AdS_3/CFT_2 and integrability

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Based on work with
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Motivation

- Holography a profound insight into quantum physics
- Can we understand highly-quantum aspects of holography?
- Sometimes, yes: Δ of Konishi operator $\mathrm{Tr}(\bar{\phi}\phi)$ to silly loop order in planar $\mathcal{N} = 4$ with integrability!
- Integrability works in examples of $\mathrm{AdS}_3/\mathrm{CFT}_2$

Why AdS_3/CFT_2

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- Large moduli space including WZW point [Ooguri Maldacena]
- **Challenge:** Matching to CFT for non-protected quantities

Plan

1. **Introduction to $\mathrm{AdS}_3/\mathrm{CFT}_2$**
2. **Exact worldsheet S matrix for Green-Schwarz strings**
3. **Protected closed string spectrum**
4. **Moduli, integrability and the WZW theory**
5. **Outlook**

Introduction to AdS_3/CFT_2

$AdS_3 \times S^3 \times T^4$

D1- and D5-branes in string theory

	0	1	2	3	4	5	6	7	8	9
$N_c \times$ D1	•	•								
$N_f \times$ D5	•	•	•	•	•	•				

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Near-horizon limit: $AdS_3 \times S^3 \times T^4$ + R-R 3-form charge

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Gauge Theory:

Open string gauge theory *not* conformal - flows to CFT in IR

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  $\mathcal{N} = (4,4)$ vector: $\Phi^{\alpha\dot{\alpha}}$, $\Psi_{\mathrm{L}}^{\dot{\alpha}\dot{a}}$, $\Psi_{\mathrm{R}}^{\alpha\dot{a}}$, A_μ, $D^{\dot{a}\dot{b}}$

  $\mathcal{N} = (4,4)$ hyper: $T^{a\dot{a}}$, $\chi_{\mathrm{L}}^{\alpha a}$, $\chi_{\mathrm{R}}^{\dot{\alpha} a}$

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- D5-D5 strings: decoupled.

IR CFT

UV gauge theory has two branches of vacua:

- Coulomb branch: D1 separated from D5 $U(N_c) \to U(1)^{N_c}$;
- Higgs branch: D1 on top of D5 $\longrightarrow$ AdS_3/CFT_2. [Maldacena '97]

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Protected quantities matched between $\mathrm{Sym}^{N_c N_f}(\mathrm{T}^4)$ and sugra

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ Moduli

IIB string theory on T^4 has 25 moduli:

$$g_{ab}\,, \quad B_{ab}\,, \quad C^{(2)}_{ab}\,, \quad C^{(0)}, \quad C^{(4)}_{abcd}, \quad \phi$$

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In UV gauge theory they are θ-angle and three FI parameters.

Green-Schwarz strings

Green-Schwarz action in general background

Spacetime supersymmetric GS worldsheet Lagrangian known

[Cvetic, Lü, Pope, Stelle '99, Wulff '14]

$$L = L_{\text{bos}} + L_{\text{kin}} + L_{\text{WZ}}$$

where, for example

$$L_{\text{kin}} = -i\sqrt{h}h^{\alpha\beta}\bar{\theta}^I \not{E}_\alpha \left(\delta^{IJ} D_\beta + \frac{\sigma_3^{IJ}}{24} \not{F} \not{E}_\beta \right) \theta_J + \dots$$

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One finds a residual susy algebra $\mathcal{A}$ that commutes with gauge-fixed Hamiltonian. Fermions transform linearly under it

The algebra $\mathcal{A}$

The algebra of charges that commutes with H takes the form

$$\{\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}}, \overline{\mathbf{Q}}_{\mathrm{L}\dot{b}}\} = \frac{1}{2}\delta^{\dot{a}}{}_{\dot{b}}\,(\mathbf{H}+\mathbf{M}), \qquad \{\mathbf{Q}_{\mathrm{L}}{}^{\dot{a}}, \mathbf{Q}_{\mathrm{R}\dot{b}}\} = 0$$

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Commutators hold for physical level-matched states.

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Central extensions related to worldsheet momentum $\mathbf{P}$

$$\mathbf{C} = +i\frac{h}{2}(e^{+i\mathbf{P}}-1), \qquad \overline{\mathbf{C}} = -i\frac{h}{2}(e^{-i\mathbf{P}}-1),$$

$h \sim R^2/\alpha' + \cdots \sim \lambda$ with λ like the 't Hooft coupling cst

Fundamental worldsheet excitations

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Non-relativistic dispersion relation: Massless particles can scatter.

Determining the S matrix from $\mathcal{A}$

2-body S matrix fixed by $\mathcal{A}$ up to scalar **dressing factor**

$$\mathcal{S}_{(12)}(p,q)\,\mathbf{Q}_{(12)}(p,q)=\mathbf{Q}_{(12)}(q,p)\,\mathcal{S}_{(12)}(p,q)\,.$$

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$$x_p^+ + \frac{1}{x_p^+} - x_p^- - \frac{1}{x_p^-} = \frac{2i\,|m|}{h}, \qquad \frac{x_p^+}{x_p^-} = e^{ip}.$$

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For example,

$$\mathcal{S}_{(12)}(p,q) \;:\; |\phi_p^L\,, \psi_q^L\rangle \;\longrightarrow\; \frac{x_p^+ - x_q^+}{x_p^- - x_q^+}\;|\psi_q^L\,, \phi_p^L\rangle$$

All-loop Bethe equations for closed string spectrum

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The massive momentum-carrying roots satisfy the equations

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\j\neq k}}^{N_2} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\frac{1-\frac{1}{x_k^+x_j^-}}{1-\frac{1}{x_k^-x_j^+}}(\sigma_{kj}^{\bullet\bullet})^2$$

$$\times \prod_{j=1}^{N_1}\frac{x_k^- - y_{1,j}}{x_k^+ - y_{1,j}}\prod_{j=1}^{N_3}\frac{x_k^- - y_{3,j}}{x_k^+ - y_{3,j}}$$

$$\times \prod_{j=1}^{N_{\bar{2}}}\frac{1-\frac{1}{x_k^+\bar{x}_j^+}}{1-\frac{1}{x_k^-\bar{x}_j^-}}\frac{1-\frac{1}{x_k^+\bar{x}_j^-}}{1-\frac{1}{x_k^-\bar{x}_j^+}}(\widetilde{\sigma}_{kj}^{\bullet\bullet})^2$$

$$\times \prod_{j=1}^{N_0}\frac{x_k^+ - z_j^-}{x_k^- - z_j^+}\left(\frac{1-\frac{1}{x_k^-z_j^-}}{1-\frac{1}{x_k^+z_j^+}}\right)^{\frac{1}{2}}\left(\frac{1-\frac{1}{x_k^+z_j^-}}{1-\frac{1}{x_k^-z_j^+}}\right)^{\frac{1}{2}}(\sigma_{kj}^{\bullet\circ})^2,$$

Groundstates of the all-loop Bethe equations

The energy of a state is given by

$$E = N_2 + N_{\bar{2}} + ih\sum_{k=1}^{N_2}\Big(\frac{1}{x_k^+} - \frac{1}{x_k^-}\Big) + ih\sum_{k=1}^{N_{\bar{2}}}\Big(\frac{1}{\bar{x}_k^+} - \frac{1}{\bar{x}_k^-}\Big) + ih\sum_{k=1}^{N_0}\Big(\frac{1}{z_k^+} - \frac{1}{z_k^-}\Big)\,.$$

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Conclusion:
Protected states are massless zero-momentum magnons.

Protected states of AdS_3/CFT_2 from integrability.

We have conventional BMN groundstate

$$|(\phi^{++})^L\rangle$$

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Adding massless roots with zero momentum get two states

$$|(\phi^{++})^{L-1}\chi_{\mathrm{R}}^{+\pm}\rangle + \text{symmetric permutations},$$

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We have conventional BMN groundstate

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Similarly, for $\chi_{\mathrm{L}}^{+\pm}$. Easy check BEs satisfied for $z^{\pm} = 1$.

Next consider state with two right-moving massless fermions,

$$|(\phi^{++})^{L-2}\chi_{\mathrm{R}}^{++}\chi_{\mathrm{R}}^{+-}\rangle + \text{symmetric permutations},$$

the roots sit at $z^{\pm} = +1$ or $z^{\pm} = -1$. BEs satisfied.

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State	N_0	N_1	N_3	J_L	J_R	$J_\circ$
$(\phi^{++})^L$	0	0	0	$\frac{L}{2}$	$\frac{L}{2}$	0
$(\phi^{++})^{L-1}\chi_R^{+\pm}$	1	0	0	$\frac{L-1}{2}$	$\frac{L}{2}$	$\pm\frac{1}{2}$
$(\phi^{++})^L\ \chi_L^{+\pm}$	1	1	1	$\frac{L+1}{2}$	$\frac{L}{2}$	$\pm\frac{1}{2}$
$(\phi^{++})^{L-2}\chi_R^{++}\chi_R^{+-}$	2	0	0	$\frac{L-2}{2}$	$\frac{L}{2}$	0
$(\phi^{++})^{L-1}\chi_R^{+\pm}\chi_L^{+\pm}$	2	1	1	$\frac{L}{2}$	$\frac{L}{2}$	± 1
$(\phi^{++})^{L-1}\chi_R^{+\pm}\chi_L^{+\mp}$	2	1	1	$\frac{L}{2}$	$\frac{L}{2}$	0
$(\phi^{++})^L\ \chi_L^{++}\chi_L^{+-}$	2	1	1	$\frac{L+2}{2}$	$\frac{L}{2}$	0
$(\phi^{++})^{L-2}\chi_R^{++}\chi_R^{+-}\chi_L^{+\pm}$	3	1	1	$\frac{L-1}{2}$	$\frac{L}{2}$	$\pm\frac{1}{2}$
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This matches sugra and Sym^N $\frac{1}{2}$-BPS states.

Moduli and Integrability

Turning on moduli in $AdS_3 \times S^3 \times T^4$

- Integrability works for backgrounds with:
  RR charges (n.h. D1/D5) [Borsato+Ohlsson Sax+Sfondrini+BS+Torrielli]
  NSNS+RR charges (n.h. D1+F1/D5+NS5)
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Turning on C_0 in NSNS $AdS_3 \times S^3 \times T^4$

Set C_0 to a non-zero constant.

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Since $H \neq 0\,,$ $F_3 \neq 0$, and all other $F = 0$

GS action same as mixed-charge background!

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NSNS string theory integrable for $C_0 \neq 0$ [Ohlsson Sax, BS] point

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Pert. long strings appear - new sector in Hilbert space.

Conclusions and Outlook

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- *Applies to NSNS theory away from "origin" of moduli space
- At origin, NSNS S matrix finite and non-diagonal. Need to understand long string sector.
- Match short strings to Maldacena Ooguri spectrum?
  Relation to low k results ?
  [Giribet+Hull+Kleban+Porrati+Rabinovici,Gaberdiel+Gopakumar]

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Find CFT_2 dual of $\mathrm{AdS}^3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$.

[Gukov *et al.* '05] , [Tong '14], [Eberhardt, Gaberdiel, Li '17]

Thank you