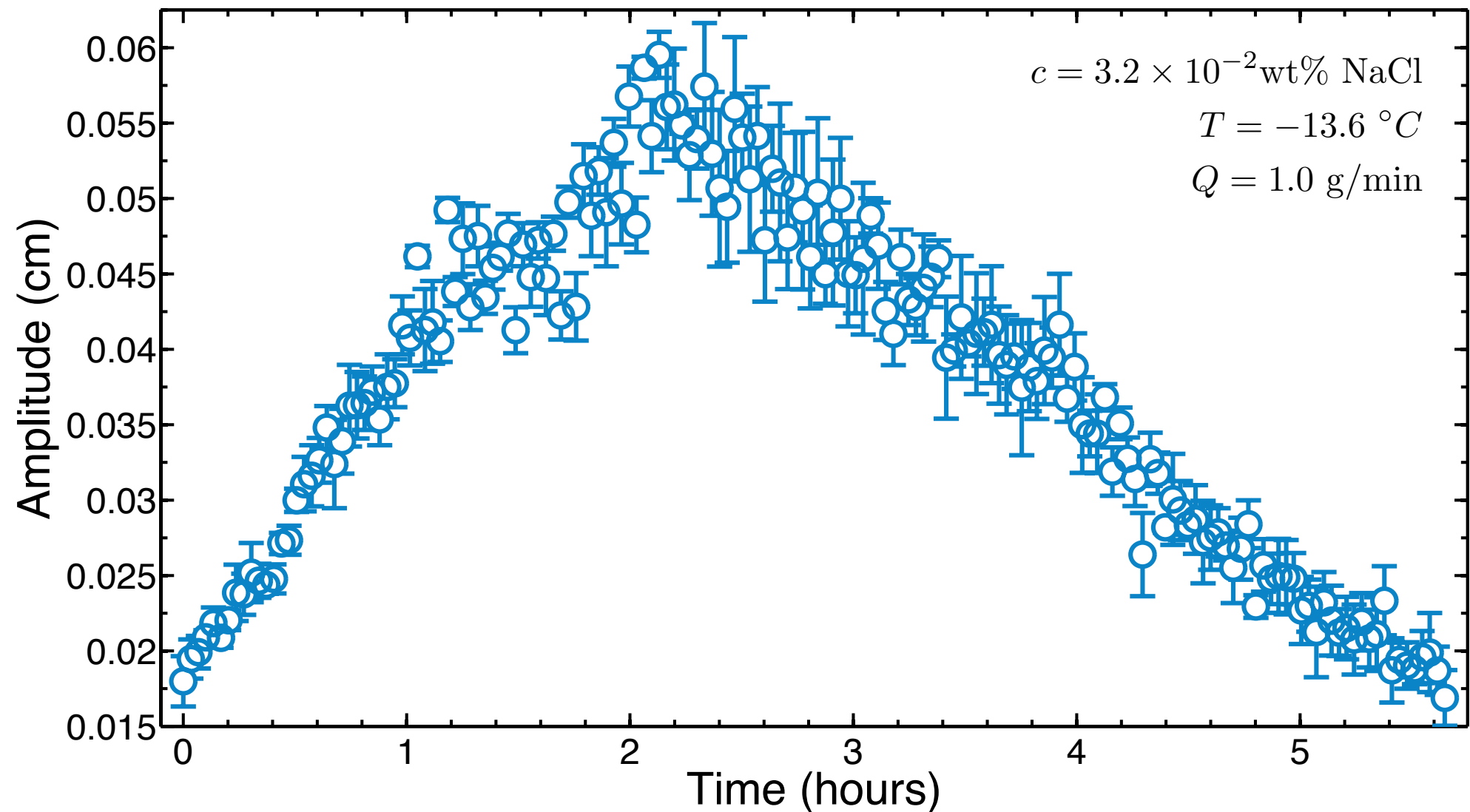


Average wavelength $\bar{\lambda} = 1.04 \pm 0.01\text{cm}$

NaCl doped icicles

Ripple evolution with cessation

NaCl doped icicles



Ripples grow at first, but smooth
out after cessation

Ripple motion

The ripples *climb* the icicle for low salt concentration

The ripples *descend* the icicle for high salt concentration

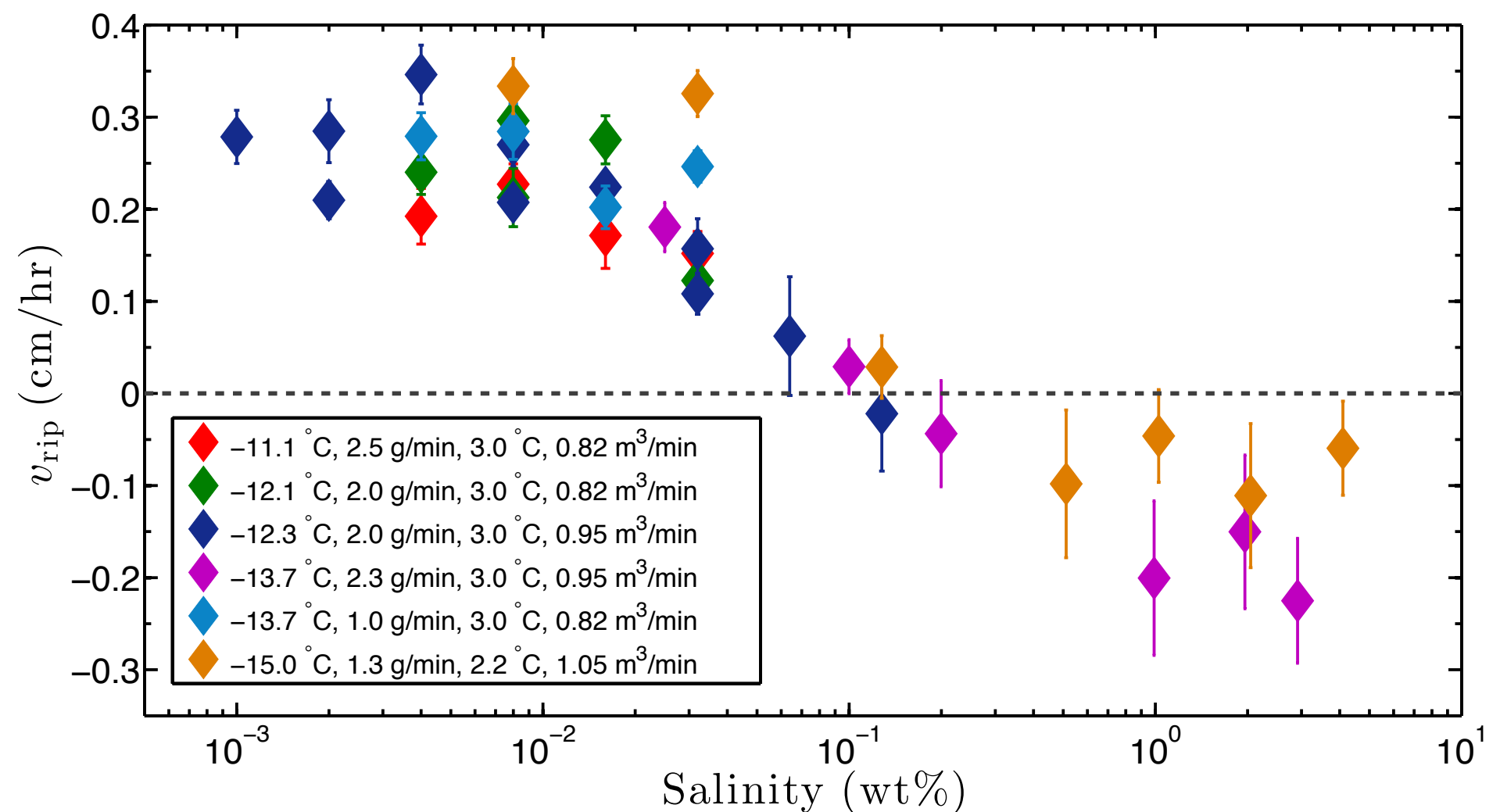
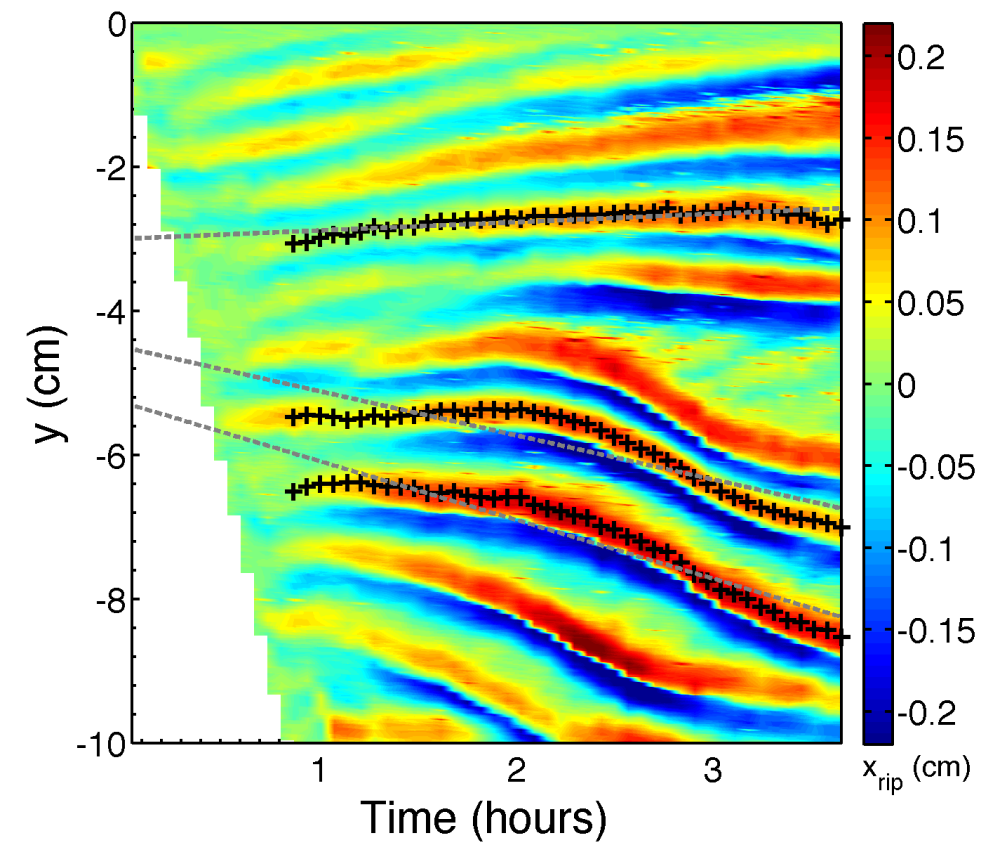
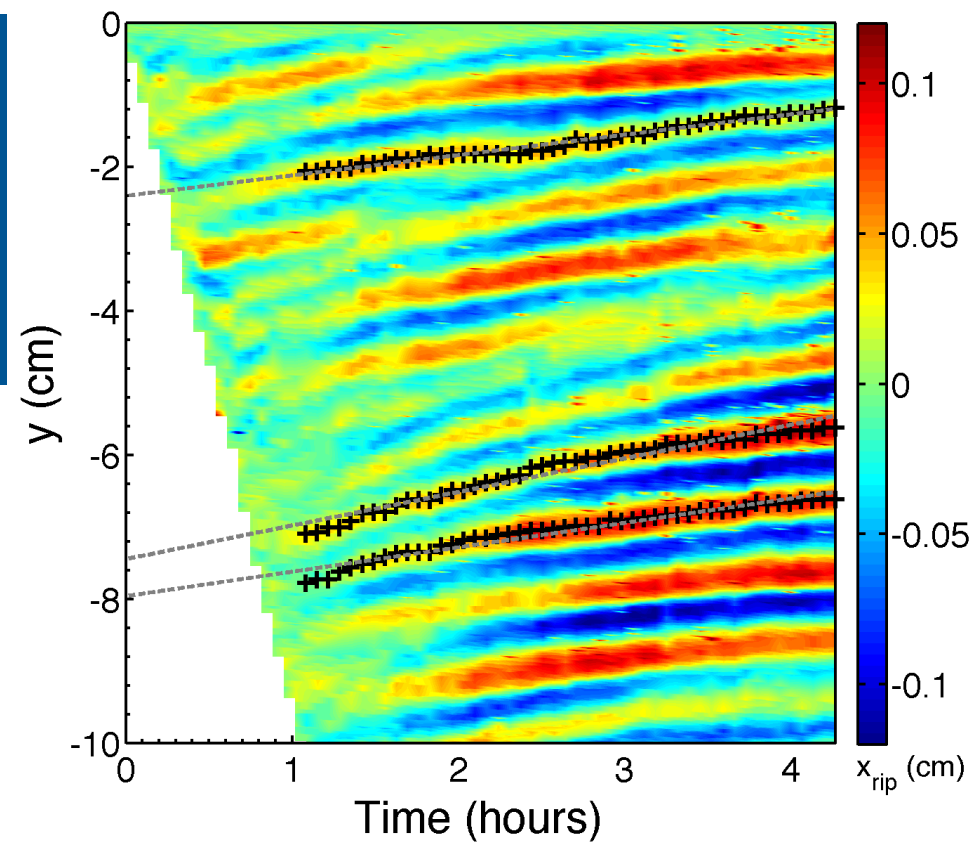
Low salt

High salt

Traveling speed vs concentration

Traveling speed *changes sign* with impurity concentration.

Positive is *up* the icicle, negative is *down*.



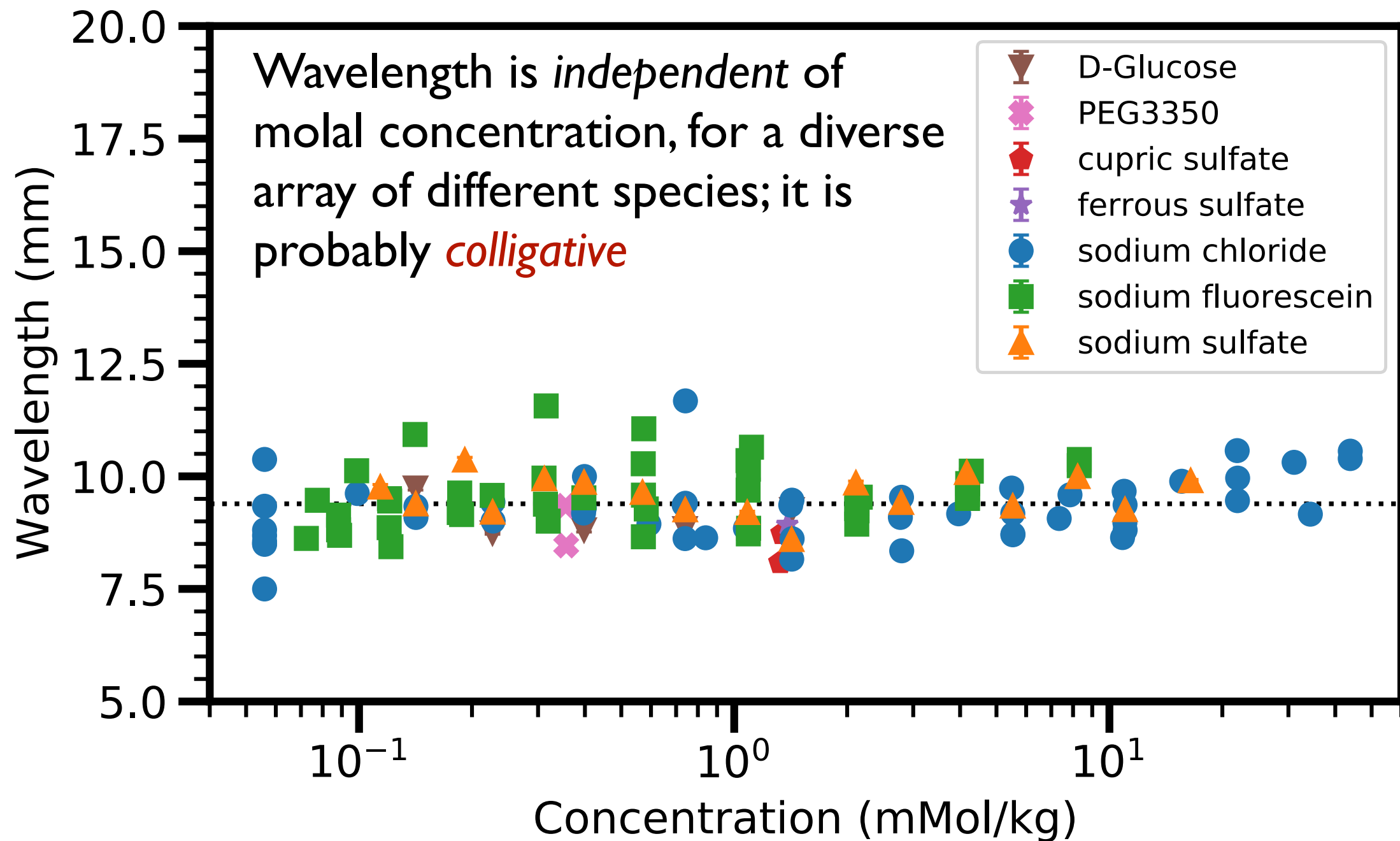
Stalagmite ripples

Ripples may also
move upward on
some cave
formations

Sectioned stalagmite by
Mirona Chirienko
*Okinawa Institute
of Science and Technology*



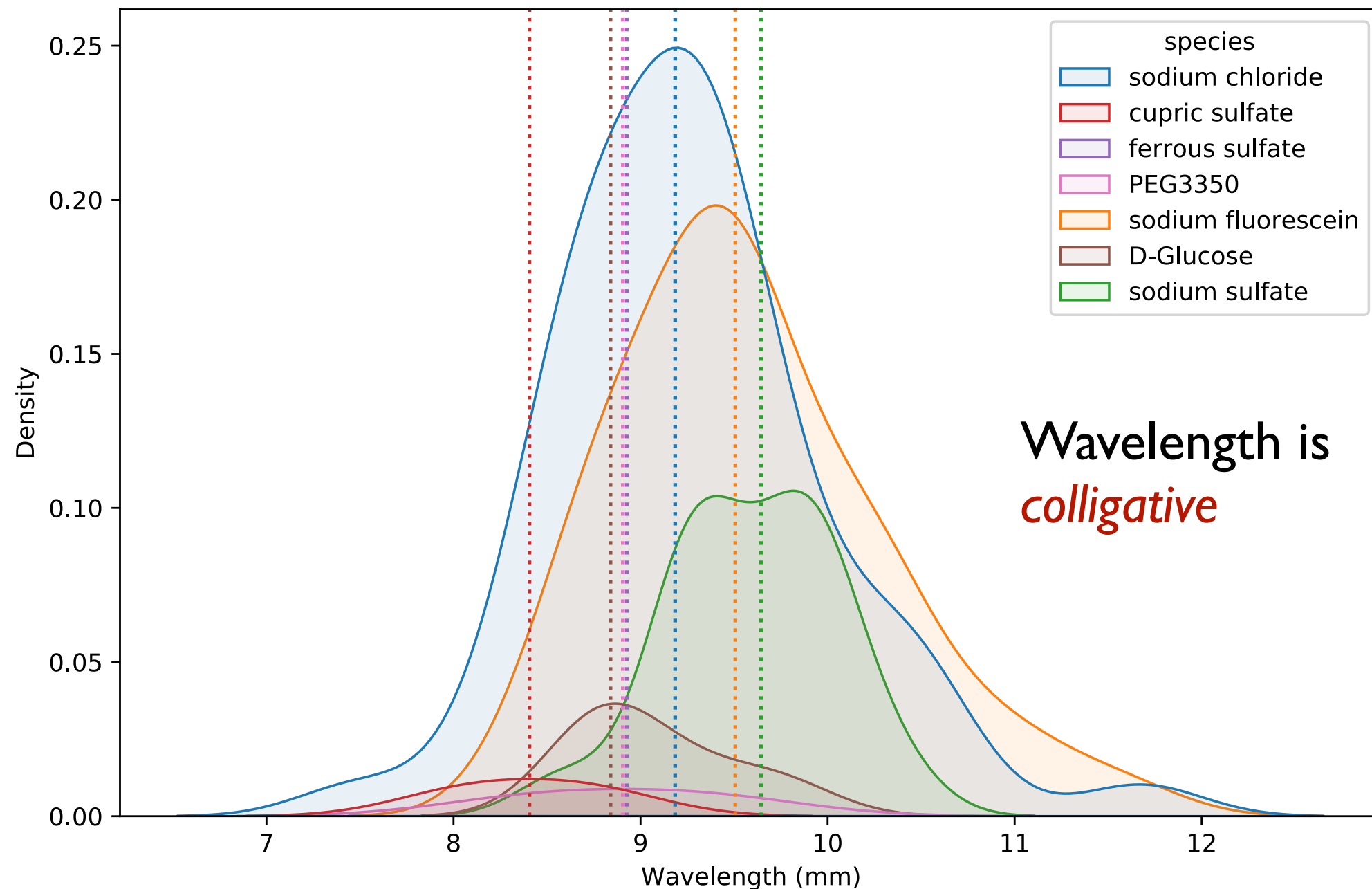
Wavelength of ripples with various impurity species



Ladan used a different “real space” peak finding algorithm to measure ripples, rather than a relying on Fourier.

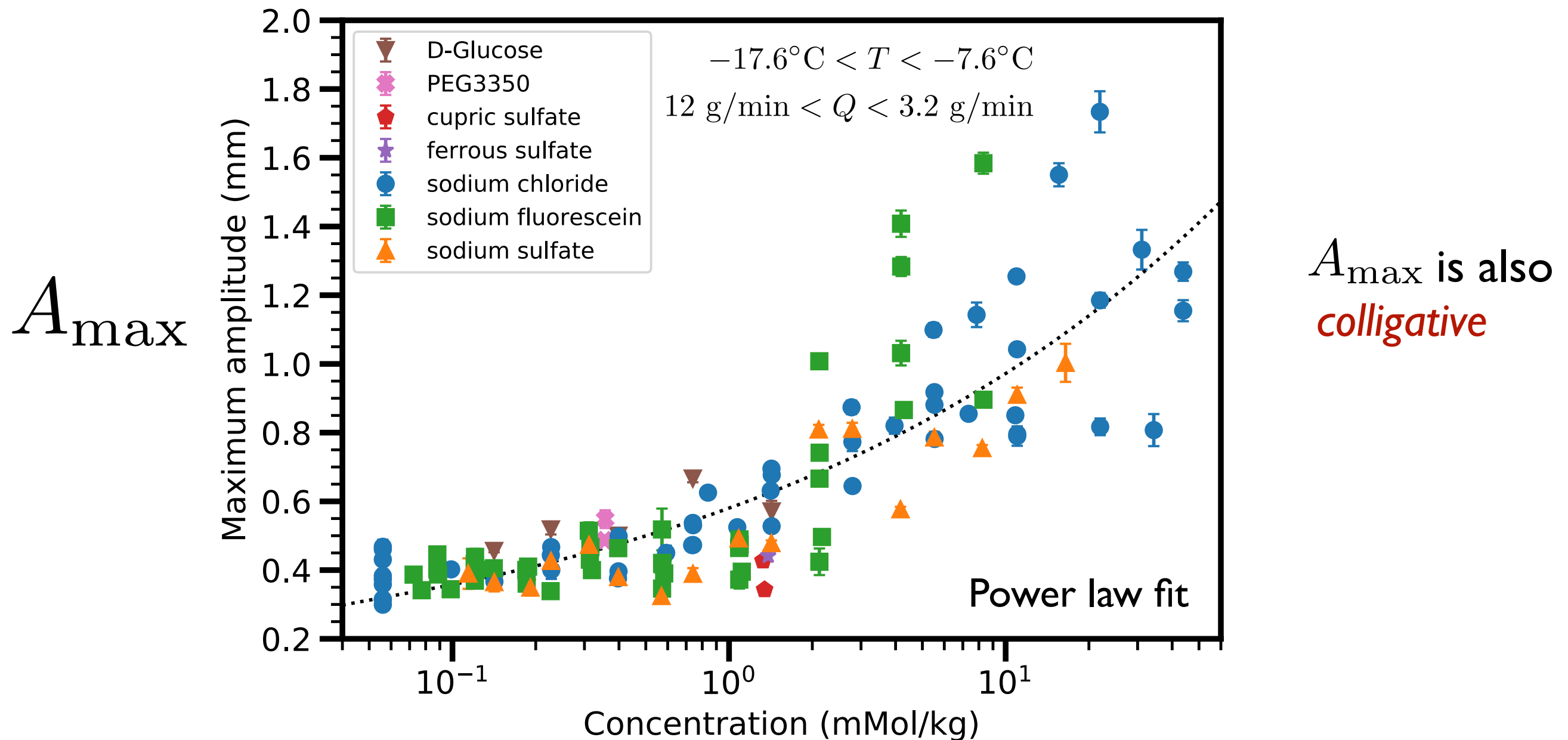
Average wavelength $\bar{\lambda} = 9.39 \pm 0.06$ mm

Wavelength of ripples with various impurity species



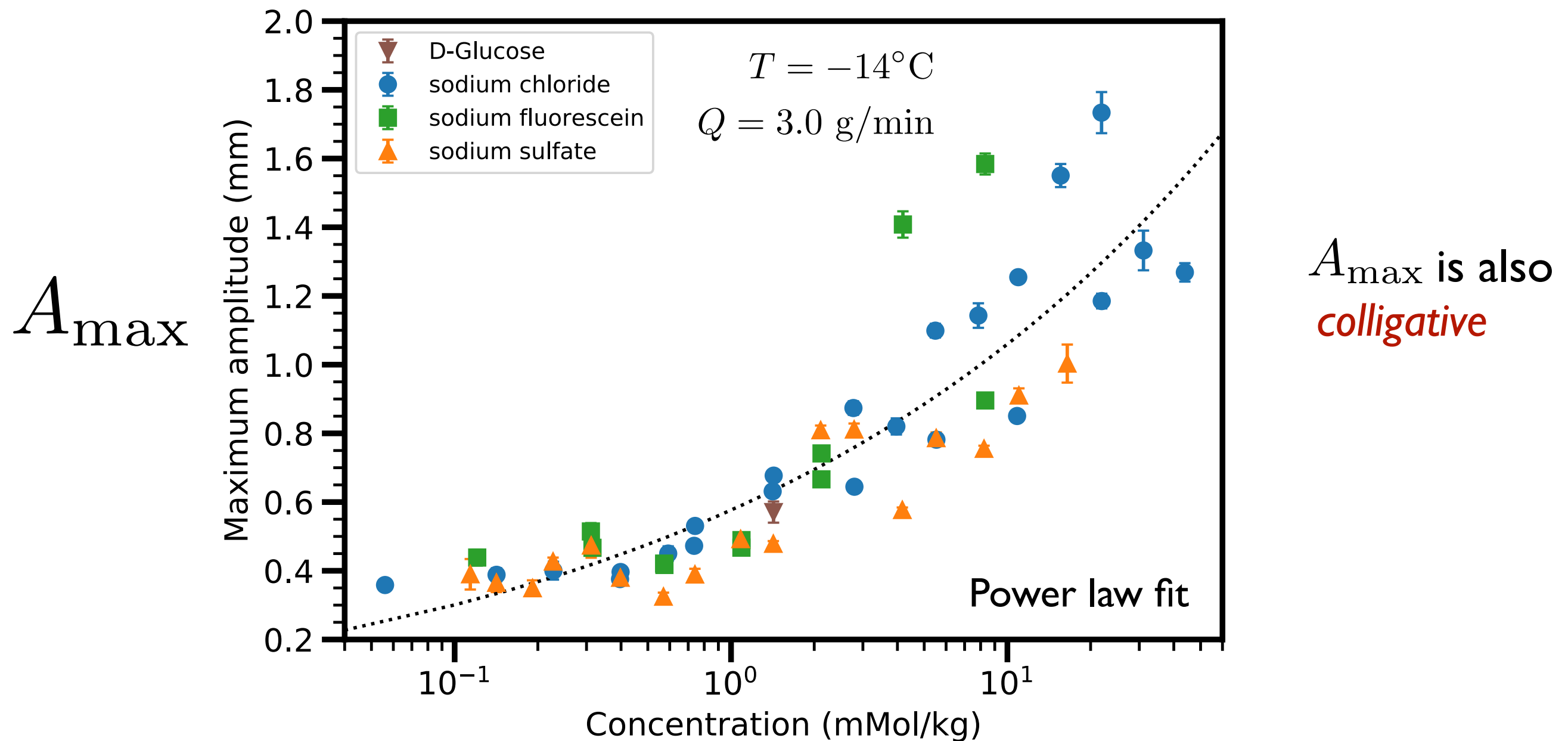
Normalized distribution of average wavelengths for various species show no significant differences.

Max amplitude of ripples with various impurity species



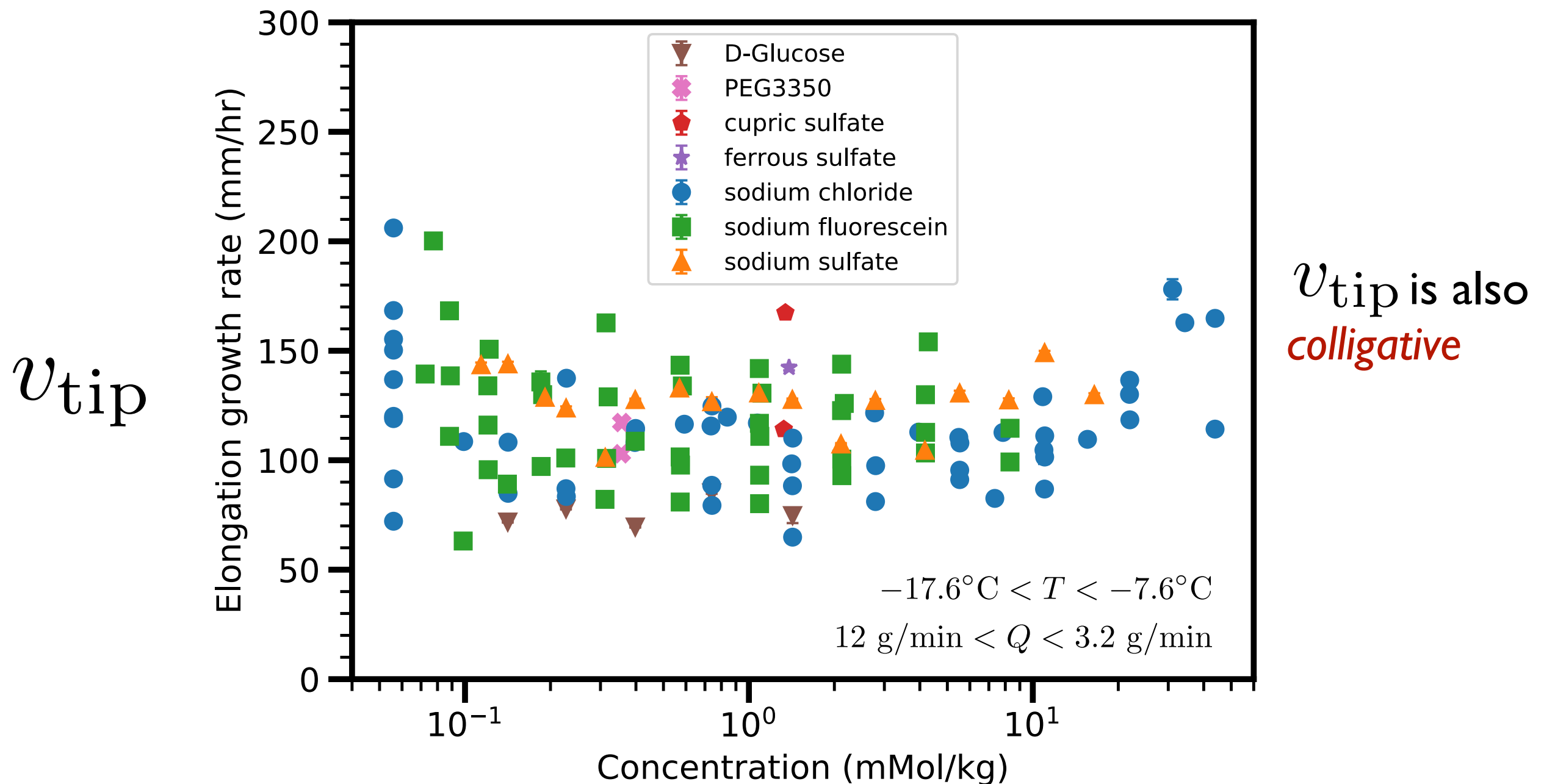
Maximum amplitude reached for all temperature and flow rate conditions and various impurity species.

Max amplitude of ripples with various impurity species



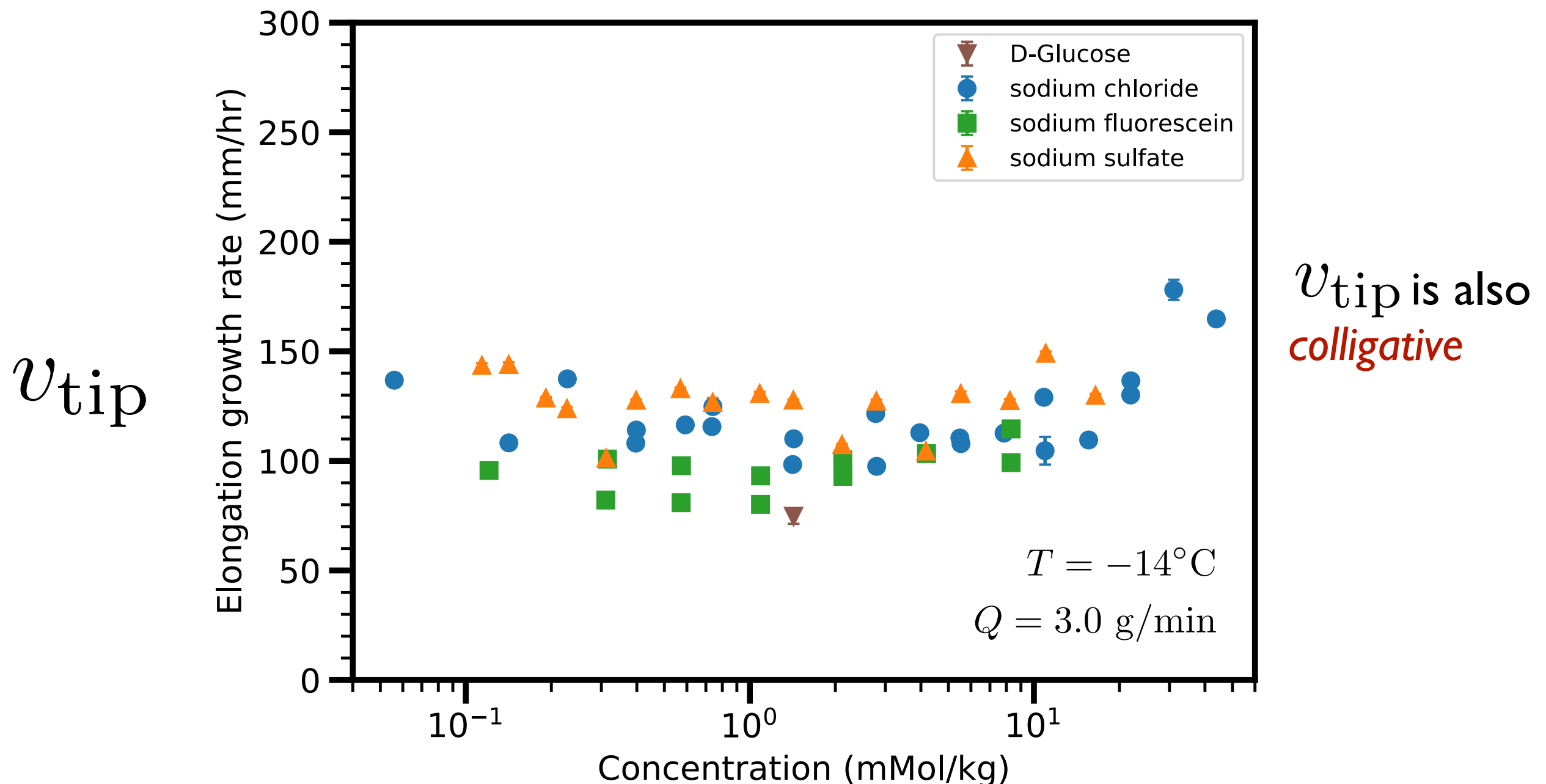
Maximum amplitude reached for $T = -14^{\circ}\text{C}$ $Q = 3.0 \text{ g/min}$ and various impurity species.

Tip growth speed with various impurity species



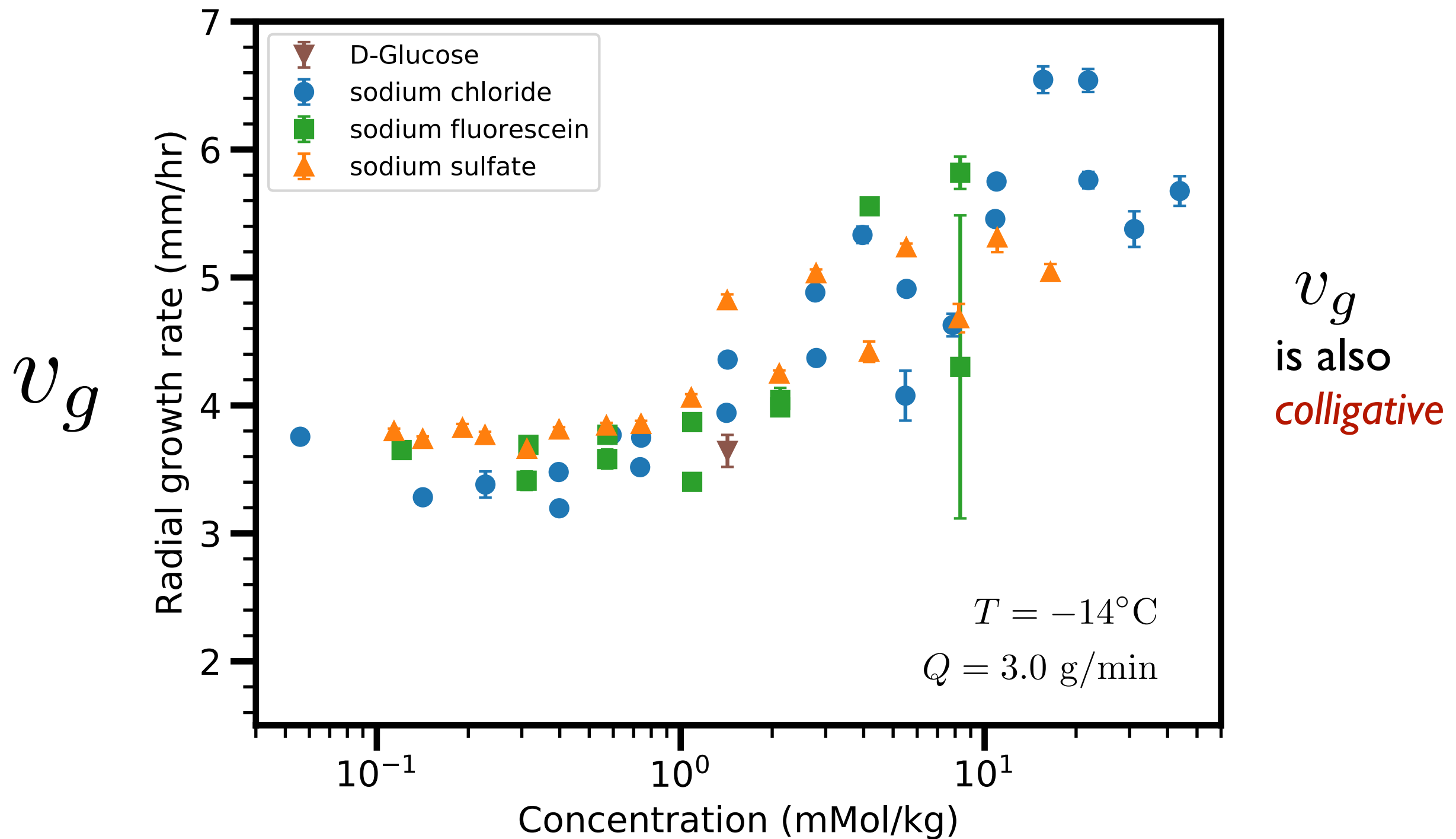
Tip growth speed is essentially independent of concentration for all temperature and flow rate conditions and various impurity species.

Tip growth speed with various impurity species



Tip growth speed is essentially independent of concentration for $T = -14^\circ\text{C}$ $Q = 3.0 \text{ g/min}$ and various impurity species.

Radial growth speed with various impurity species



Radial growth speed increases with concentration for
 $T = -14^\circ\text{C}$ $Q = 3.0 \text{ g/min}$ and various impurity species.

The effect of varying surface tension

Ripple properties, λ , $A_{\max}$, v_{tip} , v_g all appear to be (at least approximately) **colligative**.
What happens if we dope with TX-100 surfactant, which should be **non-colligative**?

Recall that adding 2×10^{-4} wt/wt of TX-100 did not by itself trigger ripples. So we made a series of **fluorescein doped** icicles with 10^{-3} wt/wt of TX-100.

This is enough to reduce the surface tension by at least 50%.

I will talk about what we can learn from the dye later. Meanwhile, I think you will agree that the effect on the **ripple wavelength** is not toooooo obvious

$T = -13.7^\circ\text{C}$
 $Q = 2.5 \text{ g/min}$
4.18 mM/kg Fl. dye
 2×10^{-2} wt% TX100



The following slide contains a **correction**. The version I presented in the lecture assumed all the error bars for the wavelengths of the TX100/Fluorescein runs were equal. In fact, they vary significantly. The new version uses properly weighted averages and the correctly weighted standard deviation of the mean.

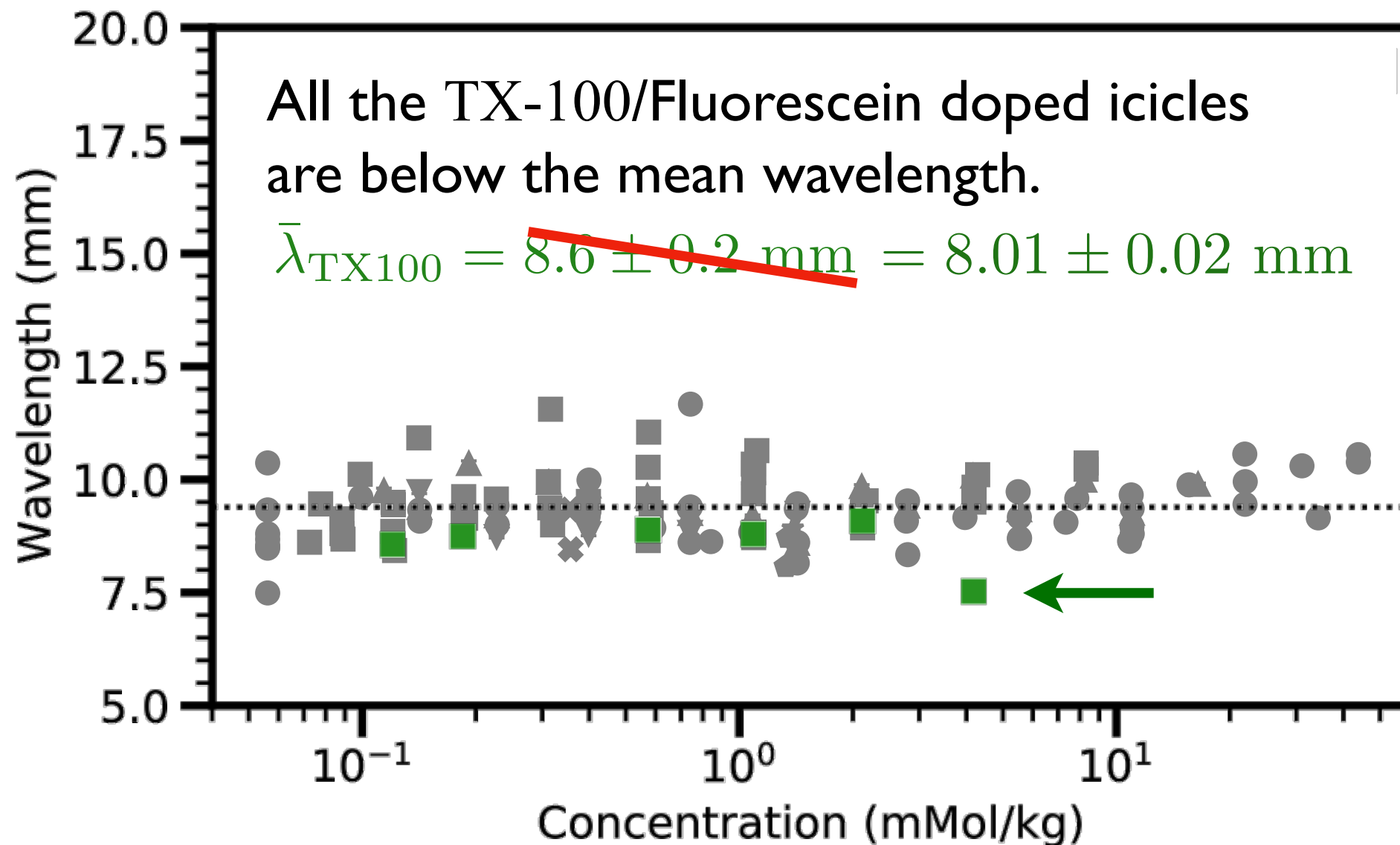
See also the discussion at the start of the third lecture.

For reference, here are the raw data:

runid	conc mM/kg	species	other	T_air deg C	flow g/min	wavelength (nm)	wavelength uncertainty (nm)
20180814_3	0.1204	sodium fluorescein	TX100	-13.9611	2.9990	8.5666	0.0968
20180814_1	0.1850	sodium fluorescein	TX100	-14.1973	3.0000	8.7454	0.1309
20180724_1	0.5723	sodium fluorescein	TX100	-13.9597	3.0005	8.8835	0.0757
20180723_1	1.0889	sodium fluorescein	TX100	-13.8637	2.9966	8.7930	0.0679
20180720_1	2.1215	sodium fluorescein	TX100	-13.8815	3.0005	9.0836	0.0743
20180719_1	4.1843	sodium fluorescein	TX100	-13.7210	2.5385	7.5261	0.0278

The effect of varying surface tension

$$\bar{\lambda} = 9.39 \pm 0.06 \text{ mm}$$
$$\sigma_{\bar{\lambda}} = 0.06 \text{ mm}$$



One data point (the one in the movie) at the highest concentration is significantly below: ~~$\lambda = 7.5 \pm 0.6 \text{ mm}$~~ $\lambda = 7.53 \pm 0.03 \text{ mm}$

Convinced!?

Need more data!

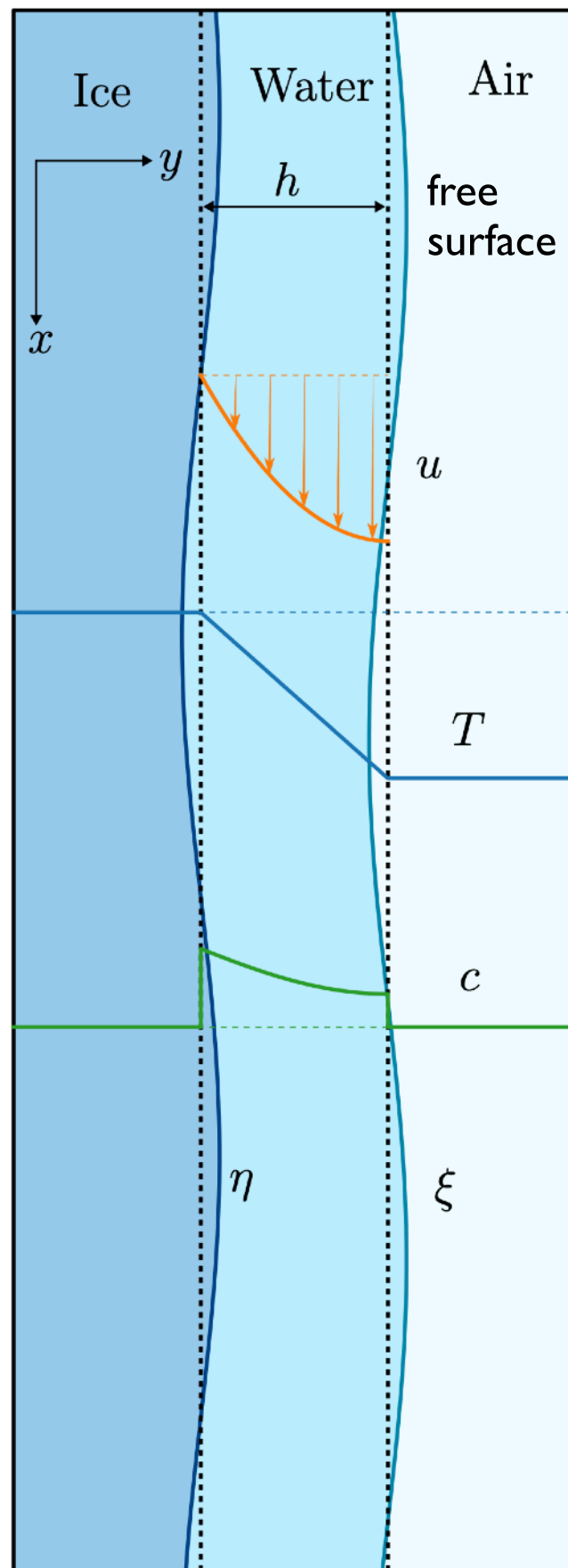
Ripple theory

Overview, history

- The first attempt to explain icicle ripples as a linear instability was [N. Ogawa and Y. Furukawa, *Phys. Rev. E* **66**, 041202 \(2002\)](#) who assumed pure water.
- This motivated K. Ueno to write a long series of papers, including [K. Ueno, M. Farzaneh, S. Yamaguchi and H. Tsuji, *Fluid Dyn. Res.*, **42**, 025508 \(2010\)](#) which criticizes and attempts to improve upon Ogawa. Ueno used the same physical model, and assumes pure water, but makes different assumptions about certain **boundary conditions**.
- Ladan [Ph.D. thesis \(2023\)](#) generalized Ueno's model to include impurities.
- Recently, [M. G. Worster, *J. Eng. Math.*, **145**, 15 \(2024\)](#) presented a critical review of Ogawa *et al* and Ueno *et al*, correcting errors and deriving new results for pure water.



Ripple theory



Base state

Velocity
 U

Temperature
 ΔT_w

Concentration
 C_0

Perturbation

All theories assume the icicle is axis-symmetric and covered by a thin flowing film of thickness $h \sim 0.1$ mm.

The water film is supercooled by a **very small** amount $\Delta T_w \sim 75$ mK.

In Ladan's generalization, an impurity concentration C is included, with input value C_0 .

The ice surface is given an **infinitesimal perturbation** $\eta = \hat{\eta} e^{ikx + \sigma t}$ where $\sigma(k)$ is a (complex) growth rate.

Solving the time independent Navier-Stokes and incompressibility equations with standard boundary conditions, gives a **4th order** Orr-Sommerfeld equation for the perturbed stream function. This can be used to find the perturbed position of the free surface $\xi = \hat{\xi} e^{ikx + \sigma t}$.

Ripple theory

Ueno pure water case

In dimensionless form, the Orr-Sommerfeld equation for the perturbed stream function depends on the **Reynolds** and **Weber** numbers:

$$\text{Re} = \frac{Uh}{\nu} \sim 0.4 \quad \text{We} = \frac{\rho U^2 h}{\gamma} \sim 10^{-4} \quad \text{where } U \text{ is the surface velocity of the base state shear and } \gamma \text{ is the surface tension.}$$

The idea is to use this flow to drive a steady-state advection diffusion equation for the temperature

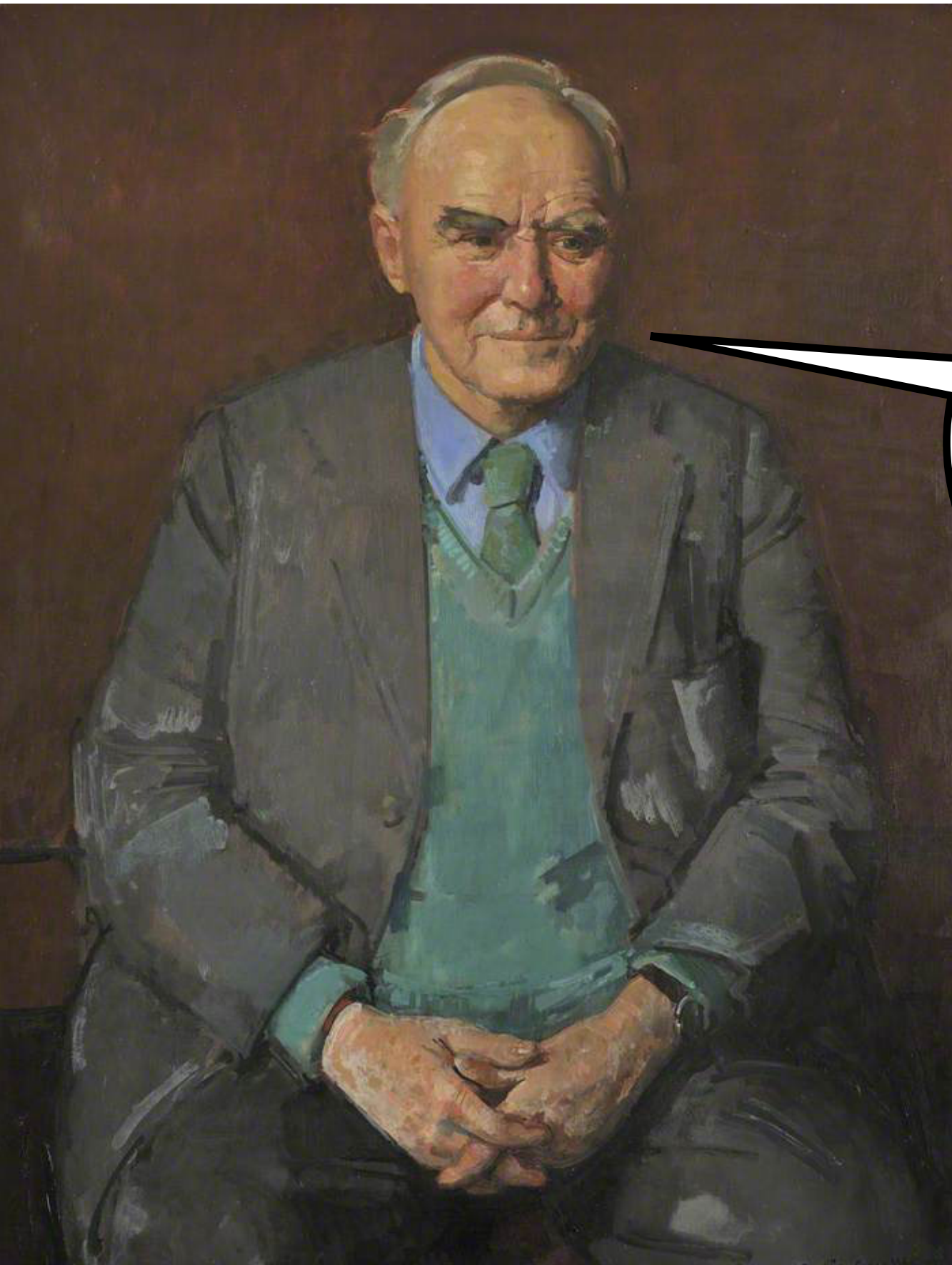
$$\vec{u} \cdot \nabla T = \frac{1}{\text{Pe}} \nabla^2 T \quad \text{where } \text{Pe} = \frac{Uh}{\kappa_w} \sim 5 \quad \text{is the } \textbf{Péclet number}.$$

$$\text{Stefan condition } \text{Pe} \mathcal{S} \left[\bar{V} + \frac{d\eta}{dt} \right] = -\nabla T|_{y=\eta} \quad \text{with} \quad \mathcal{S} = \frac{L}{C_p \Delta T_w} \sim 1000$$

the only t dependence **Stefan number**

In this picture, the ripples grow because the secondary flow carries latent heat away from the peaks, causing them to grow faster. **Nothing** happens in the air or inside the ice. It is *not* a Mullins-Sekerka type Laplace instability with the ripples sticking out into the cold outside air. Gibbs-Thompson is neglected.

Ripple theory: boundary conditions



“There are only 3 problems
in fluid mechanics:
*Boundary conditions,
boundary conditions
and boundary conditions!*”

G. I. Taylor
(who probably never said this)

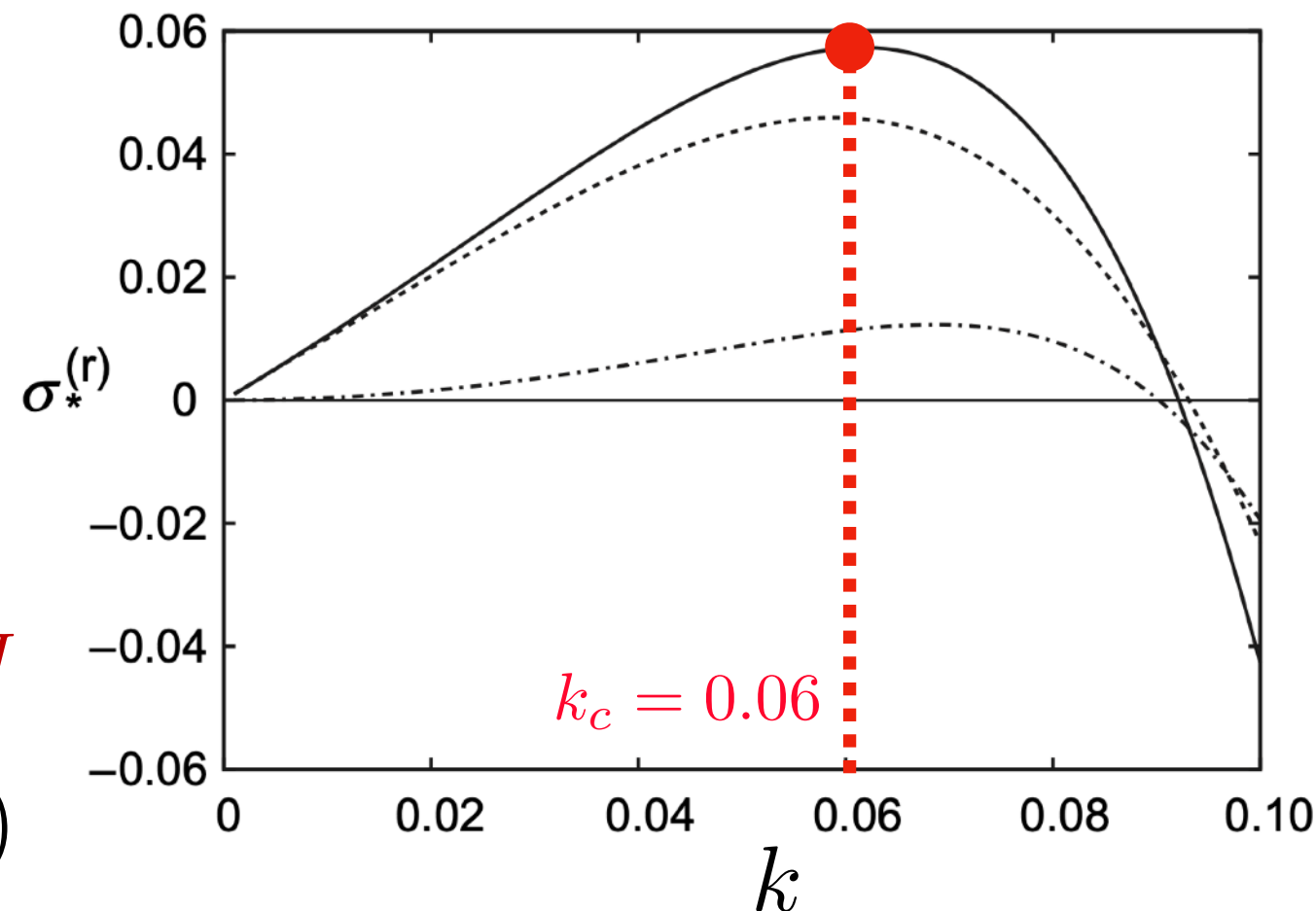
Ripple theory

The temperature equation turns out to be a second order parabolic cylinder equation. What boundary conditions shall we use for T ?

Ueno chooses to fix **both** BCs at the free water surface:

$$T|_{y=\zeta} = T_{\text{air}} \quad \text{and} \quad -(\hat{\mathbf{n}} \cdot \nabla)T|_{y=\zeta} = J$$

Then the real part of σ peaks at $\sigma_r(k_c)$ with $k_c = 0.06$ which corresponds to a fastest growing ripple at



$$\lambda = 10.5 \text{ mm.}$$

σ has an imaginary part $\sigma_i < 0$ which corresponds to an **upward traveling** ripple.

BUT: this choice leaves $T|_{y=\eta}$ *unconstrained*. Ueno shows that the ice-water interface deviates from $T_m = 0^\circ\text{C}$ by about 0.1 mK.

Ueno rationalizes that this must be a non-equilibrium effect.

Ripple theory

Ladan generalization to impure water

The only new physics needed to account for impurities is **freezing point depression**.

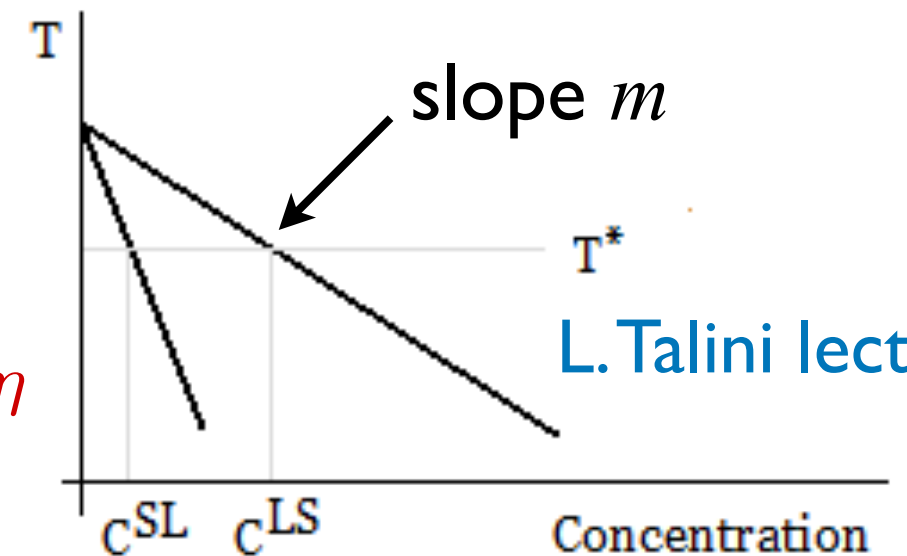
This is plausible because $mC_0/\Delta T_w \sim 1$

where m is the slope of the liquidus line.

We must have two T BCs:

ice-water: $T|_{y=\eta} = T_m - mC|_{y=\eta}$

water-air: $-(\hat{\mathbf{n}} \cdot \nabla)T|_{y=\zeta} = J$



But now J accounts for all forms of heat transfer and the BC for the perturbed temperature is a mixed **Robin** BC: $J \sim [T + \lambda(\hat{\mathbf{n}} \cdot \nabla)T]|_{y=\zeta}$

Solve:

$$\vec{\mathbf{u}} \cdot \nabla C = \frac{1}{\text{Pe Le}} \nabla^2 C$$

$$C \left[\bar{\mathbf{V}} + \frac{d\eta}{dt} \right] = -\nabla C|_{y=\eta} \text{ impurity rejection}$$

$$\text{Le} = \kappa/D \sim 10$$

Lewis number

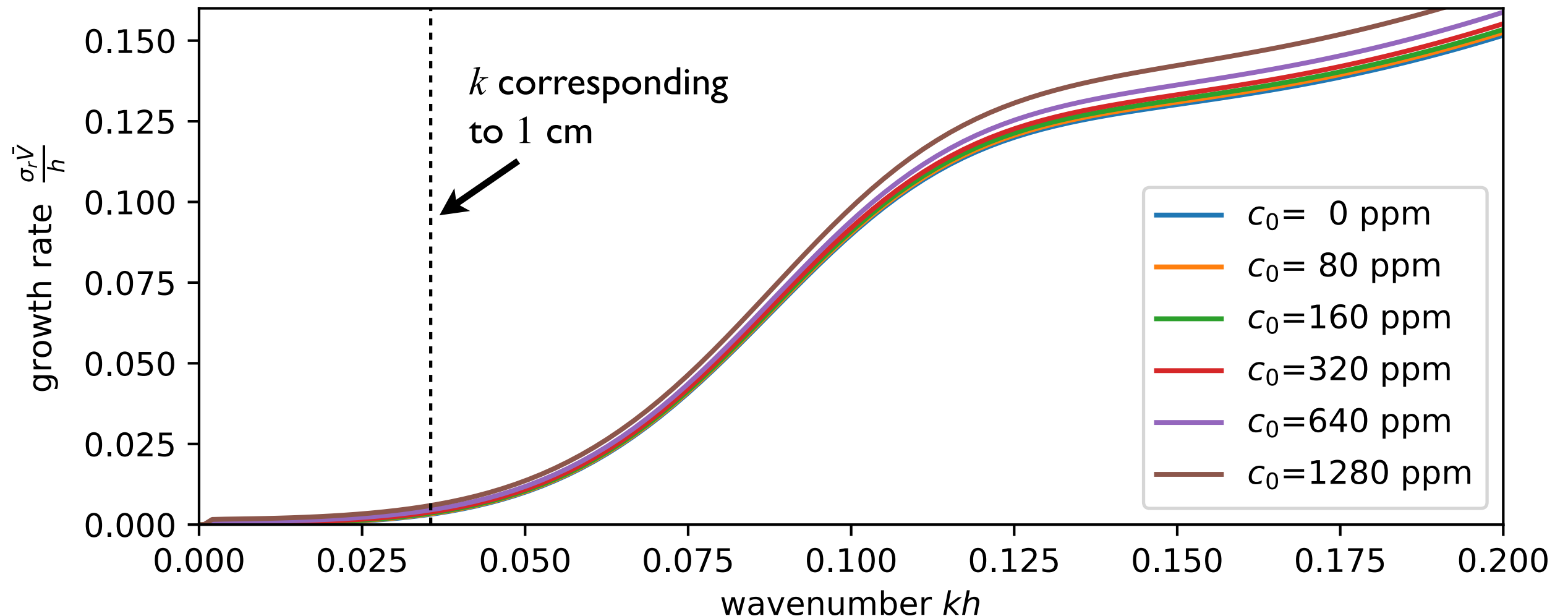
$$(\hat{\mathbf{n}} \cdot \nabla)C|_{y=\zeta} = 0 \quad \text{no flux}$$

J. Ladan *Ph.D. thesis* (2023).

J.A. Neufeld, R. E. Goldstein and M. G. Worster, *J. Fluid Mech.* **647**, 287 (2010).

Ripple theory

Ladan generalization to impure water

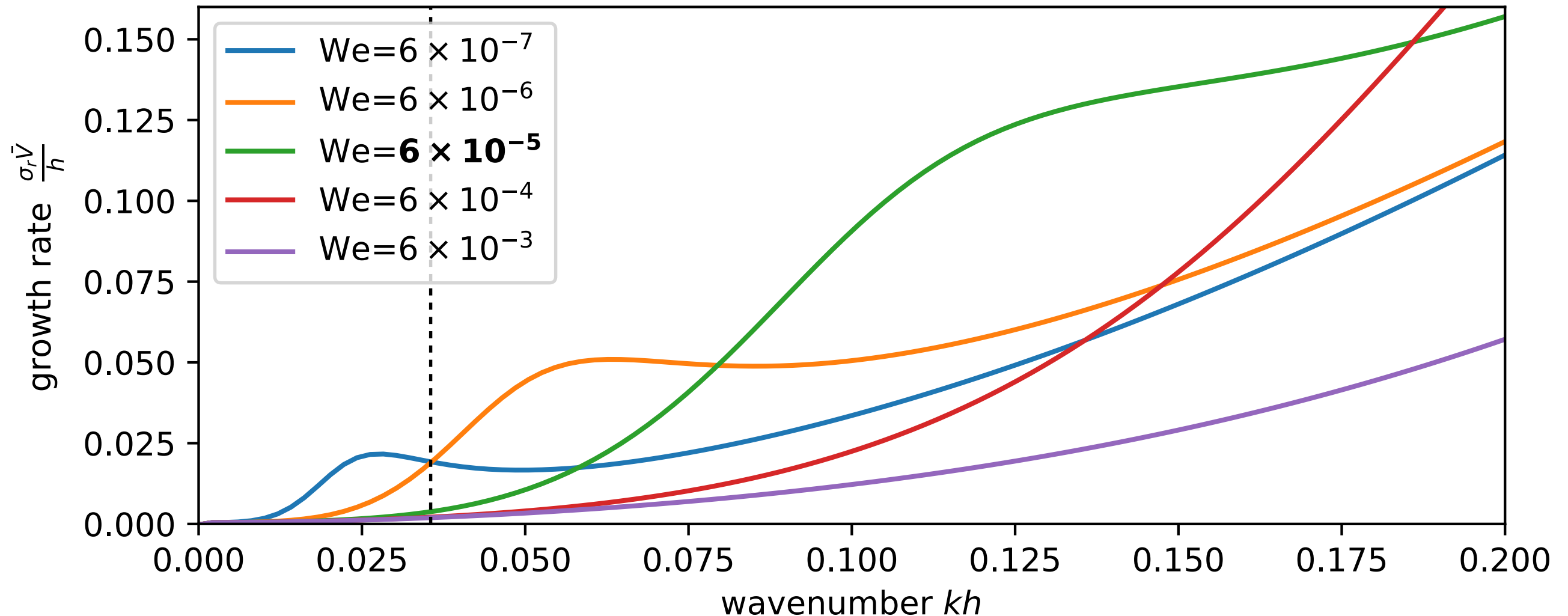


Linear theory gives a growth rate that **diverges** at large wavenumber, even for pure water. It **does not have a peak**. It is insensitive to the impurity concentration.

An “ultraviolet catastrophe” — no short wavelength cutoff.

Ripple theory

Ladan generalization to impure water



The growth rate *is* sensitive to the surface tension, but only if you change We by orders of magnitude. It seems to reach only a local peak for an unphysically large surface tension.

An “ultraviolet catastrophe” — no short wavelength cutoff.

Ripple theory

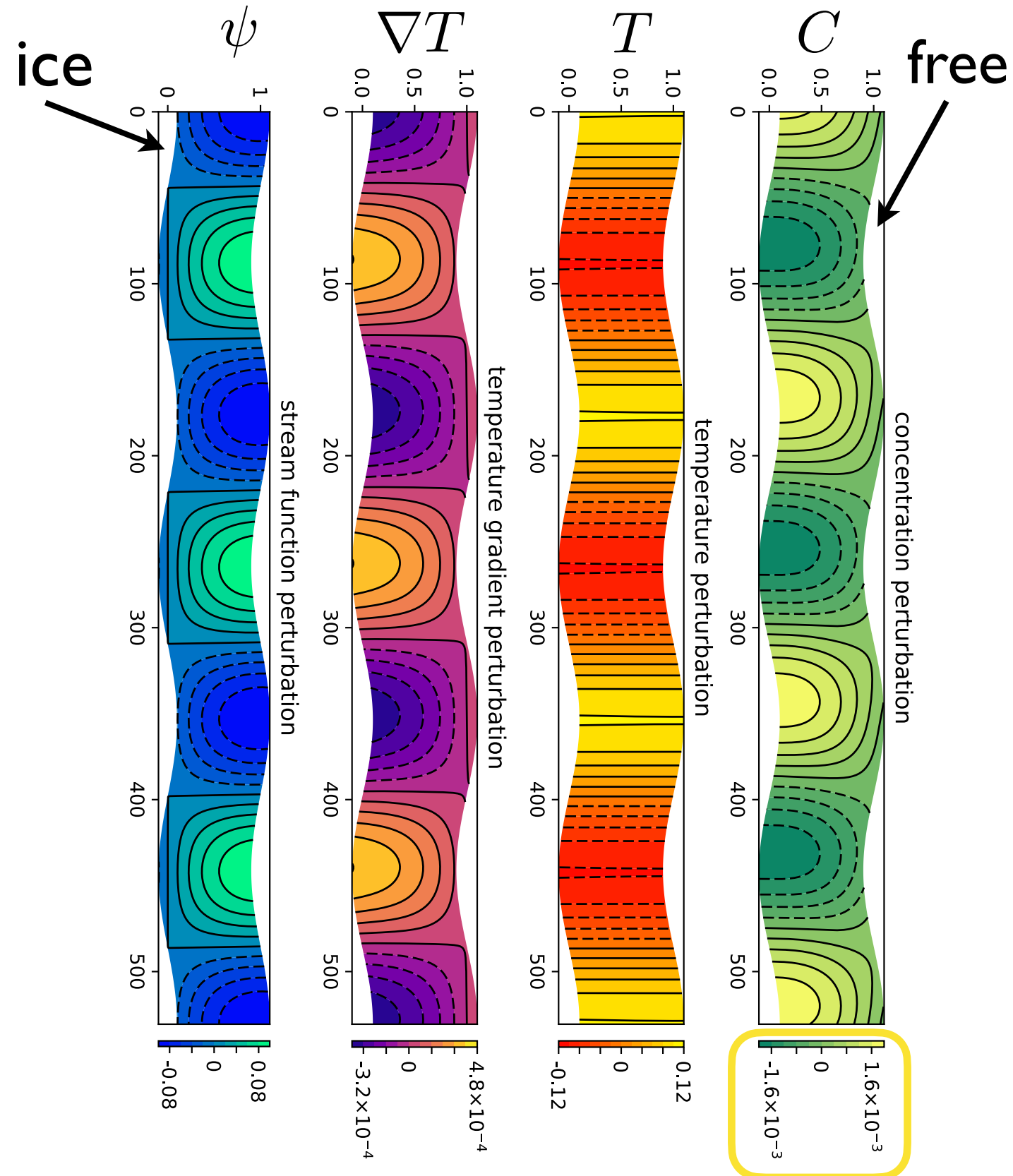
Ladan generalization to impure water

The insensitivity to impurities can be traced to the size of the coupling terms between T and C in the model.

After a careful scaling analysis, we find that the coupling scales as

$$\text{Le}/\mathcal{S} \sim 0.01$$

The effect of **freezing point depression** is intrinsically small. Too small to have much effect on the growth rate (which diverges for large k for physical boundary conditions anyway).



Ripple theory

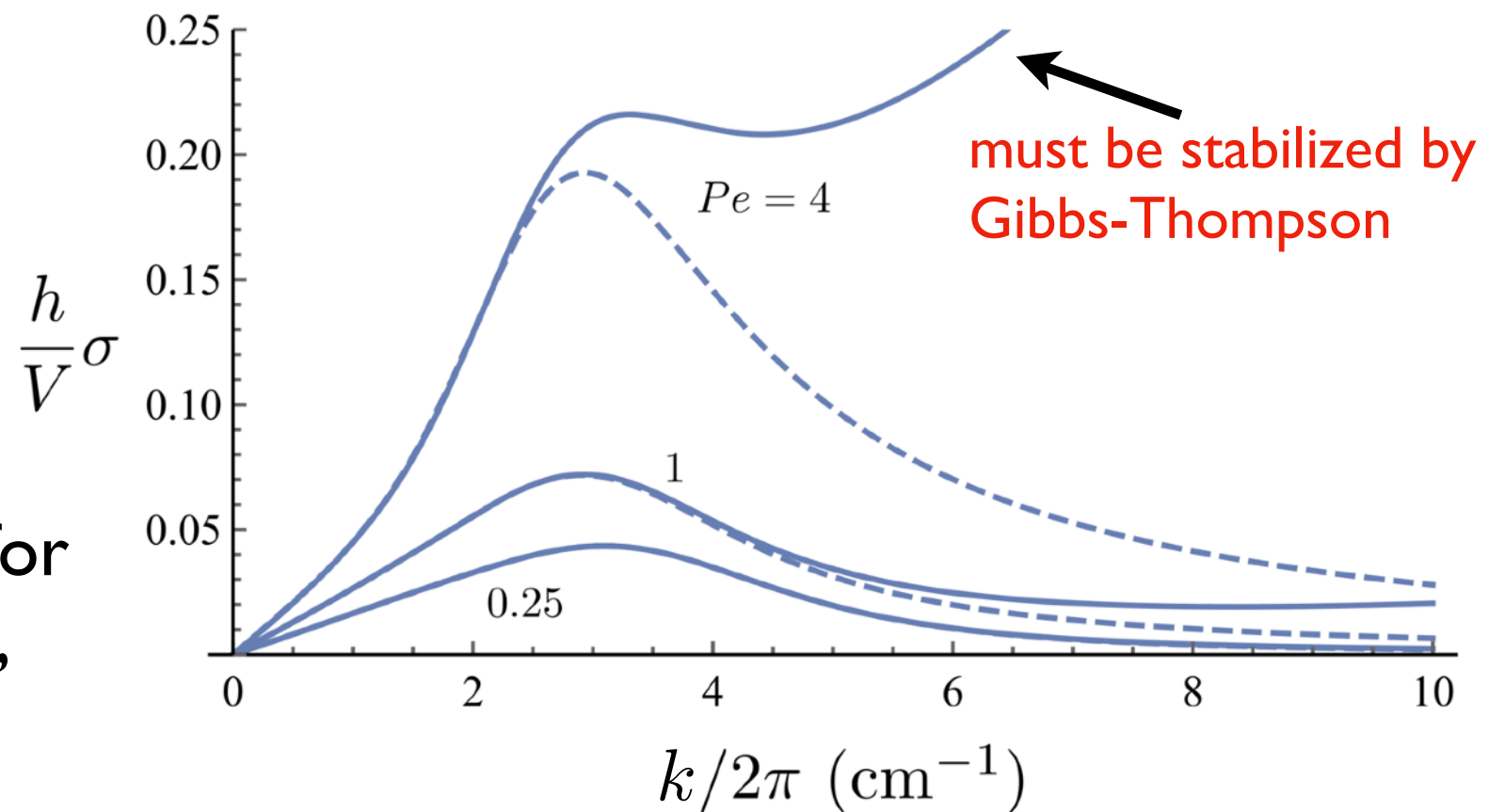
Worster reconsiders previous **pure water** models and completely reformulates the problem using lubrication and thin-film equations, avoiding Navier Stokes.

He develops a series of models for increasing influence of advection, measured by Pe .

He finds instabilities with various wavelengths, with the largest Pe only being stabilized by an “unexpectedly large” **Gibbs-Thompson** effect (which was previously thought to be negligible).

He points out Ueno’s “**unphysical, mathematical**” boundary conditions.

Worster reformulation for pure water



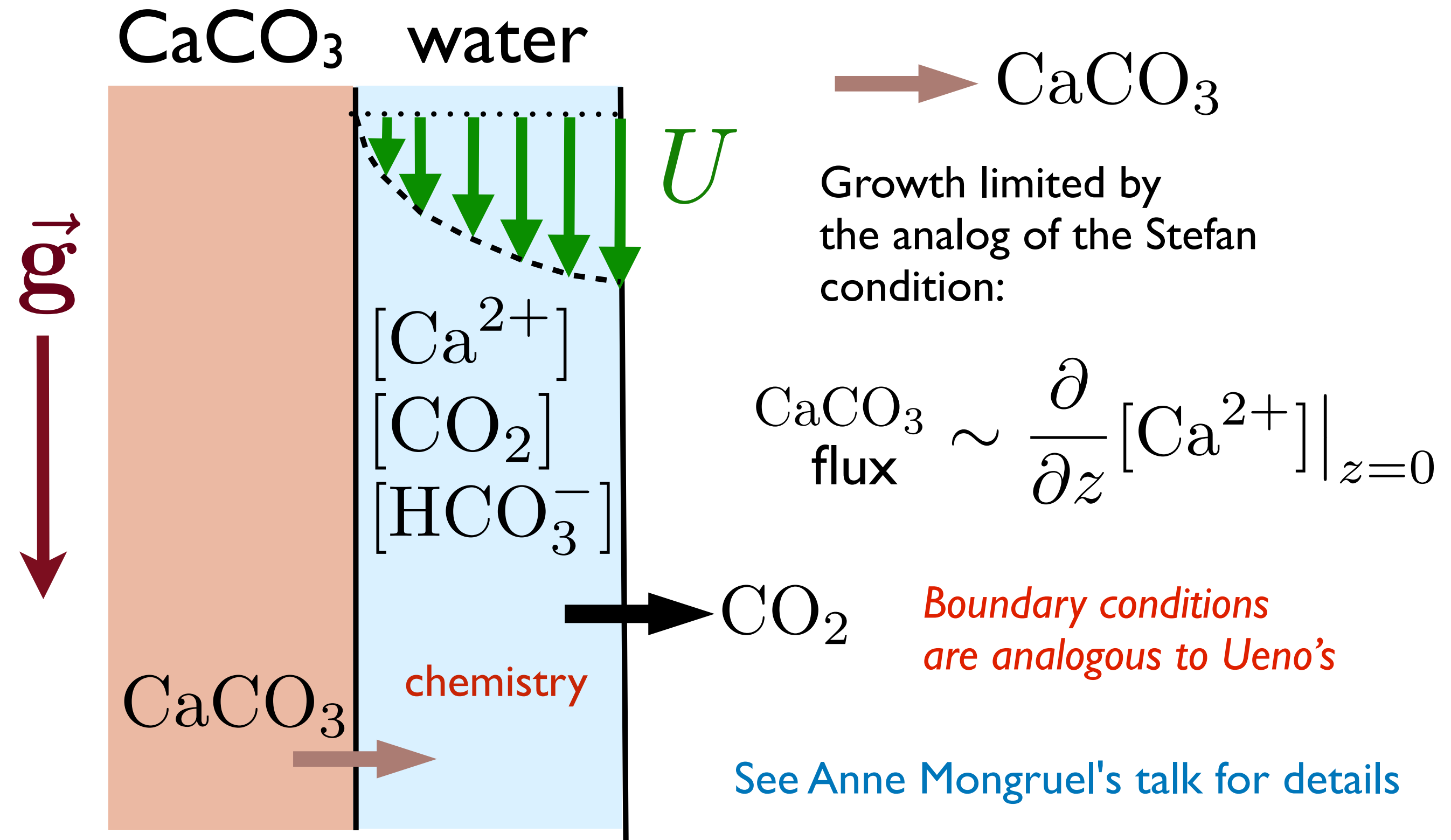
$$Pe = 0 \rightarrow \lambda \sim 5.5 \text{ mm}$$

$$Pe = 1 \rightarrow \lambda \sim 3 \text{ mm}$$

$$Pe = 4 \rightarrow \lambda \sim 2 \text{ mm}$$

Contrast with stalactite ripple theory

C. Camporeale and L. Ridolfi,
Phys. Rev. Lett., **108**, 238501 (2012).



Status of linear stability theory of ripples

We have no satisfactory stability analysis that accounts for the ripples and their dependence on impurities.

- Ueno *et al* find an instability for **pure water** with the “right” 1cm wavelength, but only by using **unphysical boundary conditions** on the temperature across the water film.
- Using physical boundary conditions and **freezing point depression** for impure water, Ladan finds a divergent growth rate and **insensitivity** to concentration.
- Worster used a thin film formulation for **pure water** and finds a series of **too-small wavelength** instabilities ultimately stabilized by Gibbs Thompson.





Fin