The fluctuation-dissipation theorem: from statistical physics to climate dynamics?

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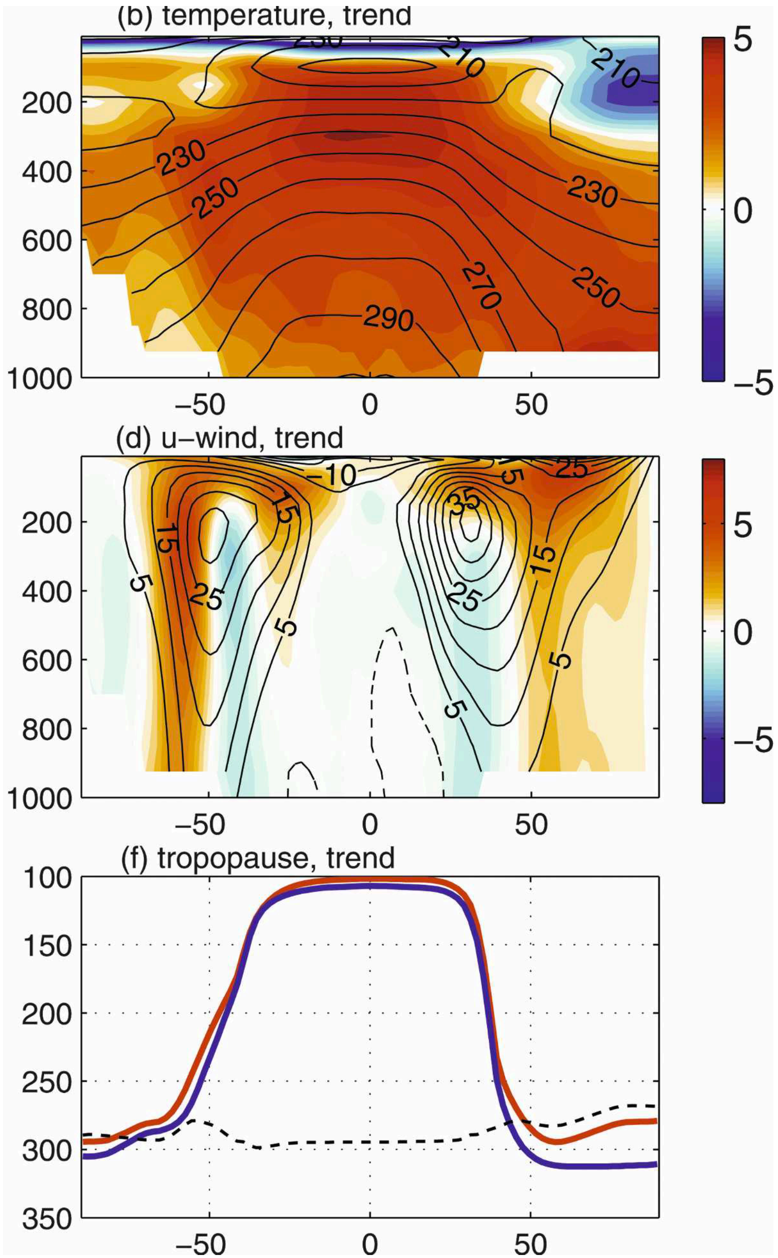
(PHH acknowledges collaborations with Fenwick Cooper and Peter Hitchcock)

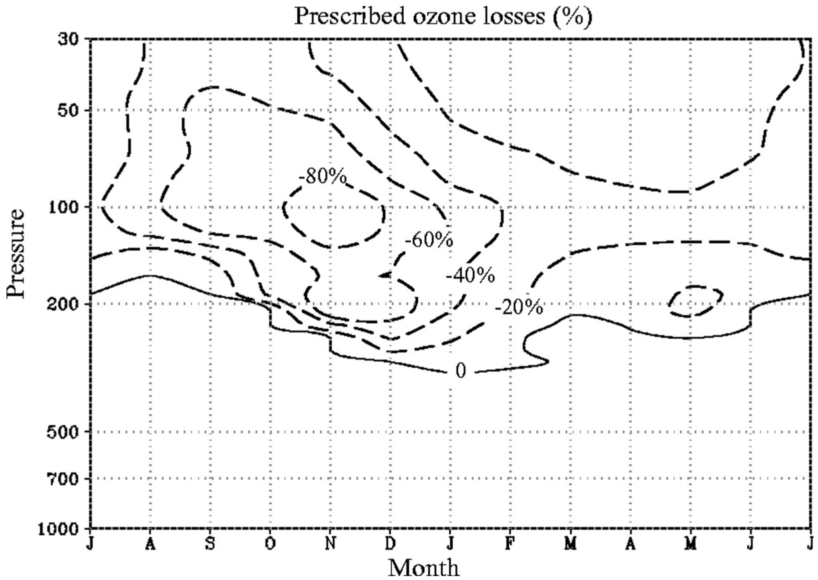
The climate response problem

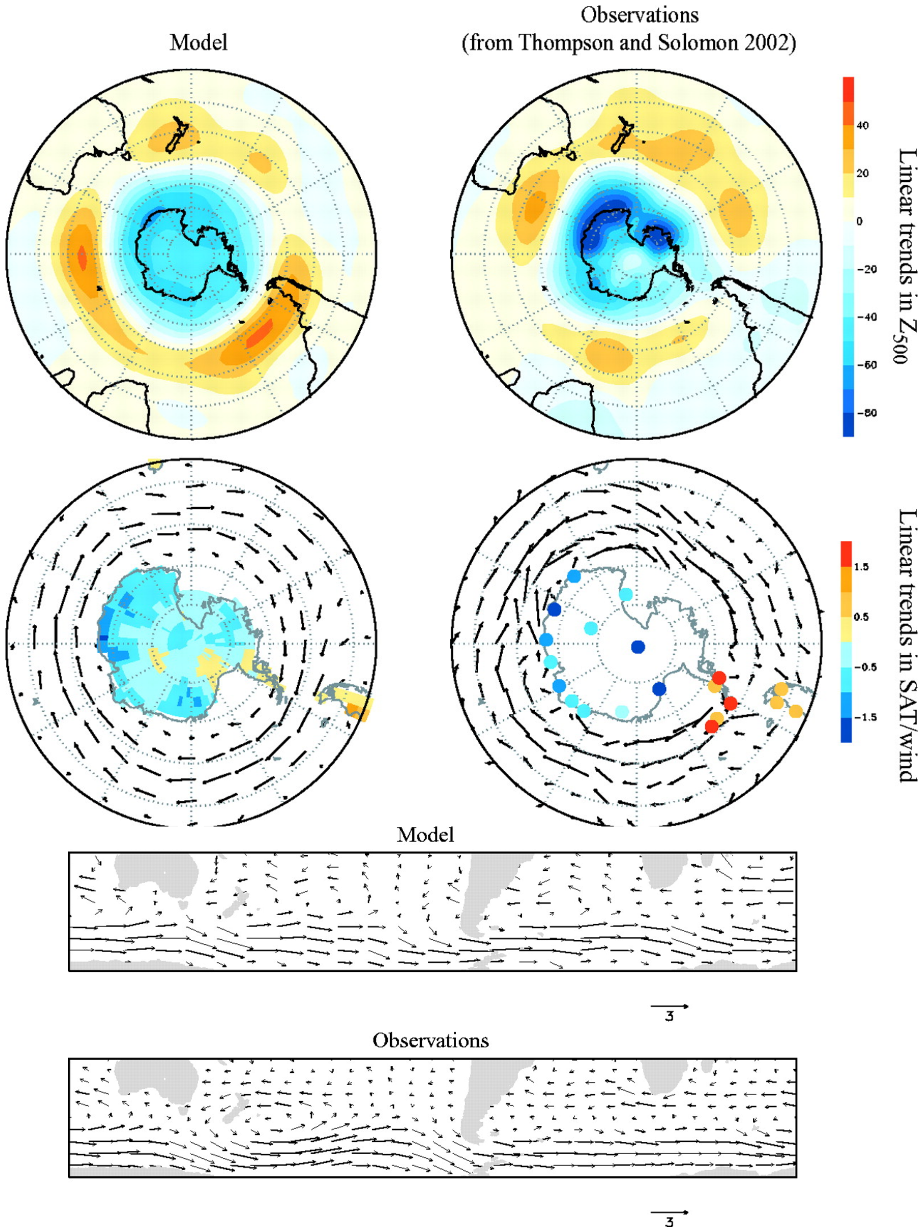
Lu et al 2008: simulations with GFDL CM2.1 model, 2081-2100 compared with 2001-2020 in A2 scenario.



Longitudinal/time average change depends on physical processes which fluctuate in longitude and time.

Changes in the SH troposphere as a dynamical response to stratospheric ozone depletion



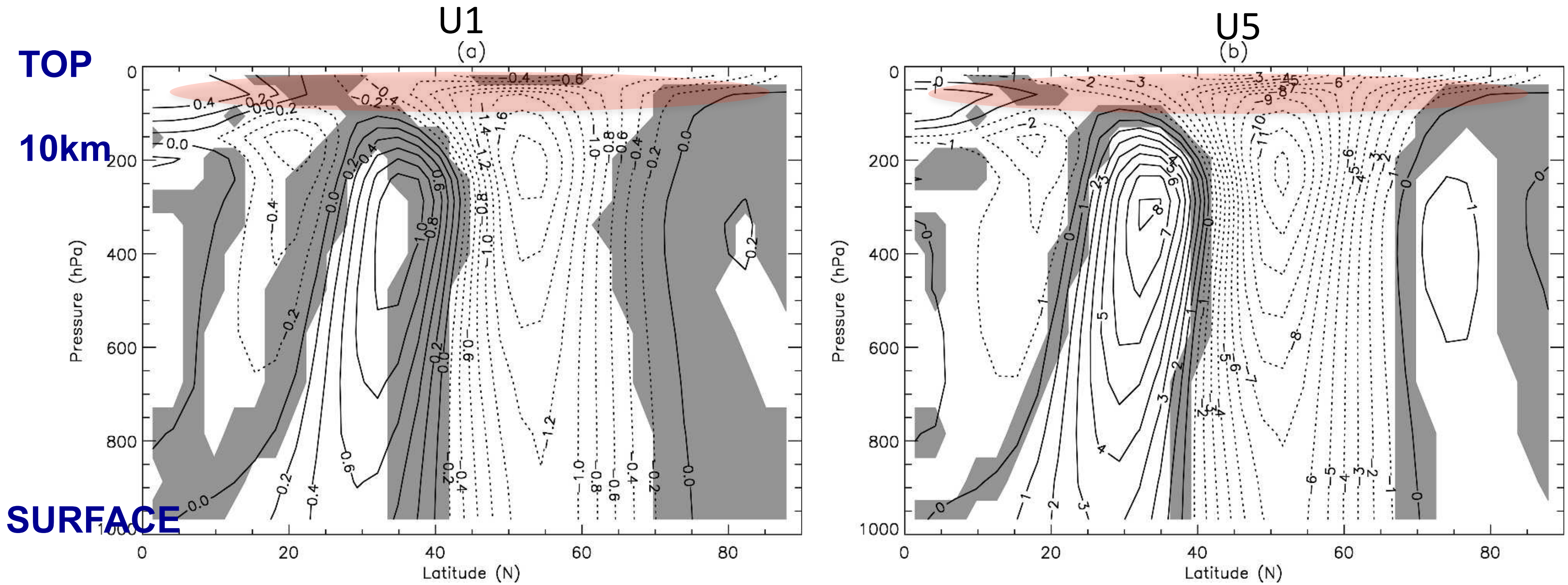


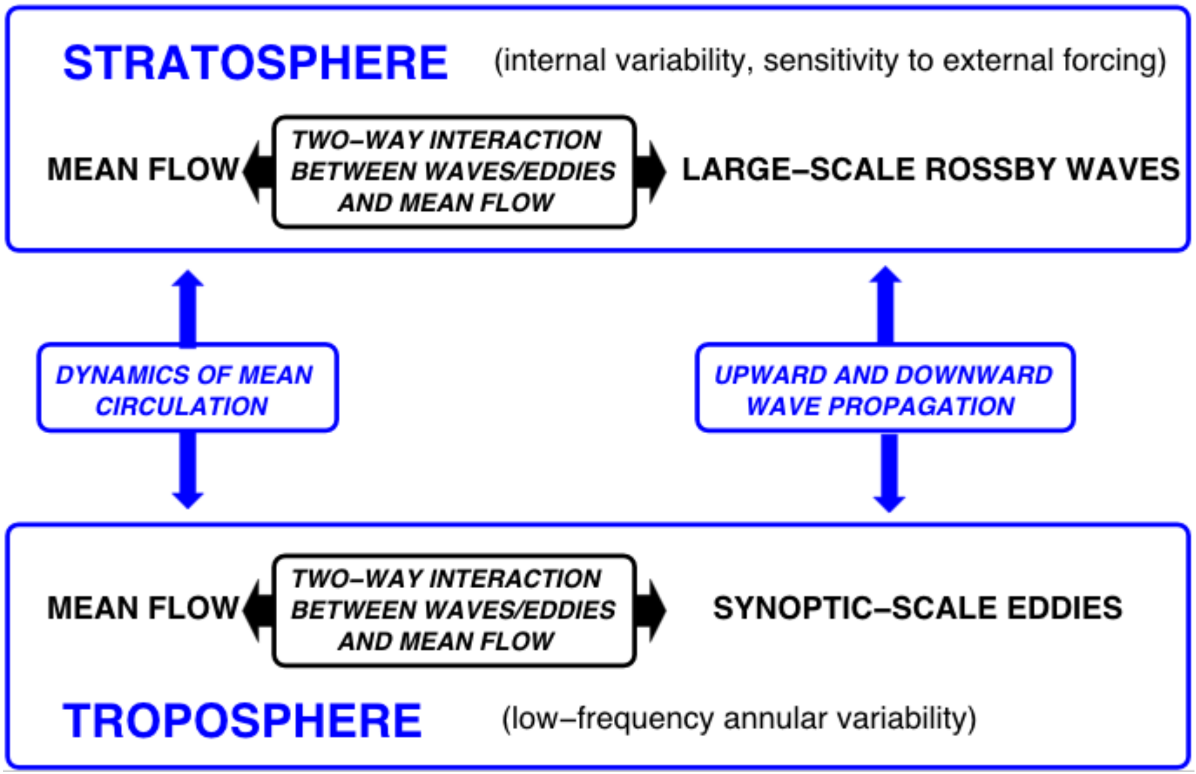
Ozone perturbation applied to AGCM

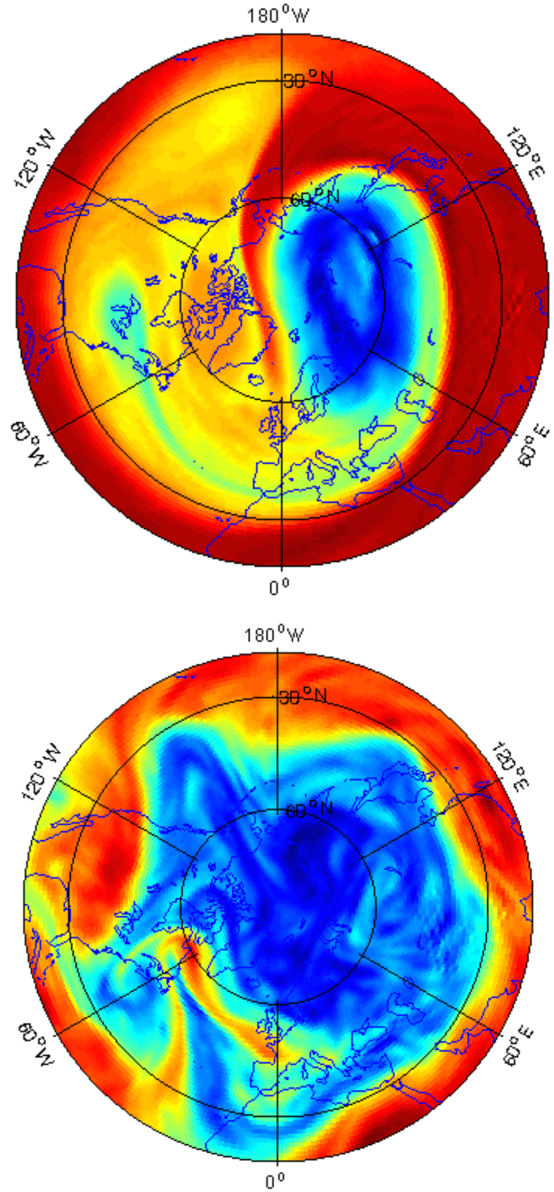
(Thompson and Solomon 2002, Gillett and Thompson, 2003)

Simple model of solar cycle effects

Haigh et al (2005): response of simple troposphere to imposed changes, e.g. uniform increase in radiative equilibrium temperature in stratosphere




STRATOSPHERE
(internal variability, sensitivity to external forcing)
MEAN FLOW
TWO–WAY INTERACTION BETWEEN WAVES/EDDIES AND MEAN FLOW
LARGE–SCALE ROSSBY WAVES
DYNAMICS OF MEAN CIRCULATION
UPWARD AND DOWNWARD WAVE PROPAGATION
MEAN FLOW
TWO–WAY INTERACTION BETWEEN WAVES/EDDIES AND MEAN FLOW
SYNOPTIC–SCALE EDDIES
TROPOSPHERE
(low–frequency annular variability)


180°W
30°N
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180°W
30°N
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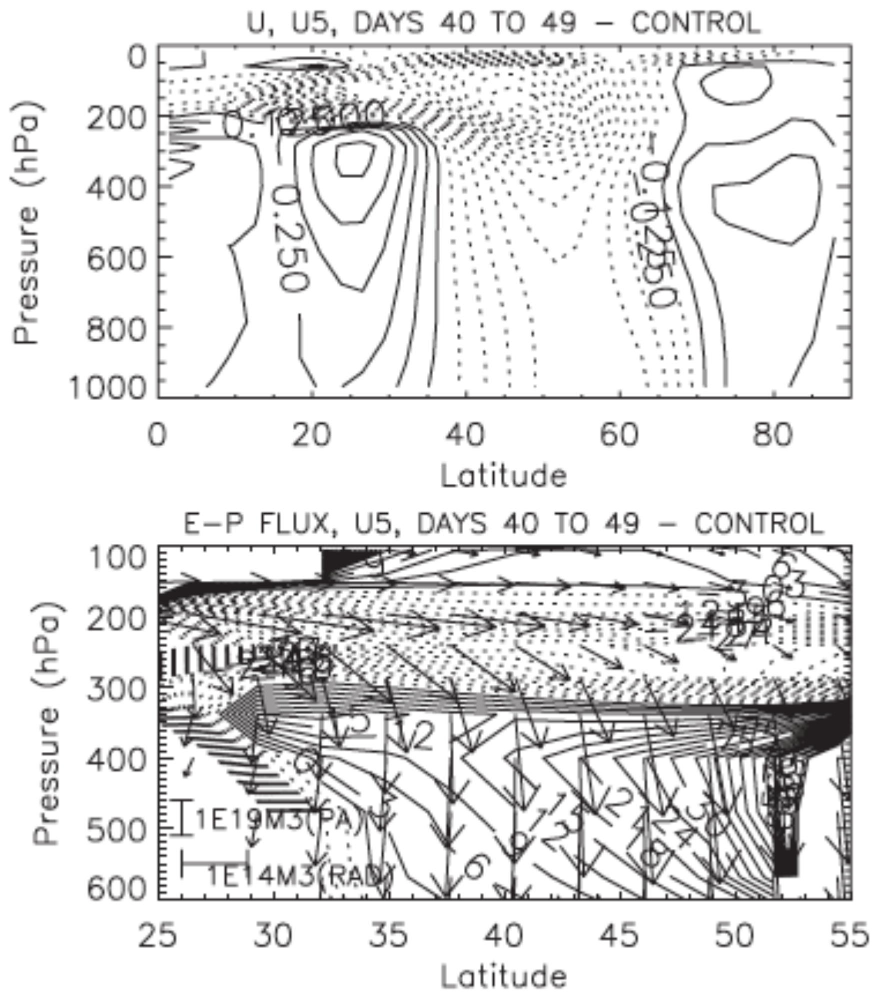
60°W
60°E
0°

Simpson et al (2009)

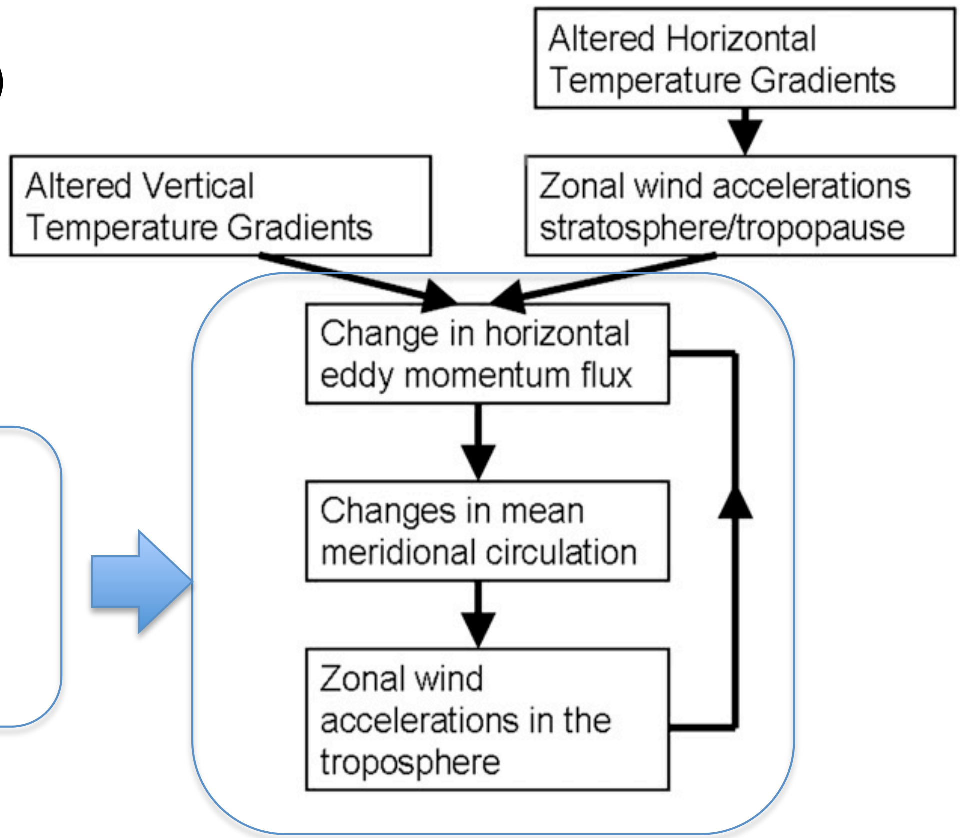
Changes in eddy fluxes are a vital part of the tropospheric dynamical response (possibility of ‘amplification’)



Can we make *predictions* about the response of the tropospheric circulation?

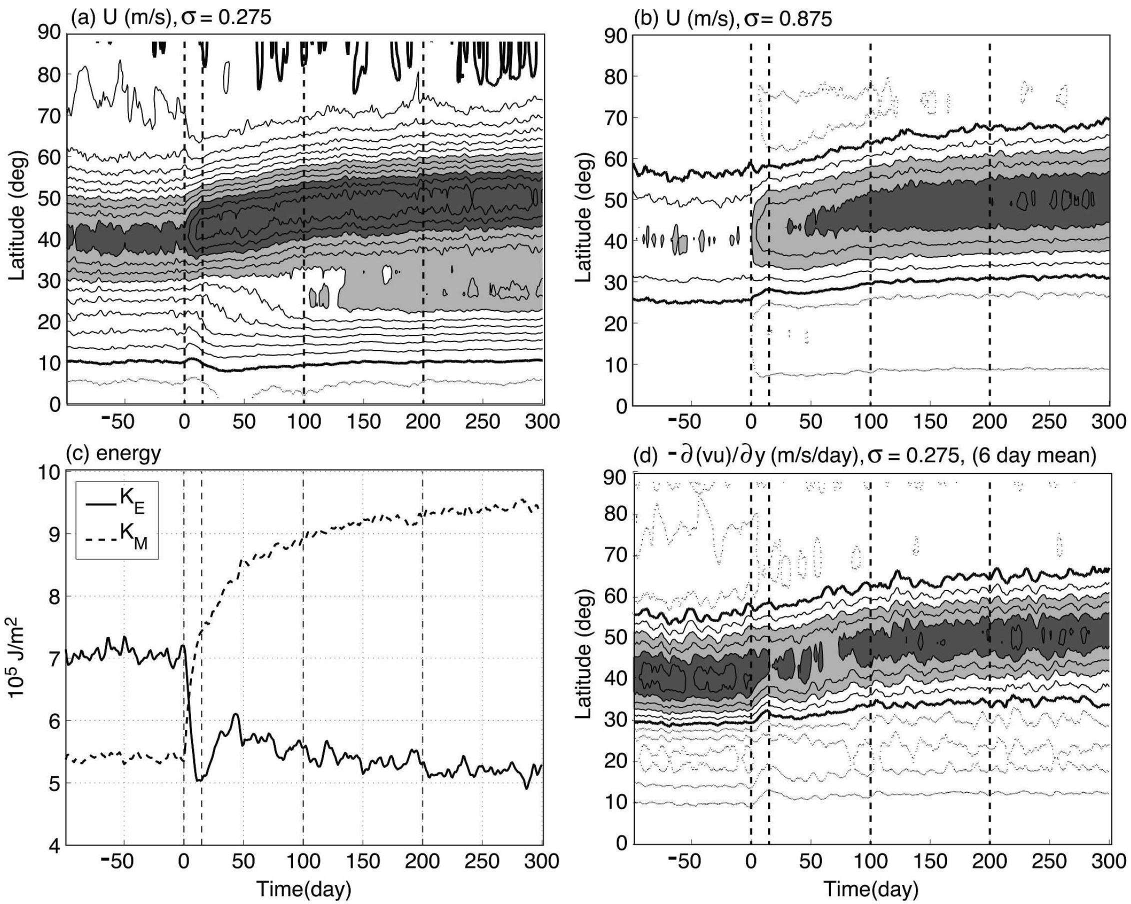
Diagram from
Simpson et al (2009)

Can this tropospheric box really be 'decomposed'?



Analogous tropospheric response problems: ozone hole (Gillett and Thompson 2003), stratospheric perturbation (Polvani and Kushner 2002, Song and Robinson 2004), surface friction (Chen et al 2007), tropospheric heating (Butler et al 2010)

Reduction in surface friction in simple circulation model



Chen et al (2007): two-stage adjustment (short-time in jet strength followed by longer term change in jet position)

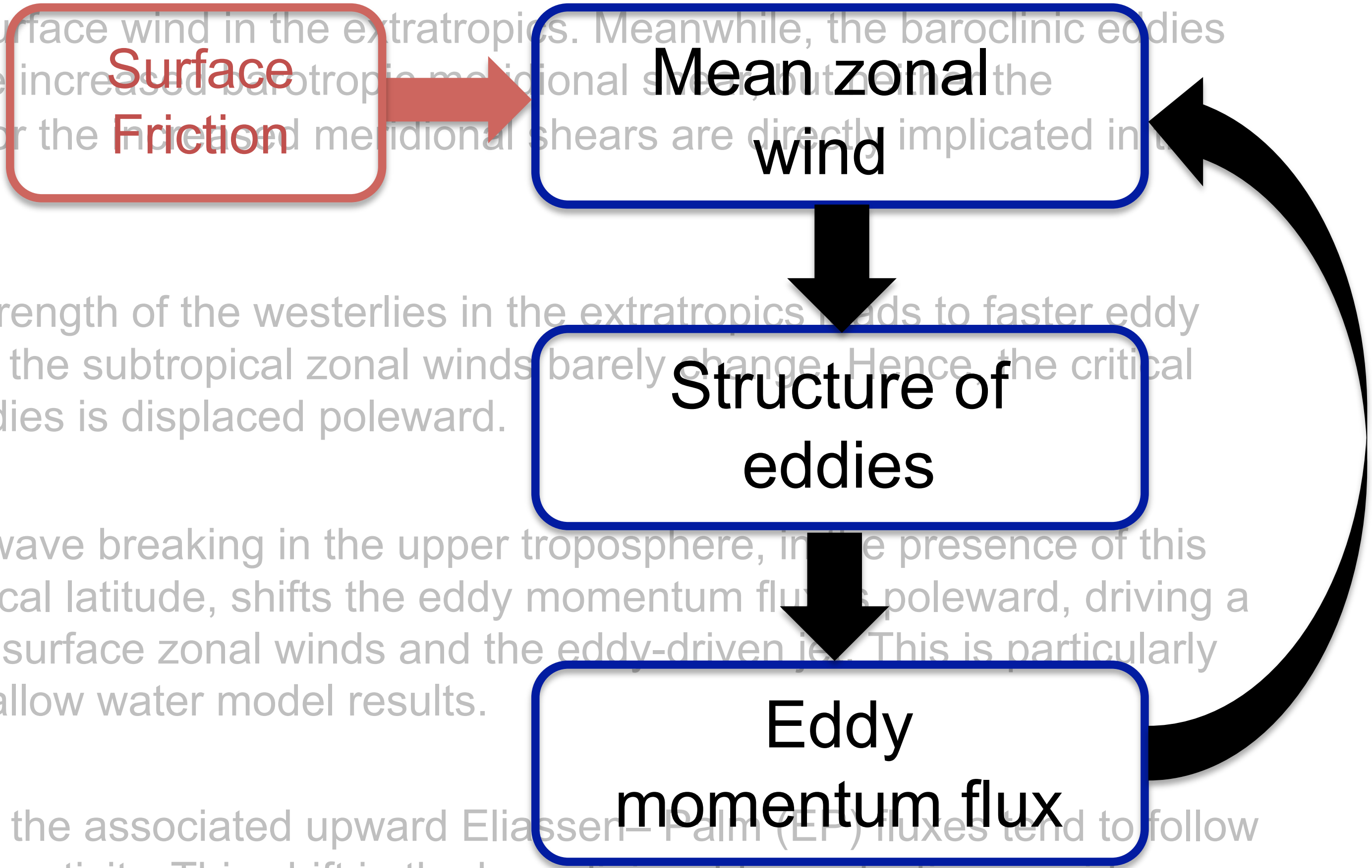
Chen et al (2007)

1) As the surface drag is reduced, the zonal wind acceleration is barotropic and proportional to the surface wind in the extratropics. Meanwhile, the baroclinic eddies are weakened by the increased barotropic meridional shear, but neither the weakening eddies nor the increased meridional shears are directly implicated in the poleward shift.

2) The increase in the strength of the westerlies in the extratropics leads to faster eddy phase speeds, while the subtropical zonal winds barely change. Hence, the critical latitude for these eddies is displaced poleward.

3) The dynamics of the wave breaking in the upper troposphere, in the presence of this poleward shift in critical latitude, shifts the eddy momentum fluxes poleward, driving a poleward shift in the surface zonal winds and the eddy-driven jet. This is particularly supported by the shallow water model results.

4) Eddy heat fluxes, and the associated upward Eliassen–Palm (EP) fluxes tend to follow this upper-level eddy activity. This shift in the baroclinic eddy production provides some positive feedback on the upper-level shift.



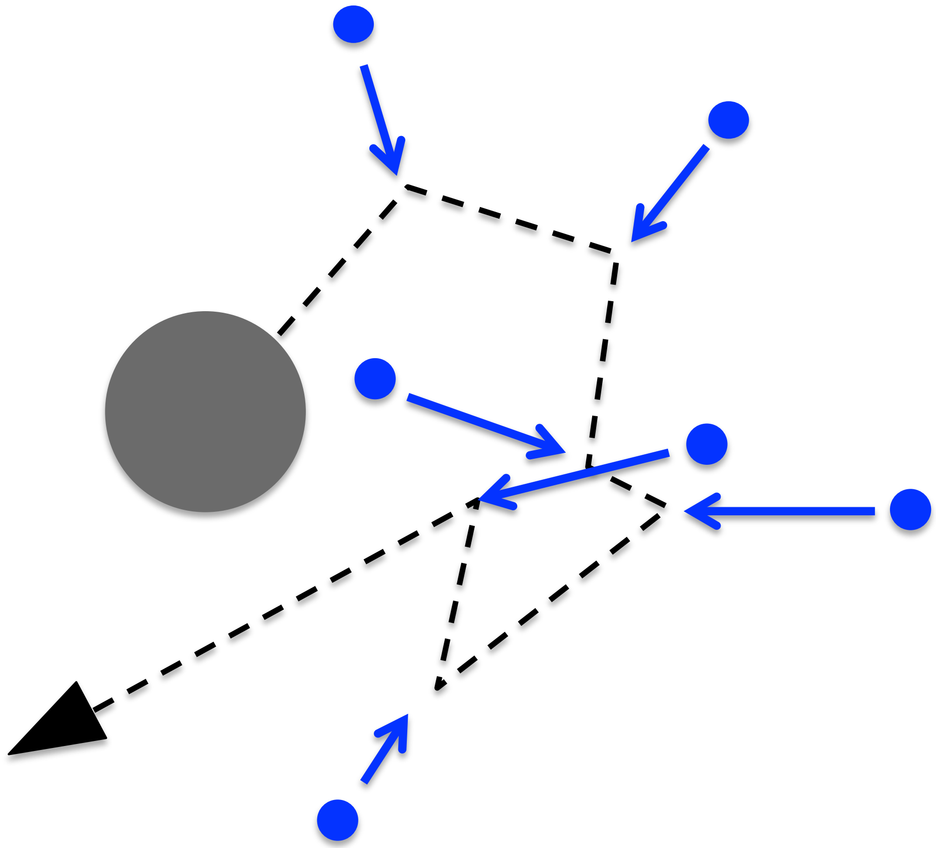
Questions

- What is relation between spatial pattern of forcing and the amplitude and spatial pattern of response? ('preferred response', 'most effective forcing')
- Will different models overpredict or underpredict response (and correct pattern of response) relative to real atmosphere?

Seek a 'unified' approach to quantitative prediction of tropospheric response, rather than post-hoc explanation of each special case

Brownian motion

Einstein (1905,1906)
Smoluchowksi (1906)



'Observable' Diffusivity of Particle:

$$D_x = \langle V_x^2 \rangle \tau_{corr}$$

Equipartition of kinetic energy: $\frac{1}{2}m\langle V_x^2\rangle = \frac{1}{2}\frac{RT}{N_A}$

Stokes law for viscous drag force: $F_S = -k_S V = -6\pi\mu a V$

$$m\frac{dV_x}{dt} = -k_S V_x + f_R(t)$$

$$\langle f_R(t_1) f_R(t_2)\rangle = C\delta(t_1 - t_2)$$

$$\langle V_x(t_1) V_x(t_2)\rangle = \langle V_x^2\rangle \exp(-k_S|t_1 - t_2|/m) \qquad \tau_{corr} = m/k_S$$

$$D = \frac{RT}{6N_A\pi\mu a}$$

EINSTEIN RELATION

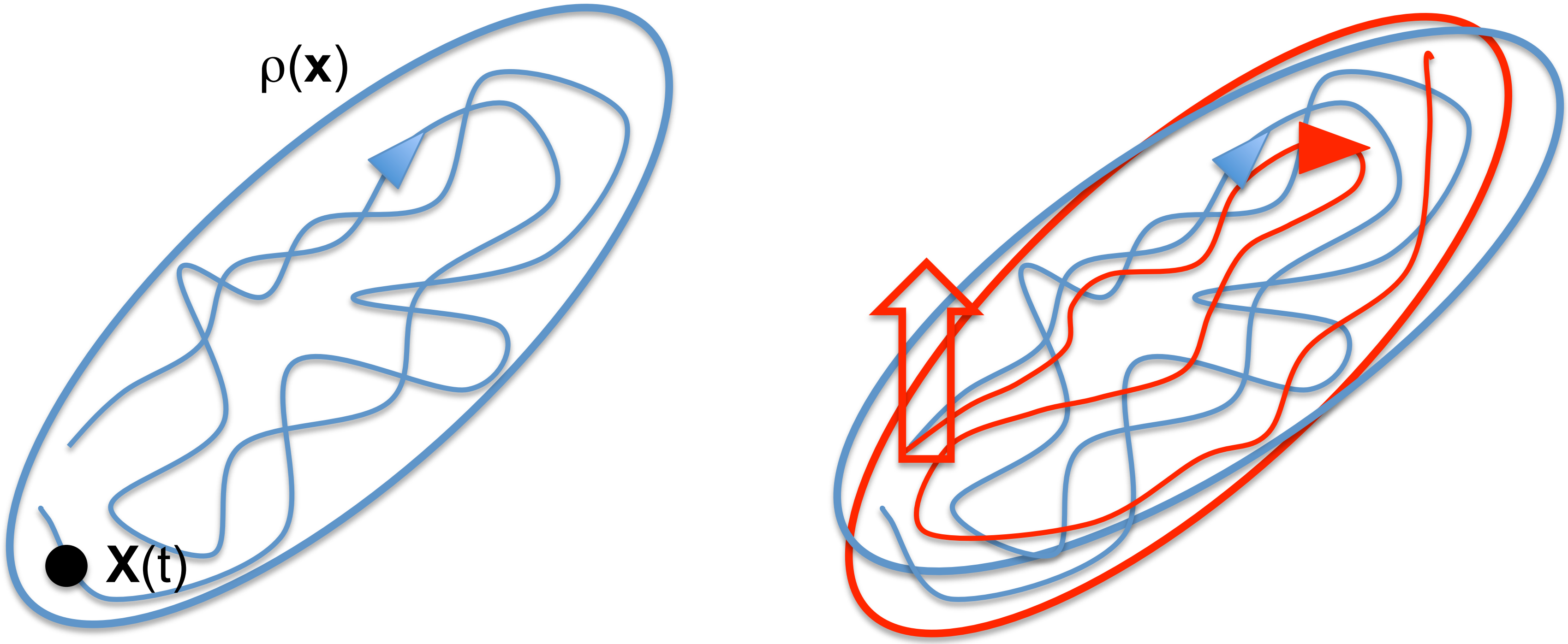
Time scale of fluctuations $\tau_{corr} = m/k_S$

$$\frac{\text{Velocity response to applied force}}{\text{Applied force}}$$

FLUCTUATION -- DISSIPATION

“Dynamical systems” approach

Calculation of change in statistical measure of chaotic/random system due to applied perturbation



UNDISTURBED SYSTEM

DISTURBED SYSTEM

Evolution equation $$\frac{d\mathbf{X}}{dt} = \mathbf{U}(\mathbf{X}, t)$$

$\mathbf{U}(\mathbf{X}, t)$ is usually nonlinear and could contain explicit randomness

Equilibrium statistical properties described by probability density function $\rho(\mathbf{x})$

Perturb evolution equation $$\frac{d\mathbf{X}}{dt} = \mathbf{U}(\mathbf{X}, t) + \Delta\mathbf{F}(\mathbf{X}, t)$$

What is new $\rho(\mathbf{x})$?

[Perturbations to individual trajectories are large – perturbations to overall statistics are small.]

Consider small applied forcing $\Delta \mathbf{F} = \mathbf{f}(\mathbf{x})\delta(t)$

At t=0: $\rho(\mathbf{x}) \to \rho_+(\mathbf{x}) \simeq \rho(\mathbf{x}) - \nabla_{\mathbf{x}}.(\mathbf{f}(\mathbf{x})\rho(\mathbf{x}))$

$$\langle \phi(\mathbf{X}(\tau))\rangle_{\mathbf{f}} = \int d\mathbf{x} \int d\mathbf{y} \phi(\mathbf{y}) \mathcal{P}(\mathbf{X}(\tau) = \mathbf{y} | \mathbf{X}(0) = \mathbf{x}) \rho_+(\mathbf{x})$$

Compare identity

$$\langle \phi(\mathbf{X}(\tau))\psi(\mathbf{X}(0))\rangle = \int d\mathbf{x} \int d\mathbf{y} \phi(\mathbf{y}) \mathcal{P}(\mathbf{X}(\tau) = \mathbf{y} | \mathbf{X}(0) = \mathbf{x}) \psi(\mathbf{x}) \rho(\mathbf{x})$$

Hence $\langle \phi(\mathbf{X}(\tau))\rangle_{\mathbf{f}} \simeq \langle \phi(\mathbf{X}(\tau)) \frac{\rho_+(\mathbf{x})}{\rho(\mathbf{x})}|_{\mathbf{x}=\mathbf{X}(0)}\rangle$ and

$$\Delta\langle \phi(\mathbf{X}(\tau))\rangle = -\langle \phi(\mathbf{X}(\tau)) \frac{\nabla_{\mathbf{x}}.(\mathbf{f}(\mathbf{x})\rho(\mathbf{x}))}{\rho(\mathbf{x})}|_{\mathbf{x}=\mathbf{X}(0)}\rangle$$

Fluctuation-Dissipation Theorem

For steady **x**-independent $\Delta\mathbf{F}$

$$\Delta\langle\mathbf{X}\rangle = -\int_0^\infty d\tau \langle \mathbf{X}(\tau) \frac{\nabla_\mathbf{x} . \rho(\mathbf{x})}{\rho(\mathbf{x})} |_{\mathbf{x}=\mathbf{X}(0)} \rangle . \Delta\mathbf{F} \quad = \mathbf{L}\Delta\mathbf{F}$$

A: General

If **X** is Gaussian then $\Delta\langle\mathbf{X}\rangle = \int_0^\infty d\tau \mathbf{C}(\tau)\mathbf{C}(0)^{-1} . \Delta\mathbf{F} = \mathbf{L}\Delta\mathbf{F}$

B: Gaussian

where $\mathbf{C}(\tau) = \langle \mathbf{X}(\tau)\mathbf{X}(0) \rangle$

Linear operator $\mathbf{L}$ given in terms of properties of undisturbed system

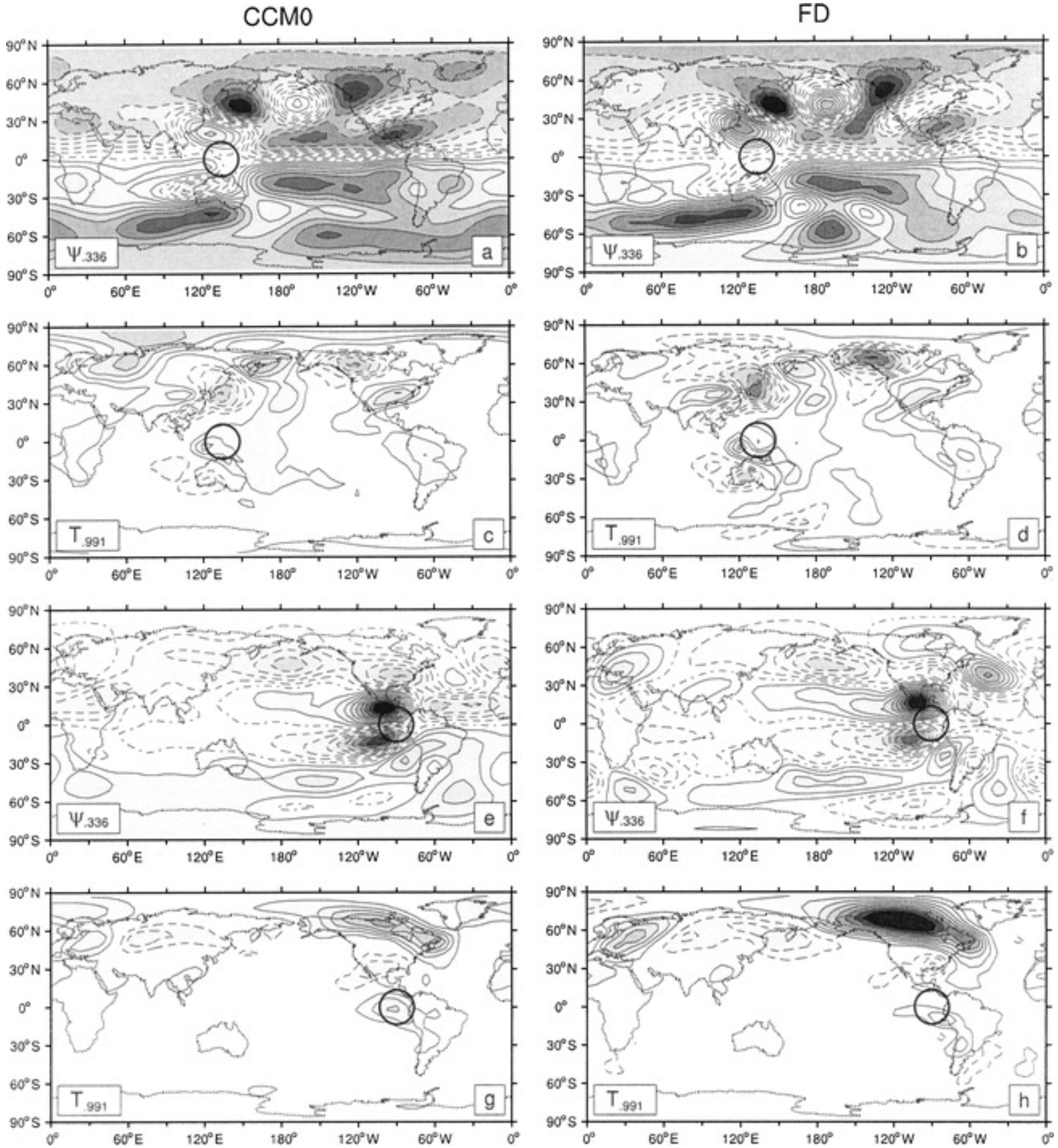
Crude approximation $\Delta\langle\mathbf{X}\rangle \sim \tau_{\mathrm{correlation}} \times \Delta\mathbf{F}$

Gritsun and Branstator (2007)

Application of Gaussian FDT to predict response to localised tropical heating in GCM

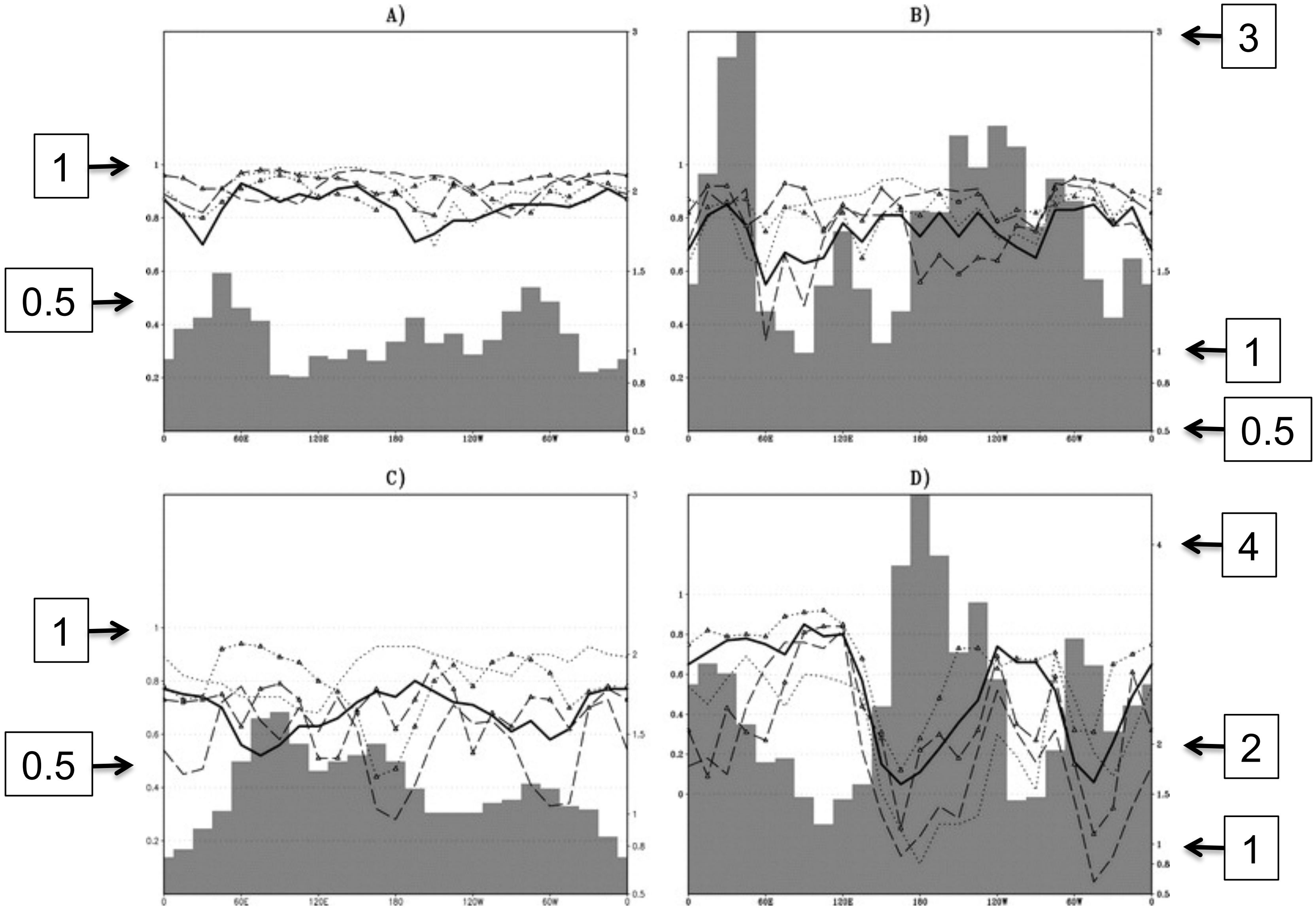
Individual AGCM integrations 40000 days

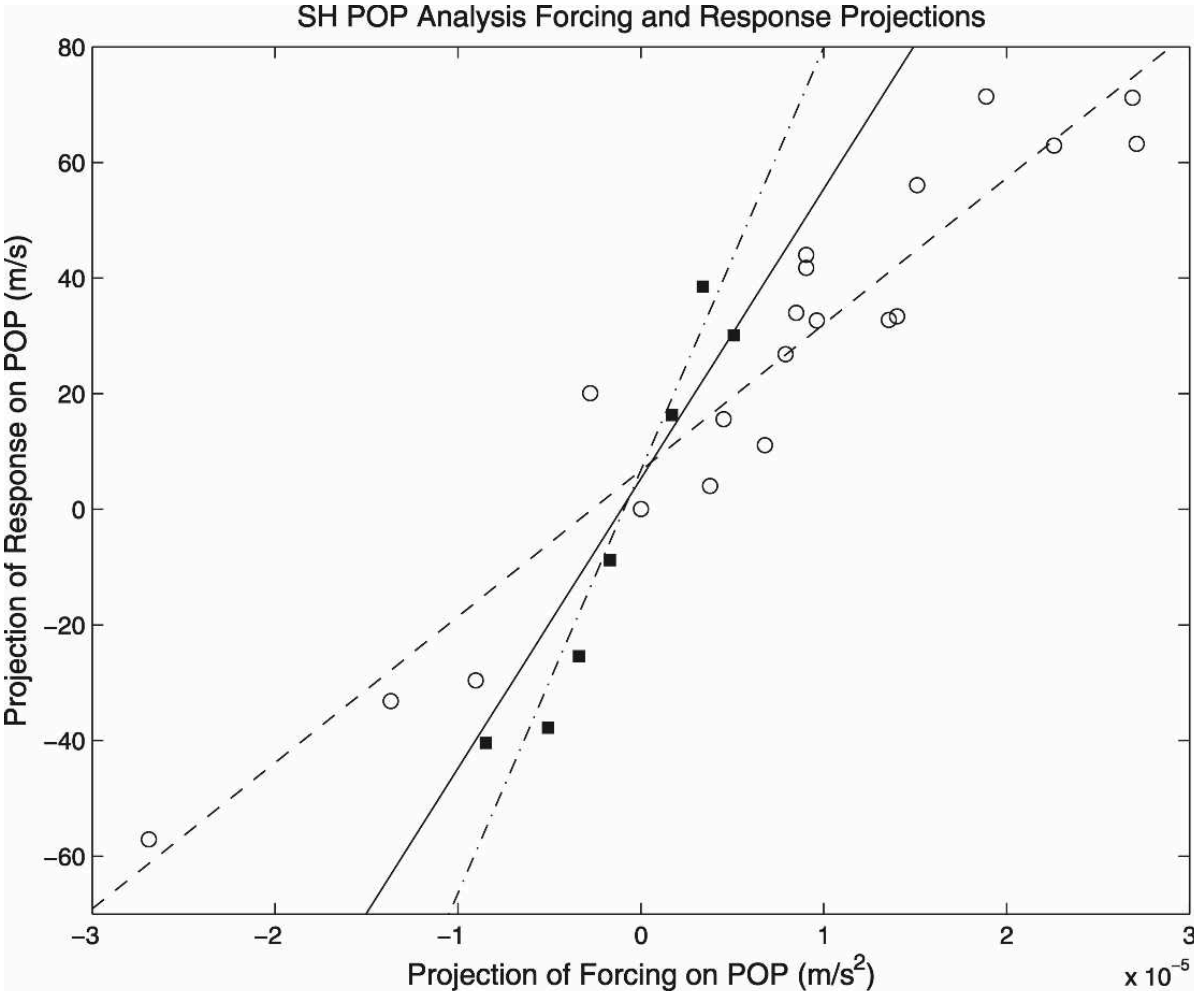
FDT estimate constructed from 4M day integration



Gritsun and Branstator (2007)

Success of FDT measured by pattern correlation and amplitude ratio.





Ring and Plumb (2008): Gaussian FDT makes incorrect prediction for response to zonally symmetric thermal (■) and mechanical (**O**) forcings

Practical issues in applying the FDT

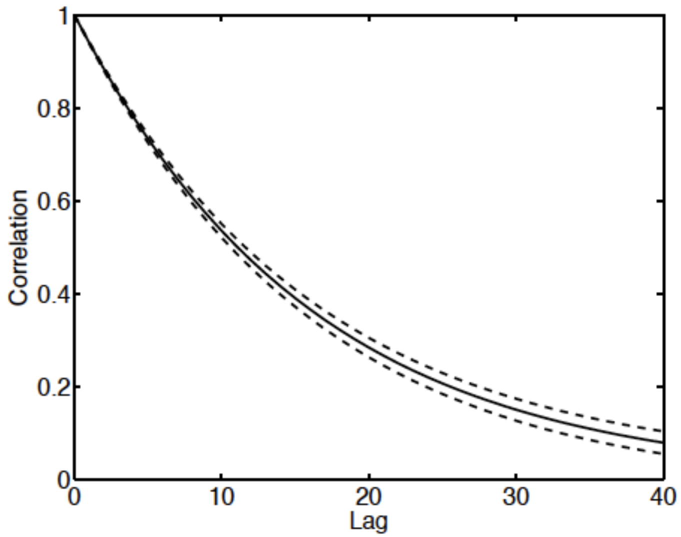
$$\mathbf{L} = \int_0^\infty d\tau \, \mathbf{C}(\tau) \, \mathbf{C}(0)^{-1}$$

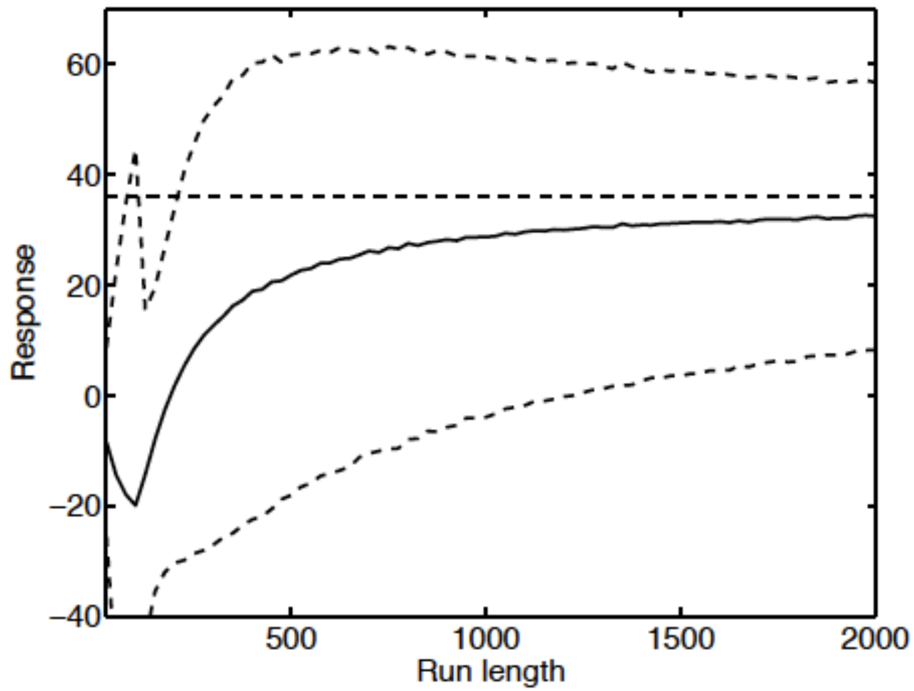
- EOFs (which diagonalise **C**(0)) are a natural choice of variable (but not the only possible choice)
- $< \mathbf{C}(\tau) \, \mathbf{C}(0)^{-1} >$ must be estimated from available data.
- $\mathbf{C}(0)^{-1}$ potentially ill-conditioned – number of useful EOFs may be restricted by length of data series
- integration from $\tau = 0$ to $\tau = \infty$ must be approximated by finite sum

Statistical requirements on application of Gaussian FDT

Cooper and H 2012

2-D linear model





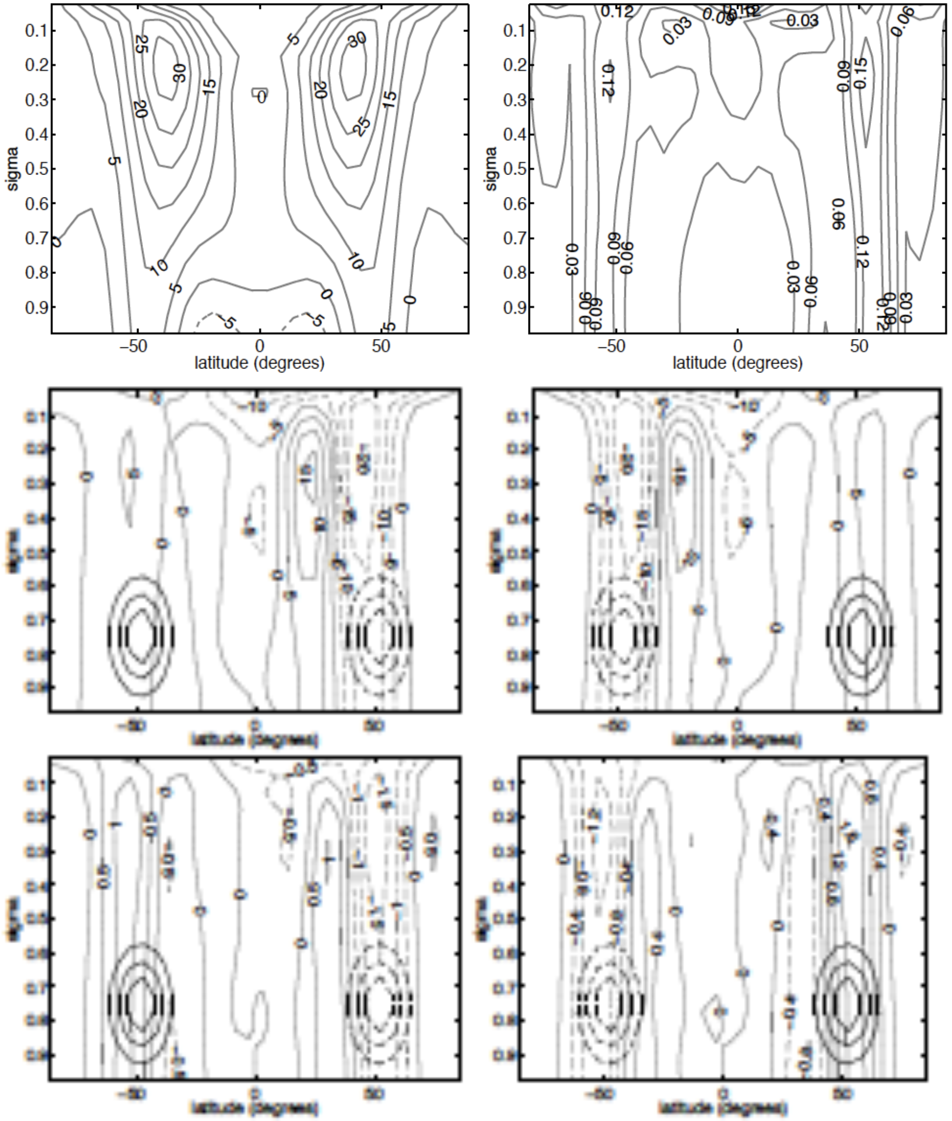
FDT prediction of response

Study based on simple T21L20 general circulation model
(Cooper and H 2012)

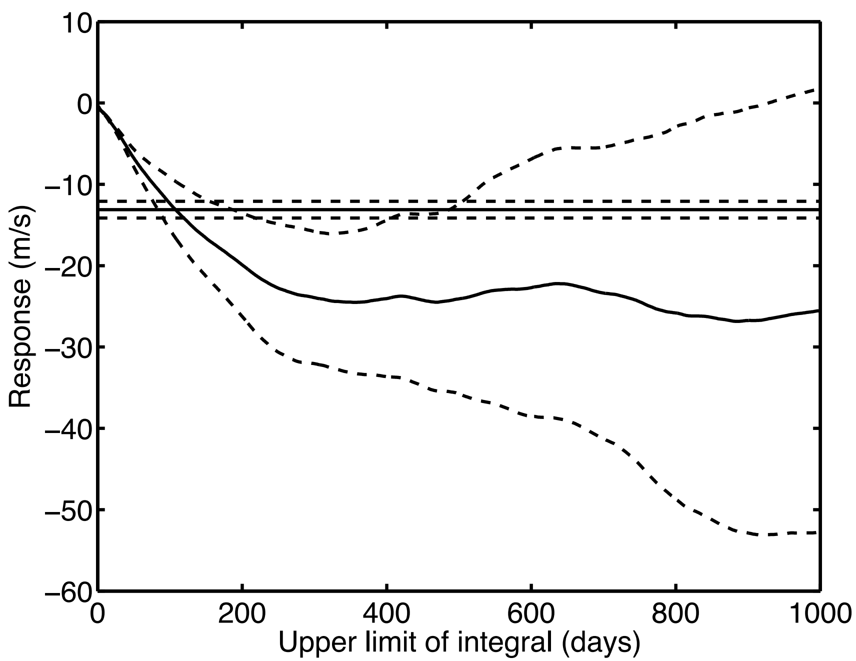
10000 day simulations, mean and variance

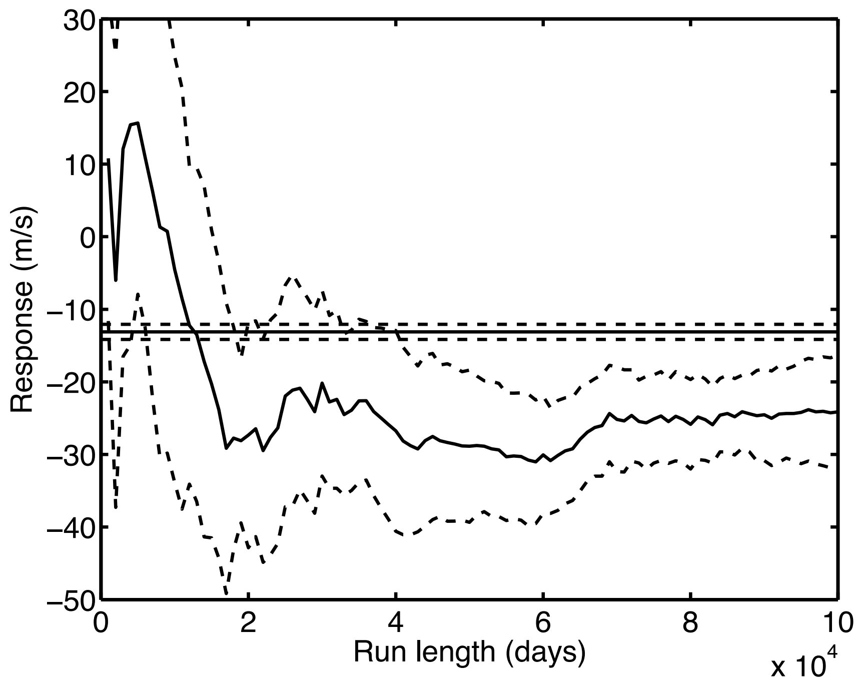
Forced response
+/- 1 m/s/day

Forced response
+/- 0.1 m/s/day



Application of the FDT to predict the response to forcing of a simple T21L20 general circulation model



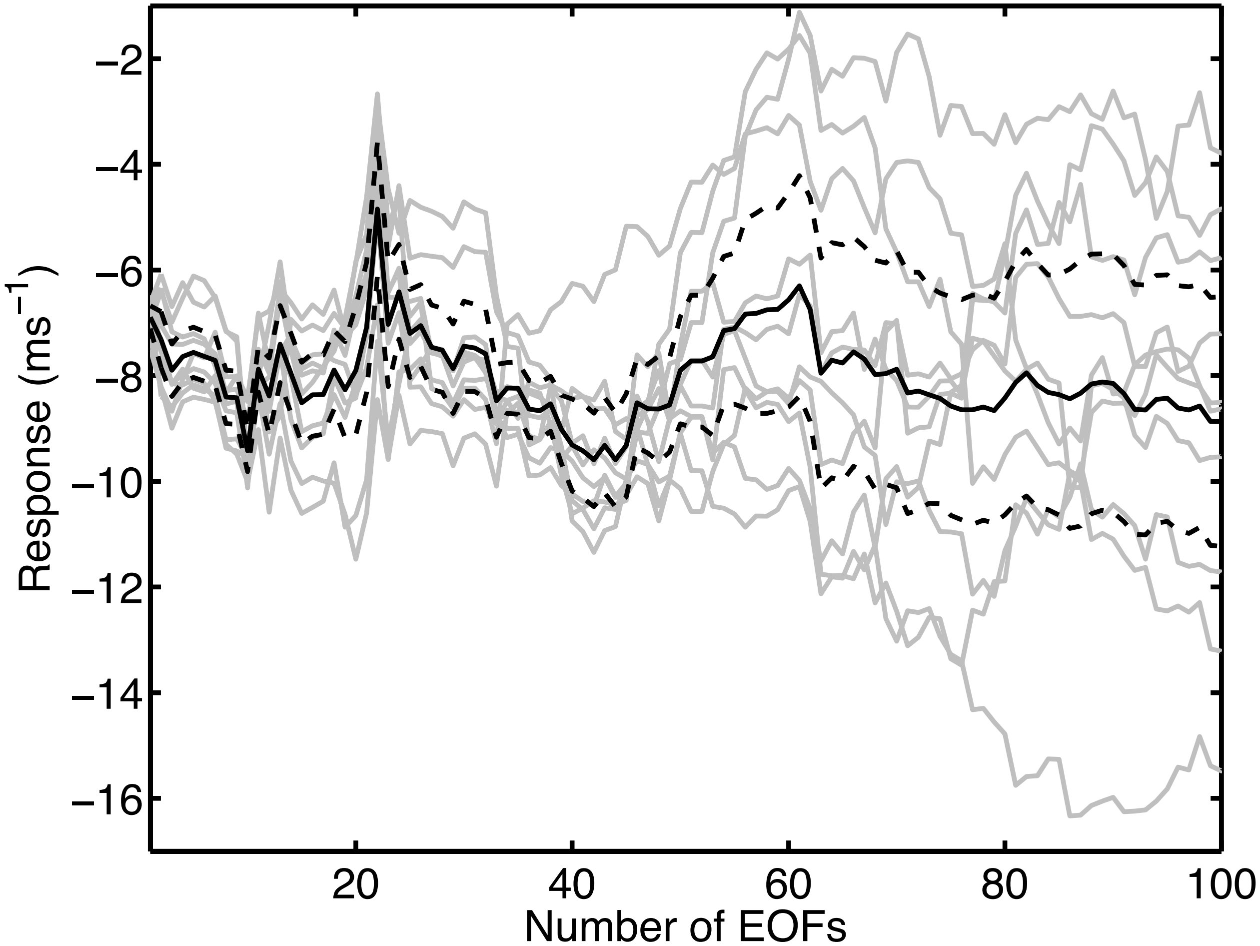


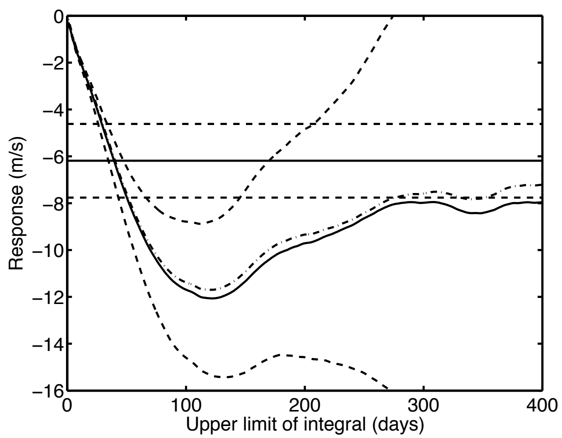
(10 x 10^5 day integrations)

(300 day upper limit to integral)

What is the optimal number of EOFs to include in the calculation?

(T31L20 model)





Specification of forcing

$$\mathbf{L}_{\text{Gaussian}} = \int_0^\infty d\tau \mathbf{C}(\tau)\mathbf{C}(0)^{-1}$$

Where did the equations go?

Implications of truncation

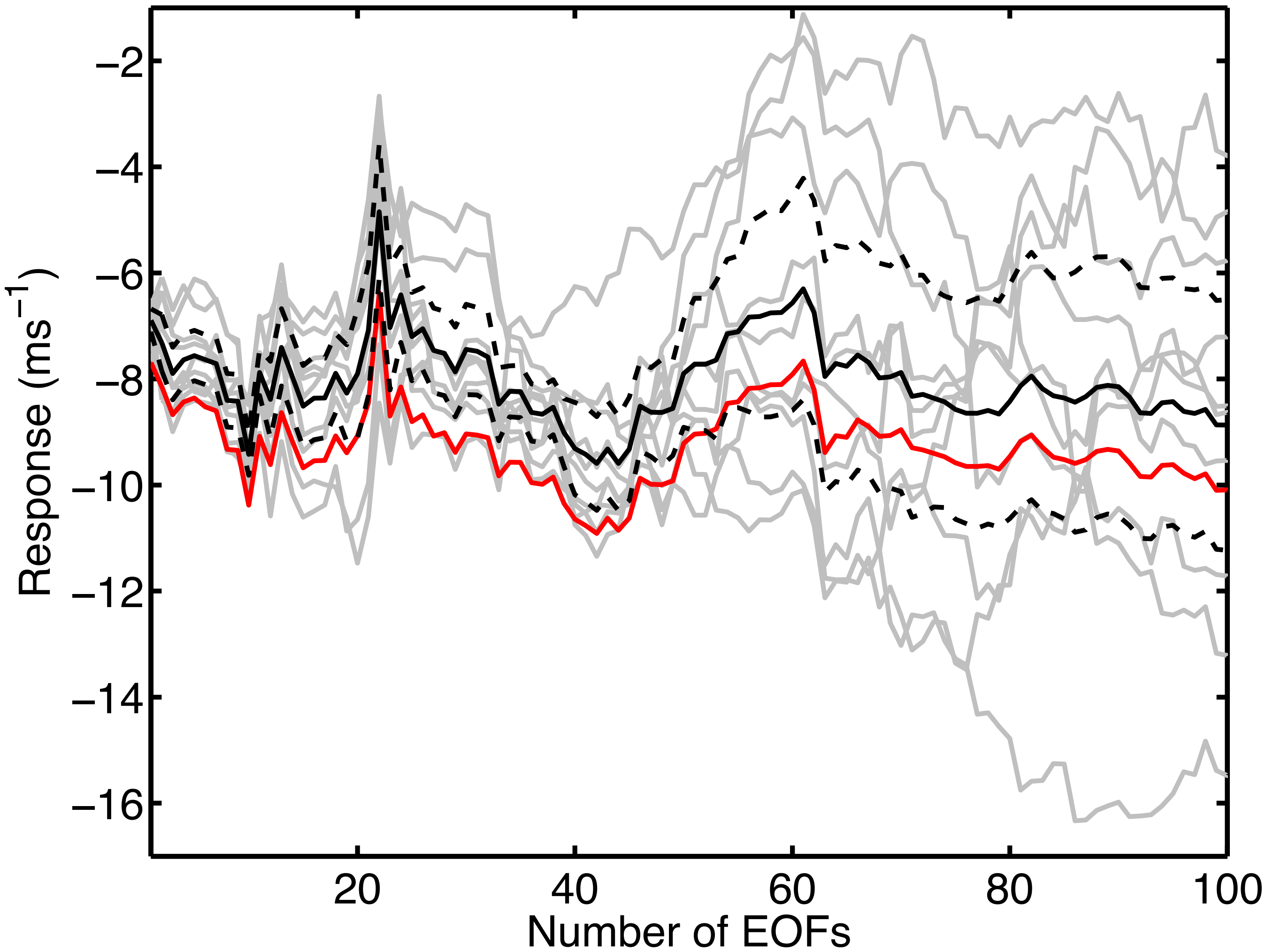
e.g. Ring and Plumb (2007)

$$\frac{\partial u}{\partial t} + \cdots = \mathcal{F} \qquad \frac{\partial T}{\partial t} + \cdots = 0$$

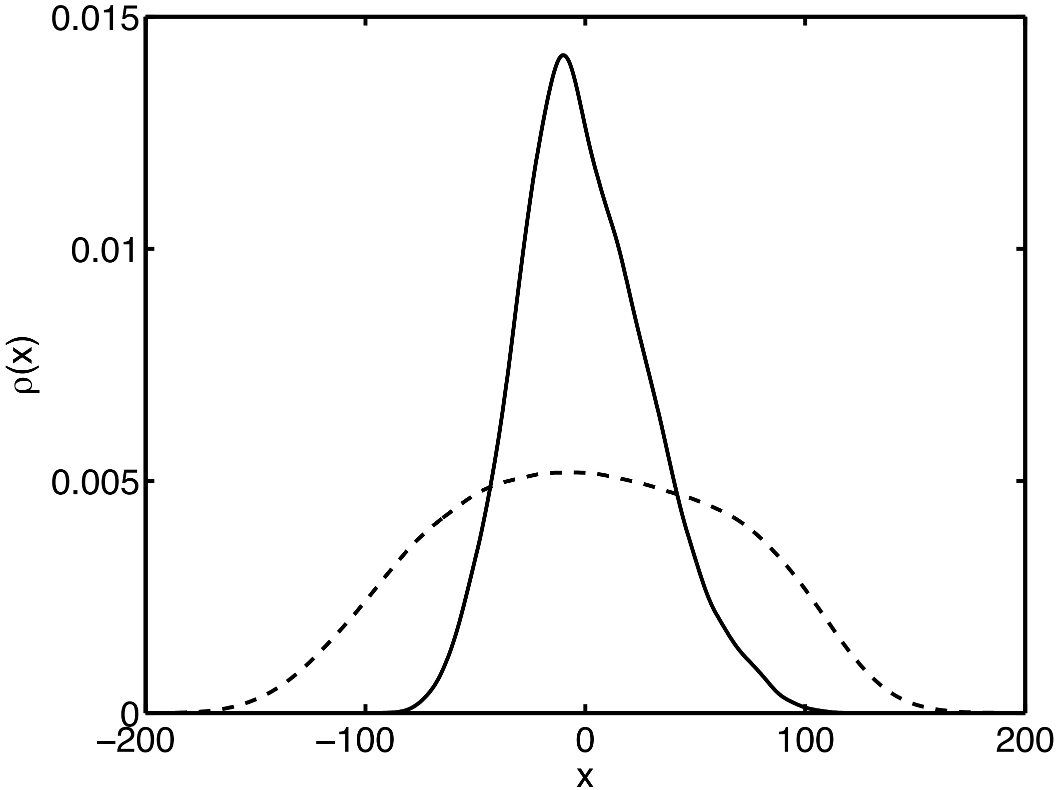
$$\frac{\partial u}{\partial t} + \cdots = \mathcal{L}(u, \mathcal{F})$$

Effect of using climatological u versus u=0 in

$$\mathcal{L}(u, \mathcal{F})$$



Usefulness of Gaussian FDT may be limited by non-Gaussianity?



pdfs of EOF1 in zonal wind for T21L20 and T31L20 simulations

The non-Gaussian case – a 'non-parametric FDT'

Cooper and H 2011

$$\Delta\langle \mathbf{X}\rangle = -\int_0^\infty d\tau \langle \mathbf{X}(\tau)\frac{\nabla_{\mathbf{x}}\rho(\mathbf{x})}{\rho(\mathbf{x})}|_{\mathbf{x}=\mathbf{X}(0)}\rangle.\Delta\mathbf{F}$$

Estimate using kernel density estimator method of non-parametric statistics

$$\hat{\rho}(\mathbf{x}; h, N) = \frac{1}{Nh^d}\sum_{i=1}^{N} K(\frac{\mathbf{x}-\mathbf{X}_i}{h})$$

$$\nabla_{\mathbf{x}}\hat{\rho}(\mathbf{x}; h, N) = \ldots$$

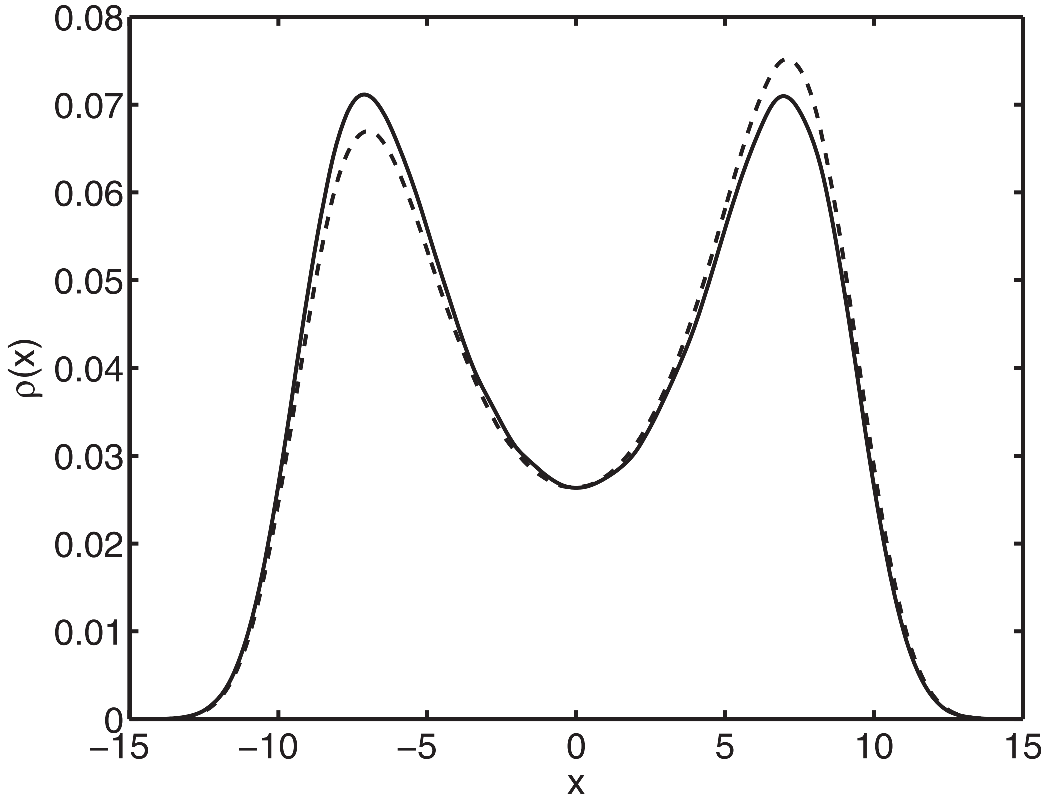
Simplest choice for K(.) is isotropic Gaussian

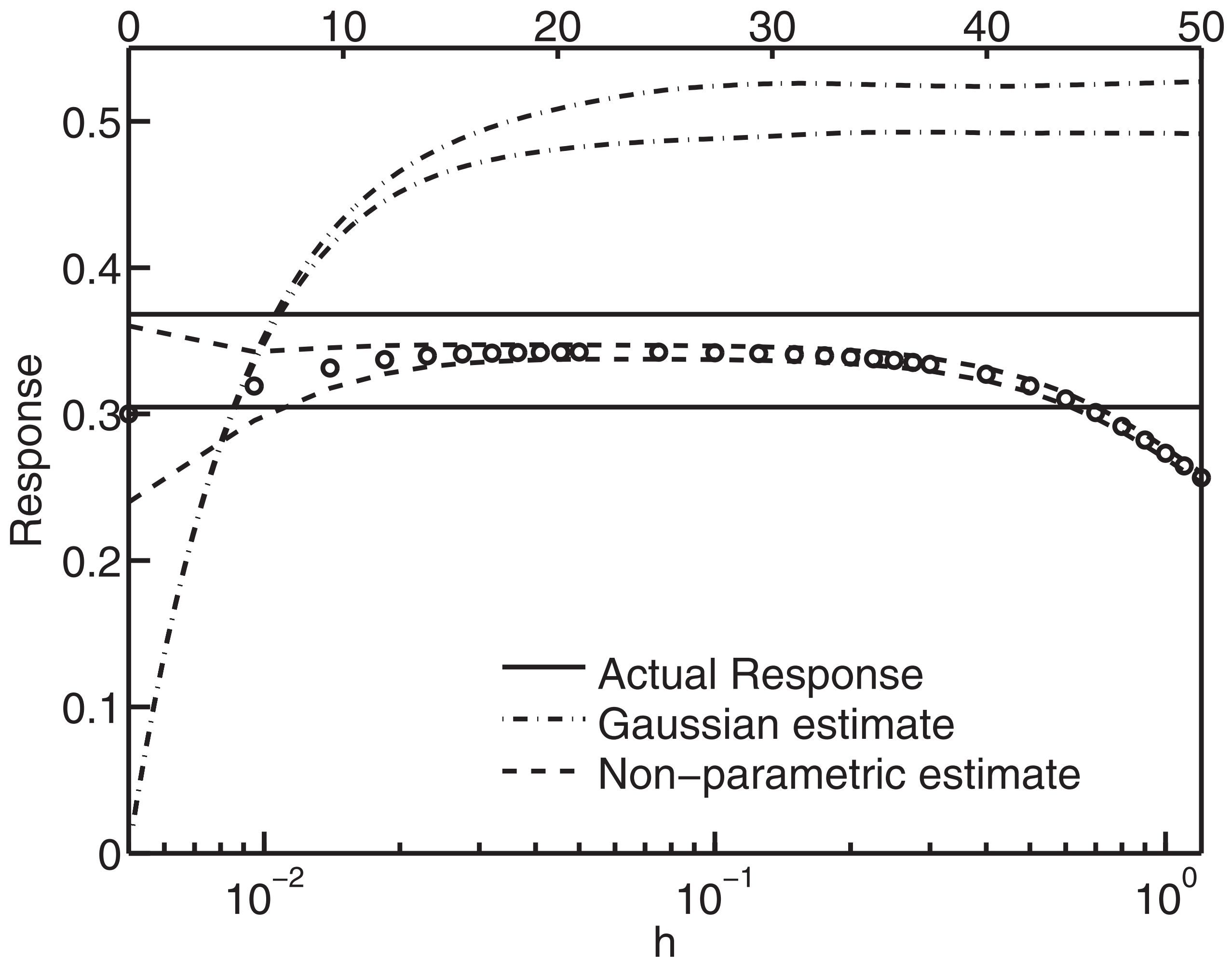
Bias and *Uncertainty* depend on *h* and *N*.

strongly non-Gaussian test case

$$\frac{dX}{dt} = b_1 X - b_2 X^3 + \xi$$

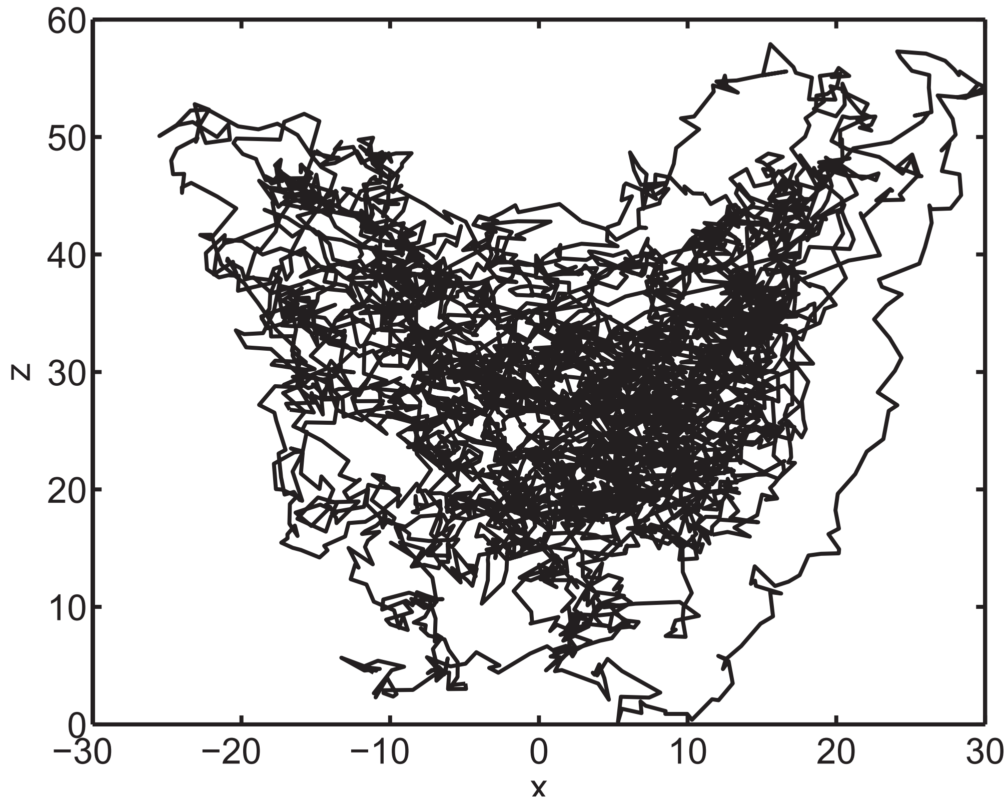
Perturbed and unperturbed pdfs





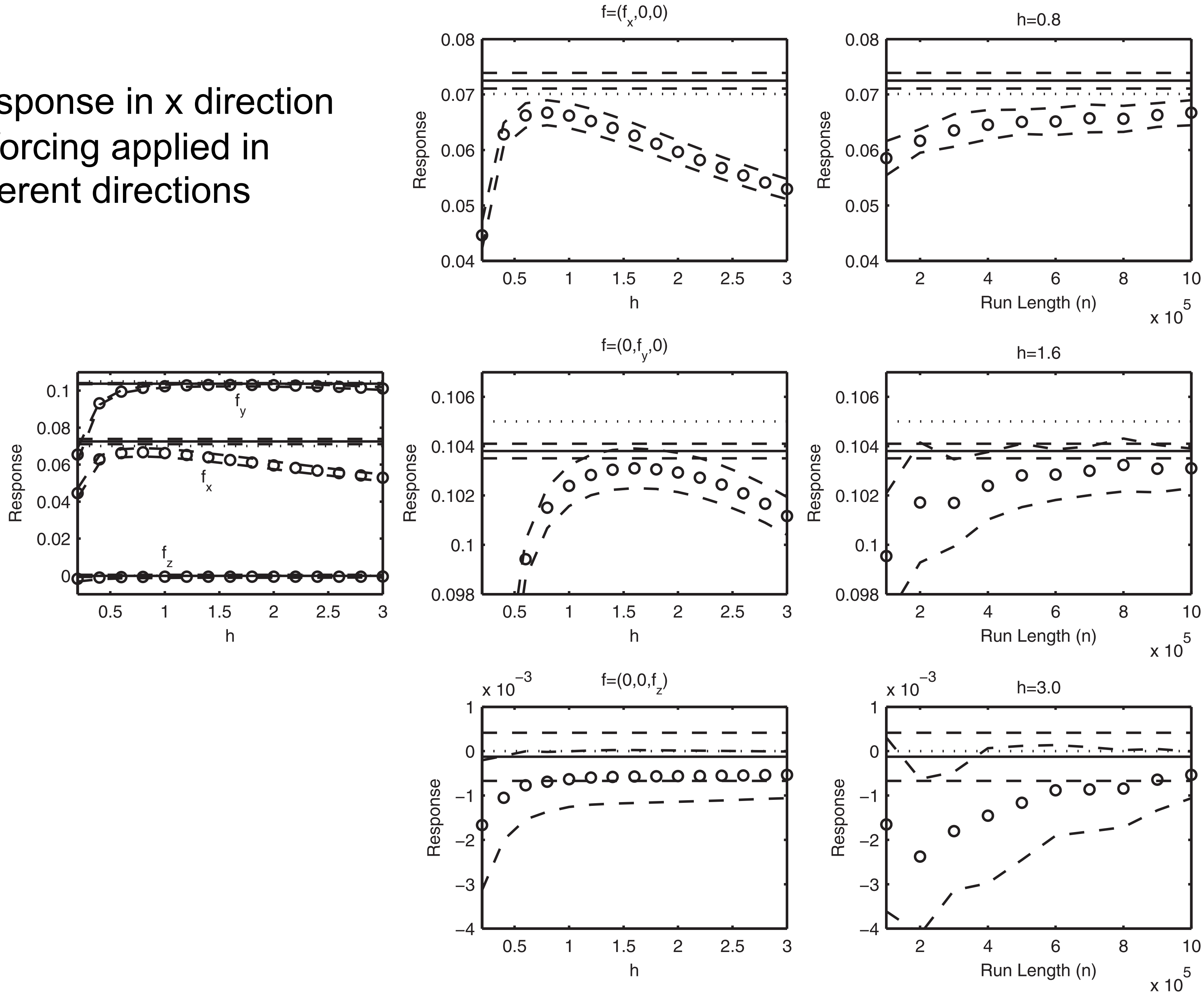
0
10
20
30
40
50
0.5
0.4
0.3
0.2
0.1
0
Response
Actual Response
Gaussian estimate
Non–parametric estimate
10^−2
10^−1
10^0
h

Application to stochastic Lorenz 1963 model

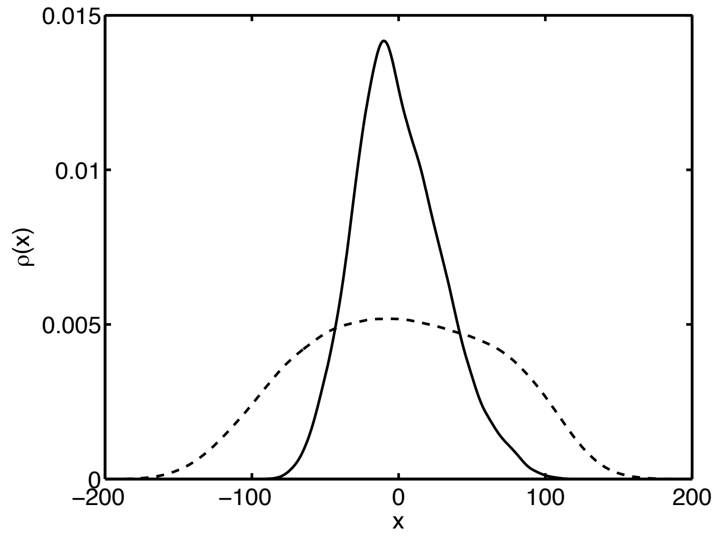


Compare with Thuburn (2005) approach of solving Fokker-Planck equation

Response in x direction to forcing applied in different directions



What is a useful measure of non-Gaussianity?



$$\mathbf{L} - \mathbf{L}_{\text{Gaussian}} = \int_0^\infty d\tau \langle \{ \langle \mathbf{X}(\tau) | \mathbf{X}(0) \rangle \{ -\rho(\mathbf{x})^{-1} \nabla_{\mathbf{x}} \rho(\mathbf{x})|_{\mathbf{x}=\mathbf{X}(0)} - \mathbf{X}(0).\mathbf{C}(0)^{-1} \} \rangle$$

Depends on structure of time correlations as well as form of pdf

Summary

•FDT potentially provides a quantitative description of tropospheric response to forcing (e.g. ozone hole, solar cycle, greenhouse gas increase) given information on statistics of unforced circulation

•If model low-frequency variability (timescales and patterns) is wrong then response to forcing will be wrong

•Typical response to forcing will be leading singular vector of response operator (providing forcing has significant projection onto leading singular vector), not necessarily the leading EOF.

•In practice can FDT do better than simple estimation of timescale of leading EOF?

•Applications? Model assessment/intercomparison?

Future lines of work? (Is the FDT a practical quantitative tool?)

•Statistical nature of FDT requires explicit information on/ estimates for bias and uncertainty

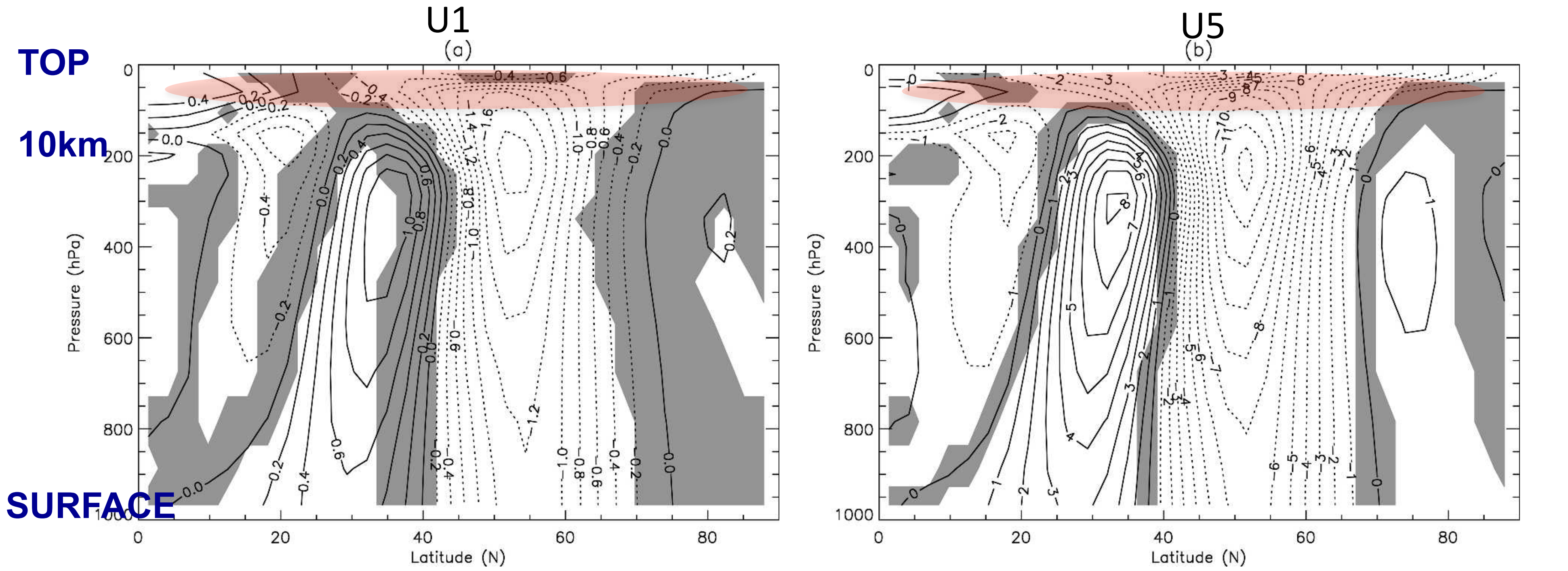
•Non-gaussian extension of FDT potentially extends validity (but there are challenges in implementation – can we escape the 'curse of dimensionality' or avoid it by working in a truncated system?)

•FDT for truncated system is non-trivial – need to consider proper 'effective forcing' on truncated system.

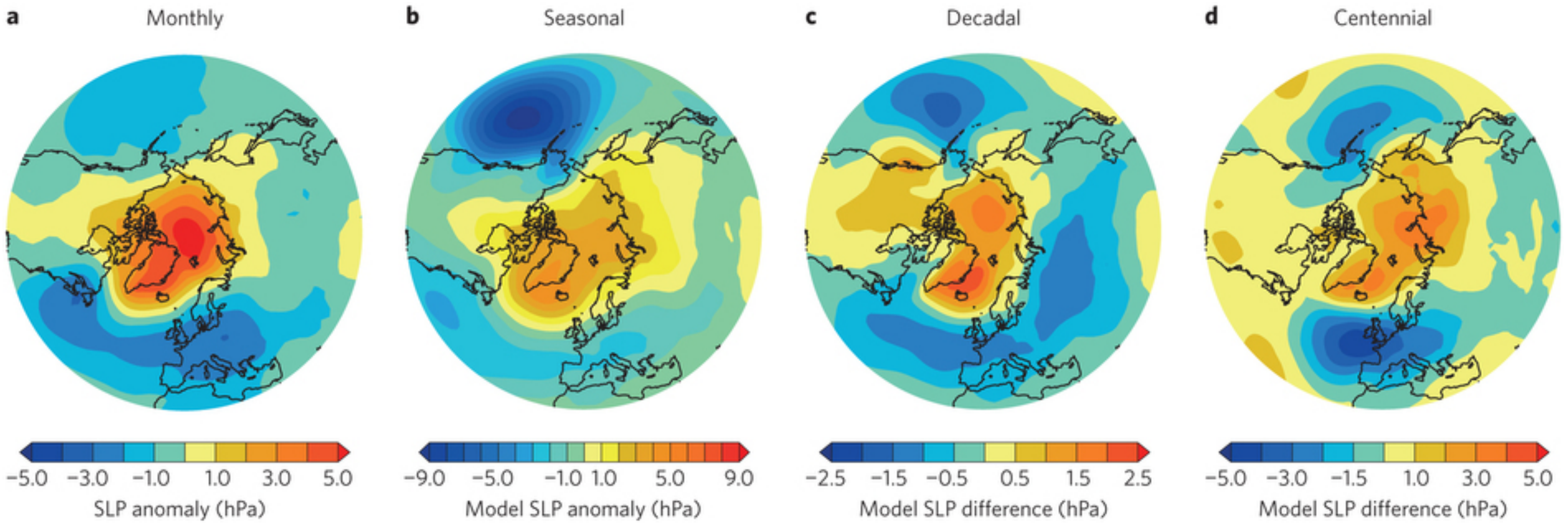
•Clearer practical guide to implementation of FDT (How long a data record is needed for required precision? How many degrees of freedom to include?)

Usefulness of linear theory?

depends on problem being considered – but recall Haigh et al (2005)



Kidston et al (2015): tropospheric response to stratosphere on different timescales



FDT – cultural differences

20th century physics: large systems, small fluctuations, FDT has been discussed/applied/interpreted in terms of macroscopic variables.

21st century physics: extension to non-equilibrium small systems with large fluctuations?

Dynamical systems: Formal derivation/justification of 'fluctuation-response' operators, conditions for applicability, can problems of non-smoothness/non-differentiability be overcome?

Climate/circulation: Evaluation of 'fluctuation-response' operator from model simulation (or from data?) is a problem in statistics of large-degree-of-freedom systems. How much data is needed for required accuracy? How can effective dimensionality be most effectively reduced?