Refining the holographic dictionary with kinematic space

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Based on ArXiv:1708.09838 with A.W Peet, published in JHEP, and ongoing work with Ian Jardine.

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Introduction

Kinematic Space has been introduced as an auxiliary space in AdS/CFT.
(Czech, Lamprou, McCandlish, Sully 1505.05515)
Reorganizes CFT degrees of freedom to reflect diff invariance of the bulk.

Provides new entries into the holographic dictionary.
(Czech, Lamprou, McCandlish, Mosk, Sully 1604.03110)

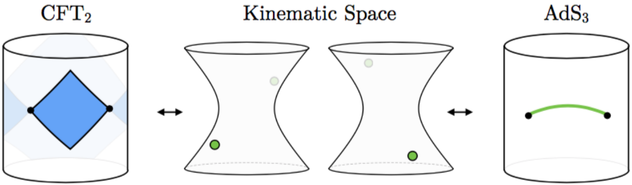
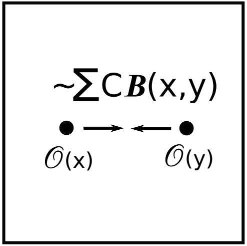


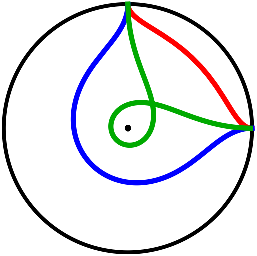
Figure: From 1604.03110 and Sully's talk last Nov.

Today we modify proposal to work in locally AdS spacetimes.

Use it to distinguish contributions of bulk geodesics to OPE.
(cf. Balasubramanian, Chowdhury, Czech, de Boer 1406.5859)

Define CFT quantity dual to bulk field integrated over geodesic.





AdS/CFT is a duality between quantum gravity in asymptotically AdS_{d+1} (Maldacena Nov. 27, 1997)

$$ds^2 = R_{\mathrm{AdS}}^2(-\cosh^2\rho\ dt^2 + d\rho^2 + \sinh^2\rho\ d\Omega_{d-1}^2), \tag{1}$$

and states of QFT with conformal symmetry in d dimensions.

The CFT lives at the timelike boundary $\rho \to \infty$.

AdS boundary has $SO(d,2)$ isometry - matches d Conformal group.

Duality at level of degrees of freedom, but organization is totally different.

Some dual relations are fairly simple:

Light field in bulk $\iff$ Local CFT operator with same spin

e.g. Massive scalar $\varphi(x)$ $\iff$ Spin zero $\mathcal{O}$ with dimension $m^2 = \Delta(d - \Delta)$.

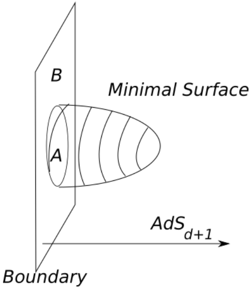
e.g. Graviton $g_{\mu\nu}$ $\iff$ CFT stress tensor $T_{\mu\nu}$, spin 2, $\Delta = d$.

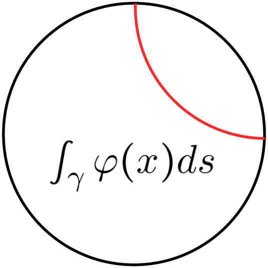
But local bulk field like $\varphi(x)$ is not a true observable - not gauge invariant.

Diff invariance needs gravitational dressing, like Wilson line in gauge theory.

Some dual relations are already diff invariant

Area of minimal surface attached to boundary region $A \quad \Longleftrightarrow$
Entanglement entropy of A with its complement, $S(\rho_A) = -\mathrm{tr}_A(\rho_A \log \rho_A)$
(Ryu & Takayanagi hep-th/0603001):



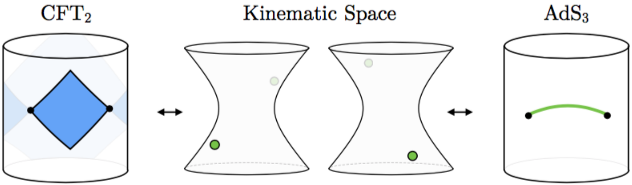


Gravitationally dressed field can be constructed by geodesic Wilson line.
Here we consider a bulk field integrated over geodesic (AdS_3/CFT_2).

What is the CFT dual of $\int_\gamma \varphi(x)ds$?

Difficult question, CFT degrees of freedom do not manifest diff invariance.

Kinematic Space reorganizes CFT degrees of freedom to reflect diff invariance of the bulk.



KS is space of boundary anchored geodesics, or pairs of CFT points.

For pure AdS, unique geodesic for each pair of spacelike boundary points.

Integral Geometry and Kinematic Space

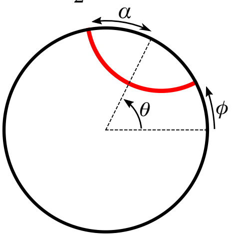
Space of boundary anchored geodesics appears in Integral Geometry.

Differential Geometry: fundamental quantity is the metric.
Integral Geometry: Crofton 2-form ω determines the geometry

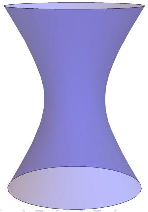
$$L(c) = \frac{1}{4}\iint_{\gamma} n_{\gamma,c}\,\omega. \qquad \text{eg. } \mathbb{H}_2: \quad \omega = -\frac{1}{\sin^2\alpha} d\alpha \wedge d\theta. \tag{2}$$

ω is the metric on the space of geodesics - "kinematic space".
Each geodesic, parameterized by angles (α, θ) is a point on KS.

For $\mathbb{H}_2$ we find kinematic space to be de Sitter$_2$



$$ds^2_{KS} = \frac{1}{\sin^2\alpha}[-d\alpha^2 + d\theta^2]$$



$$
\begin{aligned}
ds^2_{\mathbb{H}} &= d\rho^2 + \sinh^2\rho\ d\phi^2, \\
ds^2_{KS} &= \frac{1}{\sin^2\alpha}\left[-d\alpha^2 + d\theta^2\right] = \frac{1}{\sin^2(v-u)/2}dudv, \\
ds^2_{CFT} &= -dt^2 + d\phi^2 = dudv
\end{aligned}
\tag{3}
$$

Recap: Three roles for pair of boundary points (u, v)

- Uniquely identifies a bulk geodesic
- Set of null coordinates on kinematic space
- Defines an interval/subregion on CFT timeslice

Geodesics allow construction of diff invariant observables, use kinematic space to translate to CFT.

(de Boer, Haehl, Heller, Myers 1606.03307)

Towards diff-invariant dictionary entries

A pair of CFT points is a point on KS.
A CFT quantity depending on two points could be a field on KS.

Simplest option is a product of CFT operators - the OPE.

Product expanded in basis of local quasiprimary operators and descendants

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k C_{ijk}\,|x|^{\Delta_k-\Delta_i-\Delta_j}\left(1+b_1\,x^\mu\partial_\mu+b_2\,x^\mu x^\nu\partial_\mu\partial_\nu+\dots\right)\mathcal{O}_k(0). \tag{4}$$

Coefficients b_i fixed by conformal symmetry,

C_{ijk} and spectrum Δ_k of primaries are CFT data.

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k C_{ijk}\,|x|^{\Delta_k-\Delta_i-\Delta_j}\left(1 + b_1\,x^\mu\partial_\mu + b_2\,x^\mu x^\nu\partial_\mu\partial_\nu + \dots\right)\mathcal{O}_k(0). \tag{5}$$

Repackage part of expression fixed by symmetry.

Contribution to OPE from a conformal family is an OPE block

$$\mathcal{O}_i\left(x_1\right)\mathcal{O}_i\left(x_2\right) = \left|x_1 - x_2\right|^{-2\Delta_i}\sum_k C_{iik}\mathcal{B}_k\left(x_1, x_2\right). \tag{6}$$

Depends on pair of insertion points, and is candidate for field on KS.

Let's study its properties.

CFT operators are in representations of conformal group labelled by dimension Δ_k, and spin l_k.

These are eigenvalues of Casimirs C_2 built from symmetry generators $\mathcal{L}_{AB}$

$$[\mathcal{C}_2, \mathcal{O}_k(x)] = -\frac{1}{2}\mathcal{L}^{AB}\mathcal{L}_{AB}\mathcal{O}_k(x) = [\Delta_k(\Delta_k - d) - l_k(l_k + d - 2)]\mathcal{O}_k(x). \quad (7)$$

$\mathcal{C}_2$ commutes with conformal generators $\implies$ descendants satisfy (7) with same eigenvalue.

For the same reason, the OPE block of quasiprimary $\mathcal{O}_k$ satisfies

$$[\mathcal{C}_2, \mathcal{B}_k] = [\Delta_k(\Delta_k - d) - l_k(l_k + d - 2)]\mathcal{B}_k. \quad (8)$$

In terms of standard conformal generators $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

$$\mathcal{C}_2 = -2L_0^2 + \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) + (L \to \bar{L}). \quad (9)$$

Casimir equations are 2nd order DE's. Need to express the generators $L_{0,\pm 1}$ in a differential representation.

Warm-up: Scalar $\mathcal{O}_k$ dual to massive scalar field φ_k.
Conformal generators in scalar repn

$$\mathcal{L}_{-1} = \partial_\xi, \quad \mathcal{L}_0 = -\xi\partial_\xi - \frac{1}{2}\Delta_k, \quad \mathcal{L}_1 = \xi^2\partial_\xi + \xi\Delta_k, \tag{10}$$

correspond to AdS Killing vectors,

$$\eta_{-1} = \partial_\xi, \quad \eta_0 = -\xi\partial_\xi - \frac{1}{2}z\partial_z, \quad \eta_1 = \xi^2\partial_\xi + \xi z\partial_z - z^2\partial_{\bar{\xi}}. \tag{11}$$

Scale/Radius duality tells us $\Delta_k \to z\partial_z$. Casimir equation for $\mathcal{O}_k$ becomes

$$[\mathcal{C}_2, \mathcal{O}_k(x)] = \Delta_k(\Delta_k - 2)\mathcal{O}_k, \implies -\Box_{AdS}\,\varphi = -m^2\varphi. \tag{12}$$

OPE block $\mathcal{B}_k$ is in a different representation.
In this repn. $\mathcal{C}_2 = -\frac{1}{2}\mathcal{L}^{AB}\mathcal{L}_{AB}$ becomes

$$[\mathcal{C}_2, \mathcal{B}_k(x_1, x_2)] = -\frac{1}{2}(\mathcal{L}_1^{AB} + \mathcal{L}_2^{AB})(\mathcal{L}_{AB,1} + \mathcal{L}_{AB,2})\mathcal{B}_k(x_1, x_2)\,. \quad (13)$$

Leads to the differential equation

$$[\mathcal{C}_2, \mathcal{B}_k(x_1, x_2))] = -4\sin^2\alpha\left(-\partial_\alpha^2 + \partial_\theta^2\right)\mathcal{B}_k(x_1, x_2) \quad (14)$$

which is the scalar Laplacian on the dS_2 kinematic space: $\mathcal{C}_2 = -\Box_{dS}$,

$$-\Box_{dS}\,\mathcal{B}_k(x_1, x_2) = \Delta_k(\Delta_k - 2)\mathcal{B}_k \quad (15)$$

Conclusion: OPE block $\mathcal{B}_k$ is a massive scalar field on KS.
Sully's talk: $\mathcal{B}_k(\gamma) \leftrightarrow \int_\gamma \varphi_k(x)ds$
(Czech, Lamprou, McCandlish, Mosk, Sully 1604.03110)

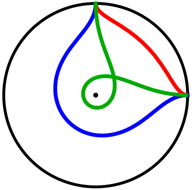
Discussion so far was entirely in pure AdS_3 dual to ground state of CFT_2

$$ds^2 = -\cosh^2\rho \; dt^2 + d\rho^2 + \sinh^2\rho \; d\phi^2. \quad (16)$$

1-to-1 correspondence of boundary intervals, bulk geodesics.

This is emphatically not the case for most geometries.
e.g. Conical defect spacetime, massive particle sitting at origin

$$ds^2 = -\cosh^2\rho \; dt^2 + d\rho^2 + \sinh^2\rho \; d\tilde{\phi}^2, \quad \tilde{\phi} \in (0, 2\pi/N) \quad (17)$$



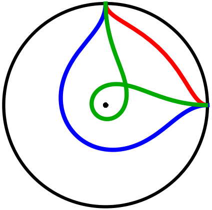
Can kinematic space still be defined consistently? What is the role of non-minimal geodesics? (cf. Asplund, Callebaut, Zukowski 1604.02687)

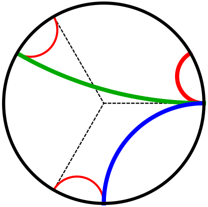
Kinematic Space for Conical Defects

$$ds^2 = -\cosh^2\rho\ dt^2 + d\rho^2 + \sinh^2\rho\ d\tilde{\phi}^2, \quad \tilde{\phi} \in (0, 2\pi/N). \qquad (18)$$

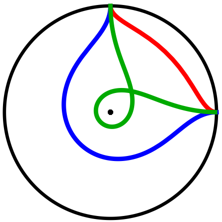
Simple quotient of AdS_3 along the angular Killing vector ∂_ϕ, $\phi = \phi + 2\pi/N$.
Breaks $SO(2,2)$ isometry group to $\partial_{\tilde{\phi}},\ \partial_t$.

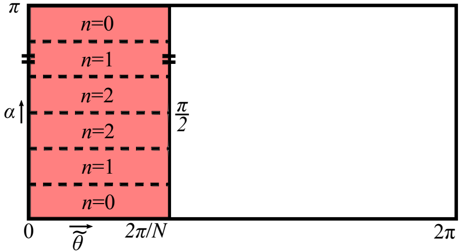
Quotient acts on geodesics through identifications as well.
Each geodesic comes from an equivalence class of geodesics in AdS_3





Identifications on geodesics change the kinematic space
(Zhang & Chen 1610.07134):





Consists of geodesics with centre angle $\tilde{\theta} \in (0, 2\pi/N)$ - angular quotient.

Includes non-minimal geodesics, winding number $n > 0$ and $\alpha > \pi/2N$.

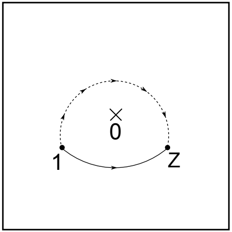
Can still say KS = space of boundary anchored geodesics.
Cannot say KS = space of pairs of CFT points.

KS Metric no longer fixed by conformal symmetry, need new strategy!
Use OPE blocks - Casimir equation implies the Laplacian on KS.

Dual of conical defect is an excited state.
(Lunin, Mathur, Saxena hep-th/0211292)
Could consider OPE in excited background.

Excited state obtained from conformal map, has branch point.

Operators can be "exchanged" in several channels. OPE sees all.
(Asplund, Bernamonti, Galli, Hartman 1410.1392)



$$\tilde{\mathcal{O}}_i(x_1)\,\tilde{\mathcal{O}}_i(x_2) = |x_1 - x_2|^{-2\Delta_i} \sum_k C_{iik} \tilde{\mathcal{B}}_k(x_1, x_2)\,. \tag{19}$$

Upshot is: this OPE block contains contributions from all geodesics connecting insertion points $\tilde{\mathcal{B}}_k(x_1, x_2) \leftrightarrow \sum_\gamma \int_\gamma \varphi_k ds$.

Can we refine this statement by isolating geodesic contributions?

Lift to covering CFT - excited state maps to ground state.
Base CFT lives on $\tilde{\phi} \in (0, 2\pi/N)$ cylinder, covering CFT on $\phi \in (0, 2\pi)$.

Theories are not the same. Base CFT has discrete gauge symmetry $\mathbb{Z}_N$.
(Balasubramanian, Bernamonti, Craps, De Jonckheere, Galli 1609.03991)

Cover CFT "ungauges" discrete symmetry. $\mathbb{Z}_N$ symmetric observables on cover are valid observables in the base.

Example: Scalar base operators $\tilde{\mathcal{O}}$ are symmetrized cover operators $\mathcal{O}$ with same dimension Δ

$$\tilde{\mathcal{O}}_k\left(t, \tilde{\phi}\right) = \frac{1}{N} \sum_{m=0}^{N-1} \exp\left(i \frac{2\pi m}{N} \frac{\partial}{\partial \phi}\right) \mathcal{O}_k\left(t, \phi\right). \tag{20}$$

Insertions are placed symmetrically around the circle, result is $\mathbb{Z}_N$ symmetric.

$\exp\left(i \frac{2\pi m}{N} \frac{\partial}{\partial \phi}\right)$ is a finite rotation, made from conformal generators.

Take $\tilde{\mathcal{O}}_i$ and form the base OPE $\tilde{\mathcal{O}}_i \tilde{\mathcal{O}}_i$. Involves cover OPE $\mathcal{O}_i \mathcal{O}_i$

OPE of base operators = double sum over OPE of cover operators. Sums can be rearranged. Define partial OPE blocks $\mathcal{B}_{k,m}$

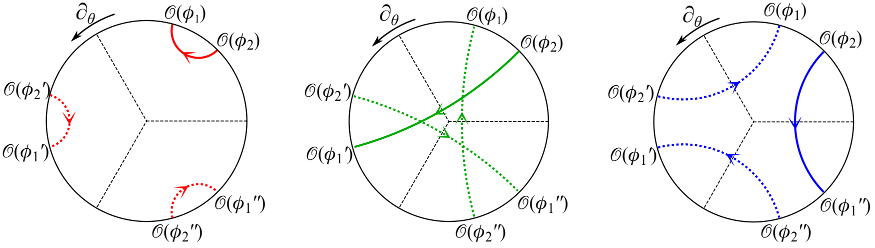
$$\mathcal{B}_{k,m}\left(t,\alpha,\theta\right) = \frac{1}{N}\left|2-2\cos(2\alpha)\right|^{-\Delta}\sum_{b=0}^{N-1}\exp\left(i\frac{2\pi b}{N}\frac{\partial}{\partial\theta}\right)\mathcal{B}_{k}\left(t,\alpha,\theta\right), \quad (21)$$

This is $\mathbb{Z}_N$ symmetric - valid observable in base CFT.

The base OPE receive contributions from all partial OPE blocks

$$\tilde{\mathcal{O}}_i(t,\tilde{\phi}_1)\tilde{\mathcal{O}}_i(t,\tilde{\phi}_2) = \sum_k C_{iik}\frac{1}{N}\sum_{m=0}^{N-1}\exp\left(i\frac{2\pi m}{N}\frac{\partial}{\partial\phi_1}\right)\mathcal{B}_{k,m}(t,\alpha_m,\theta). \quad (22)$$

$\mathcal{B}_{k,m}$ is a new finer-grained observable, but what does it represent?



(a) $\mathcal{B}_{k,m=0},\ n = 0$ (b) $\mathcal{B}_{k,m=1},\ n = 2$ (c) $\mathcal{B}_{k,m=2},\ n = 1$

$$\mathcal{B}_{k,m}\left(t,\alpha_m,\theta\right) \sim \sum_{b=0}^{N-1} \exp\left(i\frac{2\pi b}{N}\frac{\partial}{\partial\theta}\right)\mathcal{B}_k\left(t,\alpha_m,\theta\right). \tag{23}$$

Partial OPE blocks - contribution of cover operator pairs at fixed distance. Corresponds to choice of OPE channel on punctured plane.

$$\mathcal{B}_{k,m} \leftrightarrow \int_{\gamma_n} \varphi_k(x)ds$$

Bulk field integrated over single geodesic.

Cover OPE block $\mathcal{B}_k$ satisfies Casimir equation $[\mathcal{C}_2, \mathcal{B}_k] = \Delta_k(\Delta_k - 2)\mathcal{B}_k$.

$\mathcal{C}_2$ commutes with conformal generators, hence partial blocks $\mathcal{B}_{k,m} \sim \sum \exp(i\frac{\partial}{\partial \phi_1})\mathcal{B}_k$ satisfy the same equation

$$[\mathcal{C}_2, \mathcal{B}_{k,m}] = \Delta_k(\Delta_k - 2)\mathcal{B}_{k,m} \tag{24}$$

$\mathcal{C}_2$ in same representation as before, so

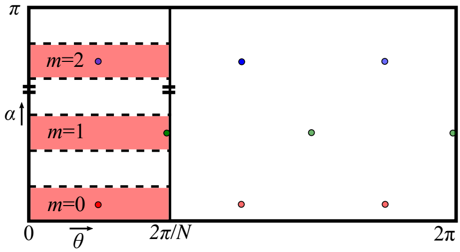
$$-\Box_{dS_2}\mathcal{B}_{k,m}(\alpha, \theta) = \Delta_k(\Delta_k - 2)\mathcal{B}_{k,m}(\alpha, \theta). \tag{25}$$

i.e. $\mathcal{B}_{k,m}$ is a scalar field on kinematic space.

But coordinates $\theta \in (0, 2\pi/N)$ and $\alpha \in (\alpha_m, \alpha_m + \pi/2N)$ are restricted.

Each $\mathcal{B}_{k,m}(\alpha_m, \theta)$ is a scalar field on a small region of dS_2

$$\Box_{dS_2} \mathcal{B}_{k,m}(\alpha_m, \theta) = m^2 \mathcal{B}_{k,m}(\alpha_m, \theta). \tag{26}$$



The base OPE is a sum over all $\mathcal{B}_{k,m}(\alpha_m, \theta)$. The resulting KS is the union of each small block.

Summary

Kinematic space reorganizes CFT degrees of freedom to reflect the diffeomorphism invariance of the bulk.

It provides a new duality in AdS/CFT between OPE blocks and geodesic operators

$$\mathcal{B}_k(\alpha,\theta) \leftrightarrow \int_{\Gamma(\alpha,\theta)} \varphi(x) ds, \tag{27}$$

Also facilitates easy re-derivations of many recent results:
Geodesic Witten diagrams ↔ conformal blocks
(Hijano, Kraus, Perlmutter, Snively 1508.00501)

Einstein equations from entanglement first law
(Faulkner, Guica, Hartman, Myers, Van Raamsdonk 1312.7856)

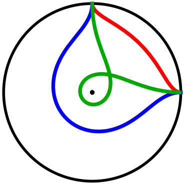
Bulk reconstruction of φ by inverting $R[\varphi](\Gamma)$
(Hamilton, Kabat, Lifschytz, Lowe hep-th/0606141)

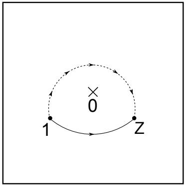
Original results only apply in pure AdS. Extended to locally AdS spacetime.

Defined a new CFT observable $\mathcal{B}_{k,m}$ in states dual to conical defects.

A defect in the bulk implies a branch point in the CFT.
Non-minimal geodesics in the bulk, OPE channels on different branches.

$\mathcal{B}_{k,m}$ isolates contributions of single OPE channel.
$\mathcal{B}_{k,m}$ is a scalar field on a small subregion of dS_2.
$\mathcal{B}_{k,m}$ is dual to $\int_\gamma ds\ \varphi_{CD}(\tilde{x})$, bulk field integrated over a single geodesic.





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